# 6.7 - The Binomial Theorem

You have actually used this theorem before...

$(a + b)^2$

$= (a+b)(a+b)$

$= a^2 + 2ab + b^2$

$(a + b)^3$

$= (a+b)(a+b)(a+b)$

$= (a^2+2ab+b^2)(a+b)$

$= a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3$

$= 1a^3 + 3a^2b + 3ab^2 + 1b^3$

Investigate the expansion of:

$(a + b)^4 = (a+b)^2(a+b)^2$

$= (a^2+2ab+b^2)(a^2+2ab+b^2)$

$= a^4 + 2a^3b + a^2b^2 + 2a^3b + 4a^2b^2 + 2ab^3 + a^2b^2 + 2ab^3 + b^4$

$= 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

In general:

| *n* value (exponent) | $(a + b)^n$ | Coefficients |
|---|---|---|
| 0 | $(a+b)^0$ | 1 |
| 1 | $(a+b)^1 = a+b$ | 1 1 |
| 2 | $(a+b)^2 = a^2+2ab+b^2$ | 1 2 1 |
| 3 | $(a+b)^3 = a^3+3a^2b+3ab^2+1$ | 1 3 3 1 |
| 4 | $(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$ | 1 4 6 4 1 |

## The Binomial Theorem

The binomial theorem is used to expand $(a + b)^n$.

The coefficients are the values of row n in Pascal's Triangle (where $n$ is the exponent to which the binomial is being raised).

$(a + b)^n$

$$= \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}a^0 b^n$$

Ex. 1 Expand each of the following :

a) $(a + b)^6$

$$=\binom{6}{0}a^6b^0+\binom{6}{1}a^5b^1+\binom{6}{2}a^4b^2+\binom{6}{3}a^3b^3+\binom{6}{4}a^2b^4+\binom{6}{5}a^1b^5+\binom{6}{6}a^0b^6$$

$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

b) $(x - 2)^4$ $\quad a = x$, $b = -2$

$$=\binom{4}{0}x^4(-2)^0+\binom{4}{1}x^3(-2)^1+\binom{4}{2}x^2(-2)^2+\binom{4}{3}x^1(-2)^3+\binom{4}{4}x^0(-2)^4$$

$$= x^4 + 4x^3(-2) + 6x^2(4) + 4x(-8) + 16$$

$$= x^4 - 8x^3 + 24x^2 - 32x + 16$$

c) The first 3 terms of

$(3x+2y)^5$ $a = 3x$, $b = 2y$

$$= \binom{5}{0}(3x)^5(2y)^0 + \binom{5}{1}(3x)^4(2y)^1 + \binom{5}{2}(3x)^3(2y)^2 + \ldots\ldots$$

$$= 1\cdot 243x^5 + 5\cdot 81x^4\cdot 2y + 10\cdot 27x^3\cdot 4y^2 + \ldots\ldots$$

$$= 243x^5 + 810x^4y + 1080x^3y^2 + \ldots\ldots$$

d) $\left(x^2 + \frac{4}{x}\right)^3$ $a = x^2$, $b = \frac{4}{x}$

$$= \binom{3}{0}(x^2)^3\left(\frac{4}{x}\right)^0 + \binom{3}{1}(x^2)^2\left(\frac{4}{x}\right)^1 + \binom{3}{2}(x^2)^1\left(\frac{4}{x}\right)^2 + \ldots\ldots$$

$$= x^6 + 3\cdot x^4\left(\frac{4}{x}\right) + 3\cdot x^2\cdot \frac{16}{x^2} + \ldots\ldots$$

$$= x^6 + 12x^3 + 48 + \ldots\ldots$$

e) The first four terms of

$$\left(\sqrt{x}-\frac{2}{\sqrt{x^3}}\right)^8$$

$a=\sqrt{x}$
$= x^{\frac{1}{2}}$

$b = -\frac{2}{\sqrt{x^3}}$
$= -2x^{-\frac{3}{2}}$

$\left\{ \sqrt{x^3} = (x^3)^{\frac{1}{2}} = x^{\frac{3}{2}} \right.$

$$=\binom{8}{0}(x^{\frac{1}{2}})^8(-2x^{-3/2})^0+\binom{8}{1}(x^{\frac{1}{2}})^7(-2x^{-\frac{3}{2}})^1+\binom{8}{2}(x^{\frac{1}{2}})^6(-2x^{-3/2})^2+\binom{8}{3}(x^{\frac{1}{2}})^5(-2x^{-3/2})^3$$

$$= x^4 + 8\cdot x^{7/2}\cdot(-2)x^{-3/2} + 28\cdot x^3\cdot 4\,x^{-3} + 56x^{\frac{5}{2}}\cdot(-8)\,x^{-\frac{9}{2}} + \ldots$$

$$= x^4 - 16x^2 + 112 - 448x^{-2} + \ldots$$

$$= x^4 - 16x^2 + 112 - \frac{448}{x^2} + \ldots$$

Ex. 2 Write $\underbrace{1 + 12x^3 + 54x^6 + 108x^9 + 81x^{12}}_{5 \text{ terms}}$ in the form $(a+b)^n$

$\therefore n=4$

1st Term

$\binom{4}{0}a^4 = 1$

$a^4 = 1$

$\therefore a = \pm 1$

Last Term

$\binom{4}{4}b^4 = 81x^{12}$

$b^4 = 81x^{12}$

$b = \pm\sqrt[4]{81x^{12}}$

$= \pm 3x^3$

Test: $a=1$, $b=3x^3$

$(1+3x^3)^4$

$\binom{4}{1}(1)^3(3x^3)^1$

$= 4\cdot 3x^3$

$= 12x^3$

✓

$(1-3x^3)^4$

$\binom{4}{1}(1)^3(-3x^3)$

$= 4\cdot 1\cdot(-3x^3)$

$= -12x^3$

✗

$\therefore (1+3x^3)^4$

**HOMEWORK**
**p. 344 #C4, 5acef, 6, 7, 19**