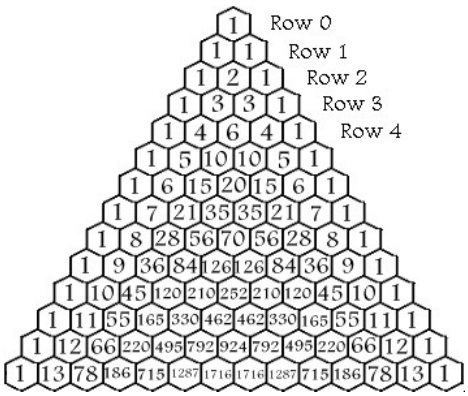
6.6 - Pascal's Triangle

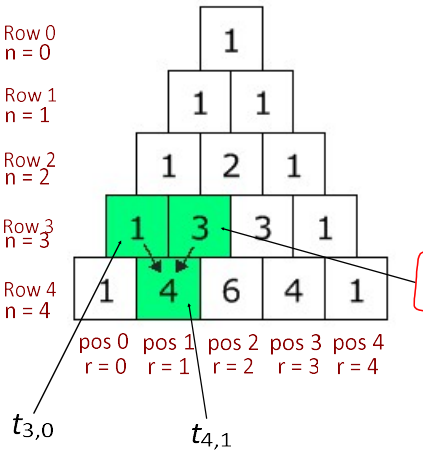


This is Pascal's Triangle.

Each term is equal to the sum of the two terms immediately above it.

The term at the beginning and end of each row is 1.

Pascal's Triangle is named after mathematician, Blaise Pascal (1623-1662).



Rows and positions in rows are numbered starting from ZERO.

Rows are denoted by the letter *n*.
Positions within rows are denoted by the letter *r*.

Notice that $t_{4,1} = t_{3,0} + t_{3,1}$

row, position

As terms we have:

$$t_{0,0}$$
$$t_{1,0} \quad t_{1,1}$$
$$t_{2,0} \quad t_{2,1} \quad t_{2,2}$$
$$t_{3,0} \quad t_{3,1} \quad t_{3,2} \quad t_{3,3}$$
$$\text{etc...}$$

$t_{n,r}$ represents the term in row *n*, position *r*.

$t_{n,r} = t_{n-1,r-1} + t_{n-1,r}$

Ex. 1 Express $t_{4,3}$ as the sum of the terms directly above it.

$t_{4,3} = t_{4-1,3-1} + t_{4-1,3}$

$t_{4,3} = t_{3,2} + t_{3,3}$

Ex. 2 The first 4 numbers in row 10 of Pascal's triangle are: 1, 10, 45, 120

a) Determine the first 4 numbers of row 11.

Row 10 1, 10, 45, 120

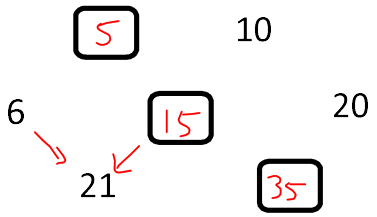
Row 11 1, 11, 55, 165

b) Determine the first 4 numbers of row 9.

Row 9 1, 9, 36, 84

Row 10 1, 10, 45, 120

Ex. 3 Fill in the blanks for this section of the Pascal's Triangle.



Ex. 4 Using the diagram of Pascal's Triangle:

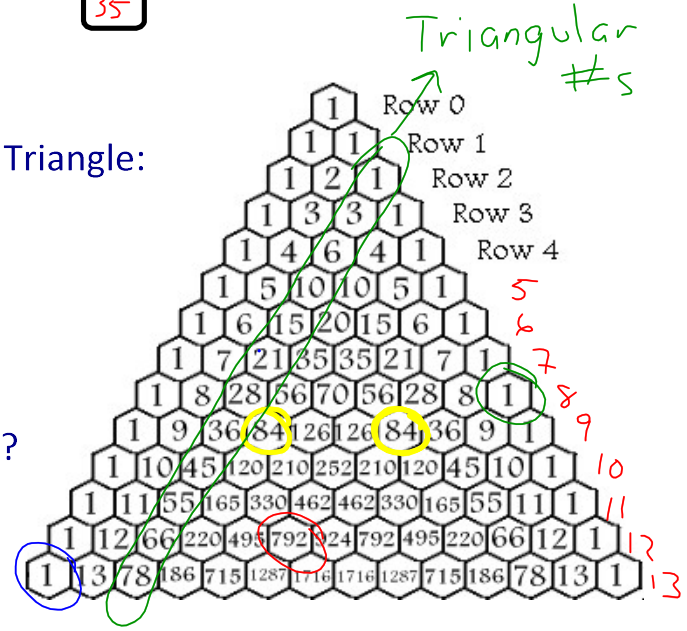
a) State the following terms:

i) $t_{12,5}$ ii) $t_{13,0}$ iii) $t_{8,8}$

792 1 1

b) Which term has a value of 84?

$t_{9,3}$ or $t_{9,6}$



Properties and Applications of Pascal's Triangle

A: ROW SUMS

The sum of the terms in any row *n* in Pascal's Triangle is 2^n.

Ex. 5 What is the sum of the numbers in row 9 of Pascal's Triangle?

$2^9 = 512$

Ex. 6 Which row in Pascal's Triangle has a sum of 4096?

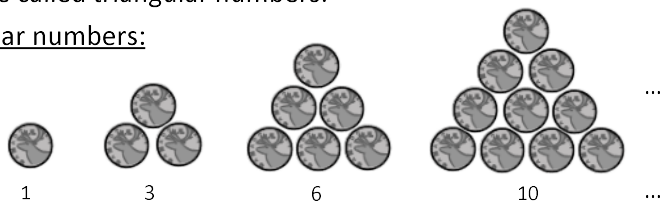
$2^n = 4096 \quad \therefore n = 12$

$2^n = 2^{12}$

B: TRIANGULAR NUMBERS

Numbers that correspond to the number of items stacked in a triangular array are called triangular numbers.

Triangular numbers:



The list of triangular numbers is in the third diagonal of Pascal's Triangle.

# of coins with *n* rows = $t_{n+1,2}$

Ex. 7 How many coins are in a triangle having 6 rows? $t_{n+1,2}$

$= t_{7,2}$

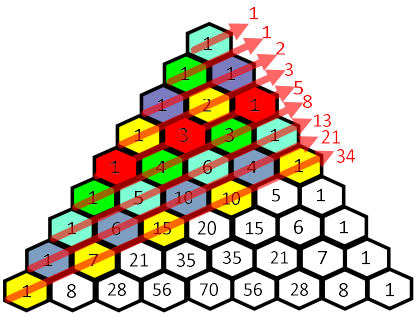
$= 21$

C: FIBONACCI NUMBERS

The Fibonacci Sequence begins with 1, 1, and then each subsequent term is found by taking the sum of the previous two terms.

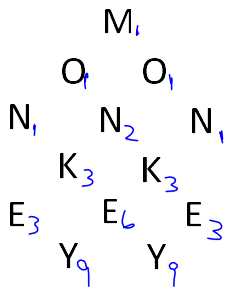
1, 1, 2, 3, 5 ,8, 13, 21, 34, 55, 89, 144, 233, 377,...

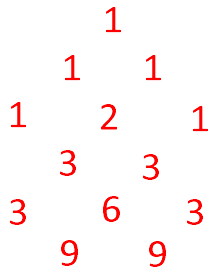
The Fibonacci Numbers are found by adding the terms of the diagonals of Pascal's Triangle as shown.



Pascal's Triangle is used to determine the number of paths between points.

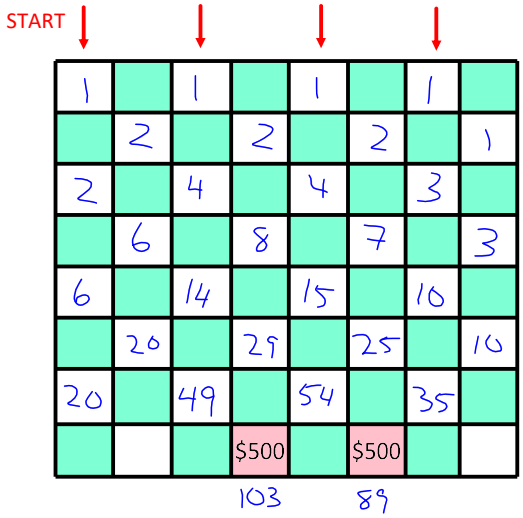
Ex. 8 How many paths will spell MONKEY by starting at the top and proceeding downwards, moving diagonally left or right?





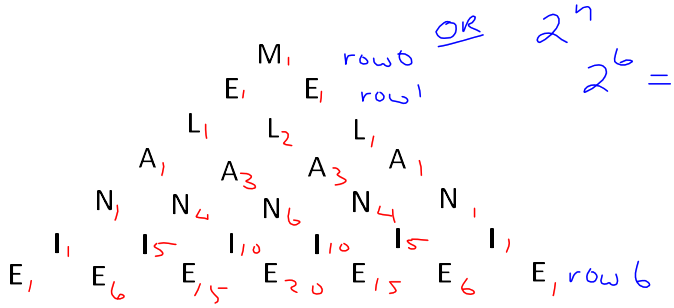
Total = 9+9 = 18 ways

Ex. 9 In Plinko, a token slides down a board. If the token cannot go through a shaded square, in how many ways could you win $500?



∴Total # of ways = 103+89
= 192

Ex. 10 How many paths will spell MELANIE by starting at the top and proceeding downwards, moving diagonally left or right?



OR 2^n

$2^6 = 64$

Total = 1+6+15+20+15+6+1
= 64

Pascal's Triangle is used in a very important theorem called the Binomial Theorem that we will study tomorrow. Consider the following as a precursor...

There must be a more efficient way of finding a specific number within Pascal's Triangle without having to draw it. The method is called Combinatorics.

Binomial Distribution Combinations Explained

Definition of a Combination $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Factorial: The product of all natural numbers less than or equal to n.

Pronounced "n choose r"

"$_nC_r$"

Evaluate:

What is the value of *5!*

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

In this course, we will use the **nCr** button on your calculator instead of the formal algebraic definition.

> In the case of Pascal's Triangle,
> **n CHOOSE r**
> means find the number that is in row **n**, position **r** .

Ex. 11 Evaluate the following combinations.

a) $\binom{7}{3}$ $_7C_3 = 35$ b) $\binom{6}{0} = 1$ c) $\binom{4}{4} = 1$ d) $_9C_7 = 36$

Ex. 12 Find the value of the term in row 4, position 3 of Pascal's Triangle.

$\binom{4}{3} = 4$

Ex. 13 Find the value of the term in the 6th row and 3rd position of Pascal's Triangle.

$n=5$ $r=2$

$\binom{5}{2} = 10$

Ex. 14 a) Write out the first four rows of Pascal's Triangle.

1
1 1
1 2 1
1 3 3 1

b) Express the rows and positions of these rows using recursive sequences with $t_{n,r}$ and create a formula for generating new terms of Pascal's Triangle sequence .

$t_{0,0}$
$t_{1,0}$ $t_{1,1}$
$t_{2,0}$ $t_{2,1}$ $t_{2,2}$
$t_{3,0}$ $t_{3,1}$ $t_{3,2}$ $t_{3,3}$

$t_{n,r} = t_{n-1,r-1} + t_{n-1,r}$

c) Express the rows as combinations.

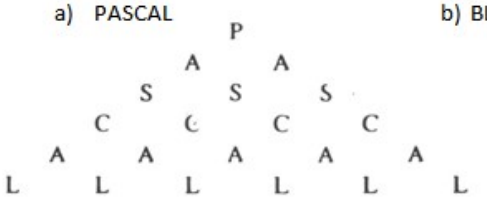
$\binom{0}{0}$
$\binom{1}{0}$ $\binom{1}{1}$
$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$
$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

HOMEWORK

p.377 # C1, C2, 1-4, 8-12, 17

2. In the arrangement of the letters given, how many different paths will spell each of the following names?

a) PASCAL



b) BLAISE

c) EULER

10. Evaluate the following:

a) $\binom{8}{6}$ b) $\binom{5}{5}$ c) $\binom{4}{0}$

11. Find the number in Pascal's Triangle given the following information about the row and position:

a) Row 12, Position 9
b) Row 1, Position 0
c) 3rd Row, 2nd Position
d) Ninth Row, Ninth Position

Answers

2. a) 32 b) 20 c) 4

10. a) 28 b) 1 c) 1

11. a) 220 b) 1 c) 2 d) 1