CHAPTER 1
Functions: Characteristics and Properties

Getting Started, p. 2

1. $f(x) = x^2 + 3x - 4$

a) $f(2) = (2)^2 + 3(2) - 4$
$= 4 + 6 - 4$
$= 6$

b) $f(-1) = (-1)^2 + 3(-1) - 4$
$= 1 - 3 - 4$
$= -6$

c) $$f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^2 + 3\left(\frac{1}{4}\right) - 4$$
$$= \frac{1}{16} + \frac{3}{4} - 4$$
$$= -\frac{51}{16}$$

d) $f(a + 1) = (a + 1)^2 + 3(a + 1) - 4$
$= (a + 1)(a + 1) + 3a + 3 - 4$
$= a^2 + 2a + 1 + 3a - 1$
$= a^2 + 5a$

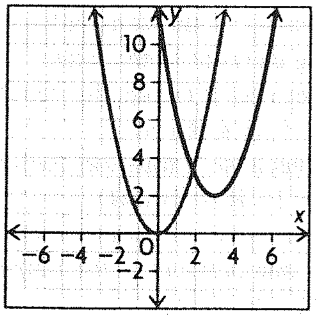
2. a) $x^2 + 2xy + y^2 = (x + y)(x + y)$

b) $5x^2 - 16x + 3 = 5x^2 - 15x - 1x + 3$
$= 5x(x - 3) + (-1)(x - 3)$
$= (5x - 1)(x - 3)$

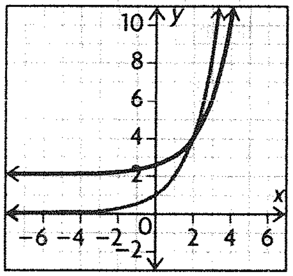
c) $(x + y)^2 - 64 = (x + y)^2 - (8)^2$
$= (x + y + 8)(x + y - 8)$

d) $ax + bx - ay - by = x(a + b) + (-y)(a + b)$
$= (a + b)(x - y)$

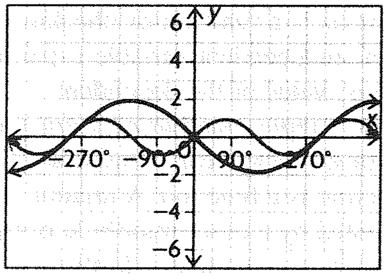
3. a) horizontal translation 3 units to the right, vertical translation 2 units up;



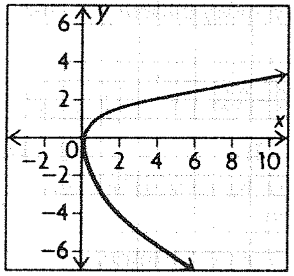
b) horizontal translation 1 unit to the right, vertical translation 2 units up;



c) horizontal stretch by a factor of 2, vertical stretch by a factor of 2, reflection across the x-axis;



d) horizontal compression by a factor of $\frac{1}{2}$, vertical stretch by a factor of 2, reflection across the x-axis;



4. a) $D = \{x \in \mathbf{R} \mid -2 \le x \le 2\}$,
$R = \{y \in \mathbf{R} \mid 0 \le y \le 2\}$

b) $D = \{x \in \mathbf{R}\}$, $R = \{y \in \mathbf{R} \mid y \ge -19\}$

c) $D = \{x \in \mathbf{R} \mid x \ne 0\}$, $R = \{y \in \mathbf{R} \mid y \ne 0\}$

d) $D = \{x \in \mathbf{R}\}$, $R = \{y \in \mathbf{R} \mid -3 \le y \le 3\}$

e) $D = \{x \in \mathbf{R}\}$, $R = \{y \in \mathbf{R} \mid y > 0\}$

5. a) This is not a function; it does not pass the vertical line test.
b) This is a function; for each x-value, there is exactly one corresponding y-value.
c) This is not a function; for each x-value greater than 0, there are two corresponding y-values.
d) This is a function; for each x-value, there is exactly one corresponding y-value.
e) This is a function; for each x-value, there is exactly one corresponding y-value.

6. a) $y = x^3$
$y = 2^3$
$y = 8$

b) $y = x^3$
$20 = x^3$
$\sqrt[3]{20} = x$
$2.71 \doteq x$

7. If a relation is represented by a set of ordered pairs, a table, or an arrow diagram, one can determine if the relation is a function by checking that each value of the independent variable is paired with no more than one value of the dependent variable. If a relation is represented using a graph or scatter plot, the vertical line test can be used to determine if the relation is a function. A relation may also be represented by a description/rule or by using function notation or an equation. In these cases, one can use reasoning to determine if there is more than one value of the dependent variable paired with any value of the independent variable.

1.1 Functions, pp. 11–13

1. a) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} | -4 \le y \le -2\}$; This is a function because it passes the vertical line test.
b) $D = \{x \in \mathbf{R} | -1 \le x \le 7\}$;
$R = \{y \in \mathbf{R} | -3 \le y \le 1\}$; This is a function because it passes the vertical line test.
c) $D = \{1, 2, 3, 4\}$; $R = \{-5, 4, 7, 9, 11\}$; This is not a function because 1 is sent to more than one element in the range.
d) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R}\}$; This is a function because every element in the domain produces exactly one element in the range.
e) $D = \{-4, -3, 1, 2\}$; $R = \{0, 1, 2, 3\}$; This is a function because every element of the domain is sent to exactly one element in the range.
f) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} | y \le 0\}$; This is a function because every element in the domain produces exactly one element in the range.

2. a) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} | y \le -3\}$; This is a function because every element in the domain produces exactly one element in the range.
b) $D = \{x \in \mathbf{R} | x \ne -3\}$; $R = \{y \in \mathbf{R} | y \ne 0\}$; This is a function because every element in the domain produces exactly one element in the range.
c) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} | y > 0\}$; This is a function because every element in the domain produces exactly one element in the range.
d) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} | 0 \le y \le 2\}$; This is a function because every element in the domain produces exactly one element in the range.
e) $D = \{x \in \mathbf{R} | -3 \le x \le 3\}$;
$R = \{y \in \mathbf{R} | -3 \le y \le 3\}$; This is not a function because $(0, 3)$ and $(0, -3)$ are both in the relation.
f) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} | -2 \le y \le 2\}$; This is a function because every element in the domain produces exactly one element in the range.

3. a) $D = \{1, 3, 5, 7\}$; $R = \{2, 4, 6\}$; This is a function because each element of the domain has exactly one corresponding element in the range.
b) $D = \{0, 1, 2, 5\}$; $R = \{-1, 3, 6\}$; This is a function because each element of the domain has exactly one corresponding element in the range.
c) $D = \{0, 1, 2, 3\}$; $R = \{2, 4\}$; This is a function because each element of the domain has exactly one corresponding element in the range.
d) $D = \{2, 6, 8\}$; $R = \{1, 3, 5, 7\}$; This is not a function because 2 is sent to both 5 and 7 in the range.
e) $D = \{1, 10, 100\}$; $R = \{0, 1, 2, 3\}$; This is not a function because 1 is sent to both 0 and 1 in the range.
f) $D = \{1, 2, 3, 4\}$; $R = \{1, 2, 3, 4\}$; This is a function because each element of the domain has exactly one corresponding element in the range.

4. a) This is a function because it passes the vertical line test; $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} | y \ge 2\}$
b) This is not a function because it fails the vertical line test; $D = \{x \in \mathbf{R} | x \ge 2\}$; $R = \{y \in \mathbf{R}\}$
c) This is a function because every element of the domain produces exactly one element in the range; $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} | y \ge -0.5\}$
d) This is not a function because $(1, 1)$ and $(1, -1)$ are both in the relation; $D = \{x \in \mathbf{R} | x \ge 0\}$; $R = \{y \in \mathbf{R}\}$

e) This is a function because every element of the domain produces exactly one element in the range; D = $\{x \in \mathbf{R} | x \neq 0\}$; R = $\{y \in \mathbf{R} | y \neq 0\}$

f) This is a function because every element of the domain produces exactly one element in the range; D = $\{x \in \mathbf{R}\}$; R = $\{y \in \mathbf{R}\}$

5. a) $y = x + 3$

b) $y = 2x - 5$

c) $y = 3(x - 2)$

d) $y = -x + 5$

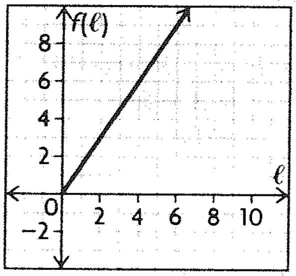
6. a) The length is twice the width.

b) Since $l = 2w$, $w = \frac{1}{2}l$

$$f(l) = l + w = l + \frac{1}{2}l$$

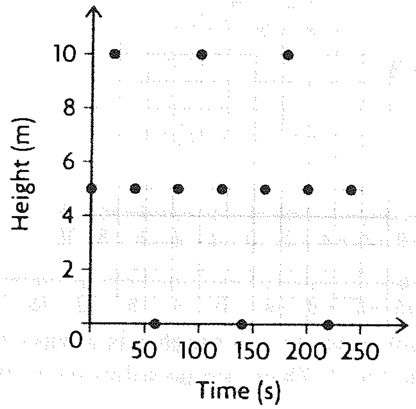
$$f(l) = \frac{3}{2}l$$

c)



d) Since $l = 2w$, the length must be 8 m and the width 4 m in order to use all 12 m of material.

7. a)

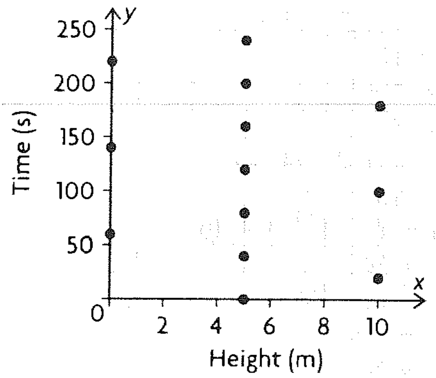


b) D = {0, 20, 40, 60, 80, 100, 120, 140, 160, 180, 200, 220, 240}

c) R = {0, 5, 10}

d) It is a function because it passes the vertical line test.

e)



f) It is not a function because (5, 0) and (5, 40) are both in the relation.

8. a) $\{(1, 2), (3, 4), (5, 6)\}$

b) $\{(1, 2), (3, 2), (5, 6)\}$

c) $\{(2, 1), (2, 3), (5, 6)\}$

9. If a vertical line passes through a function and hits two points, those two points have identical x-coordinates and different y-coordinates. This means that one x-coordinate is sent to two different elements in the range, violating the definition of *function*.

10. a)
$$\begin{aligned} d &= \sqrt{(4 - 0)^2 + (3 - 0)^2} \\ &= \sqrt{4^2 + 3^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Yes, because the distance from (4, 3) to (0, 0) is 5.

b)
$$\begin{aligned} d &= \sqrt{(1 - 0)^2 + (5 - 0)^2} \\ &= \sqrt{1^2 + 5^2} \\ &= \sqrt{26} \end{aligned}$$

$$5 \neq \sqrt{26}$$

No, because the distance from (1, 5) to (0, 0) is not 5.

c) No, because (4, 3) and (4, −3) are both in the relation.

11. a) $g(x) = x^2 + 3$

b)
$$\begin{aligned} g(3) - g(2) &= 12 - 7 \\ &= 5 \end{aligned}$$

$$\begin{aligned} g(3 - 2) &= g(1) \\ &= 4 \end{aligned}$$

So, $g(3) - g(2) \neq g(3 - 2)$

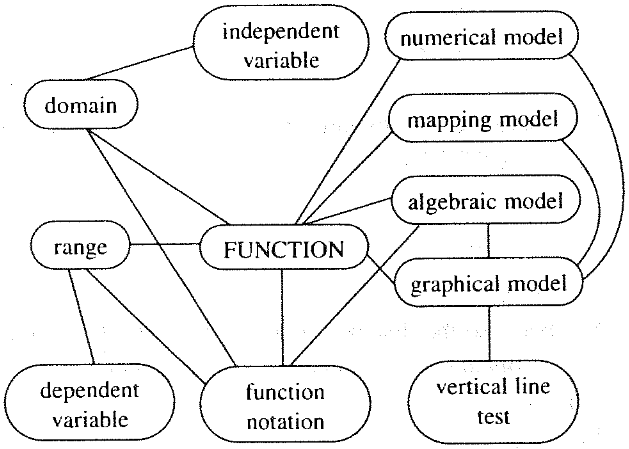
12. a) $f(6) = 1 + 2 + 3 + 6$
$= 12$
$f(7) = 1 + 7$
$= 8$
$f(8) = 1 + 2 + 4 + 8$
$= 15$

b) $f(15) = 1 + 3 + 5 + 15$
$= 24$
$f(3) \times f(5) = (1 + 3) \times (1 + 5)$
$= 4 \times 6$
$= 24$
$f(15) = f(3) \times f(5)$

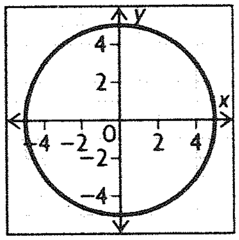
c) $f(12) = 1 + 2 + 3 + 4 + 6 + 12$
$= 28$
$f(3) \times f(4) = (1 + 3) \times (1 + 2 + 4)$
$= 4 \times 7$
$= 28$
$f(12) = f(3) \times f(4)$

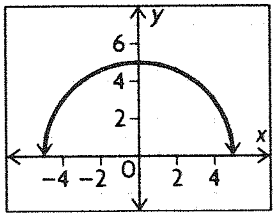
d) Yes, there are others that will work.
$f(a) \times f(b) = f(a \times b)$ whenever a and b have no common factors other than 1.

13. Answers may vary. For example:



14.





The first is not a function because it fails the vertical line test: $D = \{x \in \mathbf{R} | -5 \leq x \leq 5\}$; $R = \{y \in \mathbf{R} | -5 \leq y \leq 5\}$. The second is a function because it passes the vertical line test: $D = \{x \in \mathbf{R} | -5 \leq x \leq 5\}$; $R = \{y \in \mathbf{R} | 0 \leq y \leq 5\}$.

15. x is a function of y if the graph passes the horizontal line test. This occurs when any horizontal line hits the graph at most once.

1.2 Exploring Absolute Value, p. 16

1. $|-5| = 5$, $|20| = 20$, $|-15| = 15$, $|12| = 12$, $|-25| = 25$
From least to greatest, 5, 12, 15, 20, 25, or $|-5|$, $|12|$, $|-15|$, $|20|$, $|-25|$

2. a) $|-22| = 22$

b) $-|-35| = -35$

c) $|-5 - 13| = |-18|$
$= 18$

d) $|4 - 7| + |-10 + 2| = |-3| + |-8|$
$= 3 + 8$
$= 11$

e) $\frac{|-8|}{-4} = \frac{8}{-4}$
$= -2$

f) $\frac{|-22|}{|-11|} + \frac{-16}{|-4|} = \frac{22}{11} + \frac{-16}{4}$
$= 2 - 4$
$= -2$

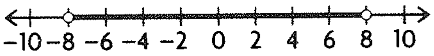
3. a) $|x| > 3$

b) $|x| \leq 8$

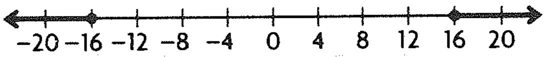
c) $|x| \geq 1$

d) $|x| \neq 5$

4. a)

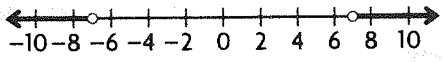


b)



c) The absolute value of a number is always greater than or equal to 0. There are no solutions to this inequality.

d)



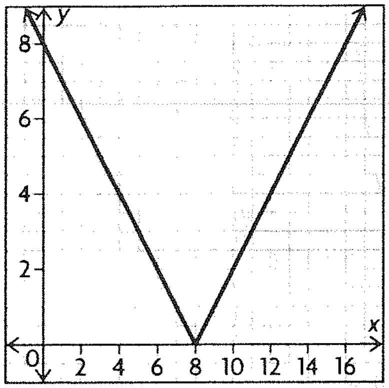
5. a) $|x| \leq 3$

b) $|x| > 2$

c) $|x| \geq 2$

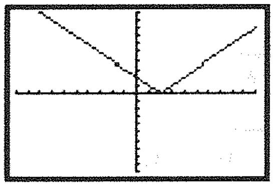
d) $|x| < 4$

6.

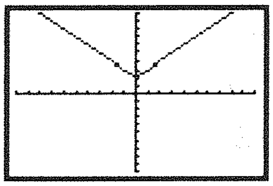


a) The graphs are the same.
b) Answers may vary. For example, $x - 8 = -(-x + 8)$, so they are negatives of each other and have the same absolute value.

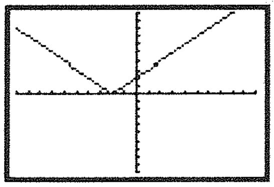
7. a)



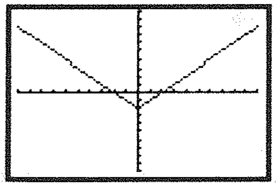
b)



c)

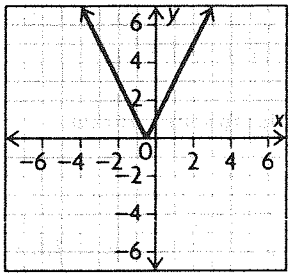


d)

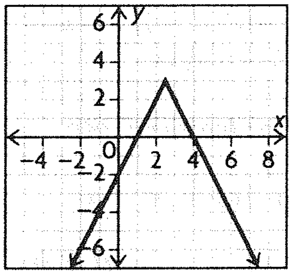


8. When the number you are adding or subtracting is inside the absolute value signs, it moves the function to the left (when adding) or to the right (when subtracting) of the origin. When the number you are adding or subtracting is outside the absolute value signs, it moves the function down (when subtracting) or up (when adding) from the origin. The graph of the function will be the absolute value function moved to the left 3 units and down 4 units from the origin.

9. This is the graph of $g(x) = |x|$ horizontally compressed by a factor of $\frac{1}{2}$ and translated $\frac{1}{2}$ unit to the left.



10. This is the graph of $g(x) = |x|$ horizontally compressed by a factor of $\frac{1}{2}$, reflected over the x-axis, translated $2\frac{1}{2}$ units to the right, and translated 3 units up.



1.3 Properties of Graphs of Functions, pp. 23–25

1. Answers may vary. For example, domain because most of the parent functions have all real numbers as a domain.
2. Answers may vary. For example, the end behaviour because the only two that match are x^2 and $|x|$.
3. Given the horizontal asymptote, the function must be derived from 2^x. But the asymptote is at $y = 2$, so it must have been translated up two. Therefore, the function is $f(x) = 2^x + 2$.
4. a) Both functions are odd, but their domains are different.
b) Both functions have a domain of all real numbers, but $\sin(x)$ has more zeros.
c) Both functions have a domain of all real numbers, but different end behaviour.
d) Both functions have a domain of all real numbers, but different end behaviour.

5. a) $f(x) = x^2 - 4$
$f(-x) = (-x)^2 - 4 = x^2 - 4$
$-f(-x) = -x^2 + 4$
Since $f(x) = f(-x)$, the function is even.

b) $f(x) = \sin(x) + x$
$f(-x) = \sin(-x) + (-x) = -\sin x - x$
$= -(\sin x + x) = -f(x)$
$-f(-x) = \sin x + x$
Since $f(-x) = -f(x)$, the function is odd.

c) $f(x) = \frac{1}{x} - x$
$f(-x) = \frac{1}{-x} - (-x) = -\frac{1}{x} + x = -f(x)$
$-f(-x) = \frac{1}{x} - x$
Since $f(-x) = -f(x)$, the function is odd.

d) $f(x) = 2x^3 + x$
$f(-x) = 2(-x)^3 + (-x) = -2x^3 - x$
$= -(2x^3 + x) = -f(x)$
$-f(-x) = 2x^3 + x$
Since $f(-x) = -f(x)$, the function is odd.

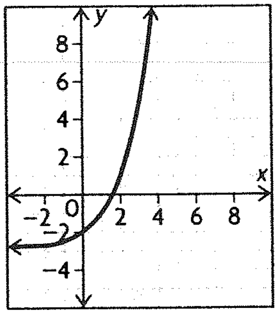
e) $f(x) = 2x^2 - x$
$f(-x) = 2(-x)^2 - (-x) = 2x^2 + x$
$-f(-x) = -2x^2 - x$
Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, the function is neither even nor odd.

f) $f(x) = |2x + 3|$
$f(-x) = |2(-x) + 3| = |-2x + 3|$
$-f(-x) = -|-2x + 3|$
Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, the function is neither even nor odd.

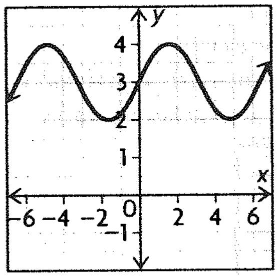
6. a) $|x|$, because it is a measure of distance from a number
b) $\sin(x)$, because the heights are periodic
c) 2^x, because population tends to increase exponentially
d) x, because there is $1 on the first day, $2 on the second, $3 on the third, etc.

7. a) $f(x) = \sqrt{x}$, because the domain of x must be greater than 0 for the function to be defined and $f(0) = \sqrt{0} = 0$
b) $f(x) = \sin x$, because the function is periodic and is at 0 at 0°, 180°, 360°, 540°, 720°, etc.
c) $f(x) = x^2$; It is even because $f(-x) = (-x)^2 = x^2 = f(x)$. The graph of the function is a smooth curve without any sharp corners.
d) $f(x) = x$, because $y = x$ in this function and, therefore, y and x have the same behaviour.

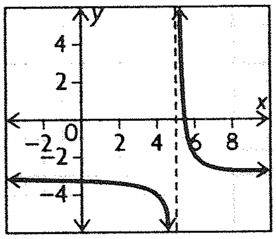
8. a) $f(x) = 2^x - 3$



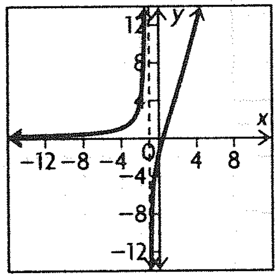
b) $g(x) = \sin x + 3$



c) $h(x) = \frac{1}{x - 5} - 3 = \frac{16 - 3x}{x - 5}$



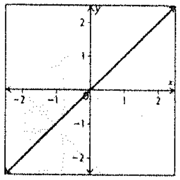
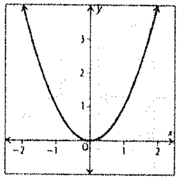
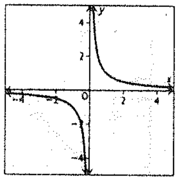
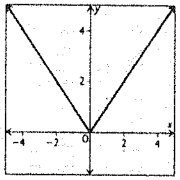
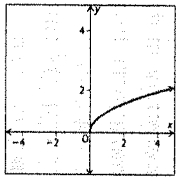
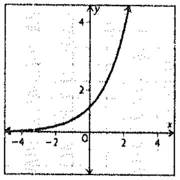
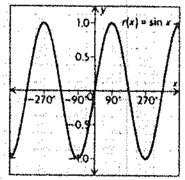
9.



10. a) The quadratic is a parabola opening upward with its vertex at (2, 0). Using the vertex form, the function would be $f(x) = (x - 2)^2$.
b) There is not only one function.
$f(x) = \frac{3}{4}(x - 2)^2 + 1$ works as well.
c) There is more than one function that satisfies the property. $f(x) = |x - 2| + 2$ and $f(x) = 2|x - 2|$ both work.

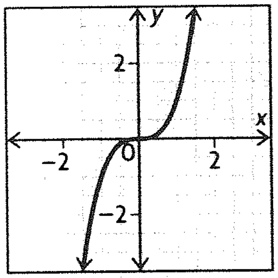
11. x^2 is a smooth curve, while $|x|$ has a sharp, pointed corner at (0, 0).

12.

Parent Function	$f(x) = x$	$g(x) = x^2$	$h(x) = \frac{1}{x}$	$k(x) = \|x\|$	$m(x) = \sqrt{x}$	$p(x) = 2^x$	$r(x) = \sin x$
Sketch							
Domain	$\{x \in \mathbf{R}\}$	$\{x \in \mathbf{R}\}$	$\{x \in \mathbf{R} \mid x \neq 0\}$	$\{x \in \mathbf{R}\}$	$\{x \in \mathbf{R} \mid x \geq 0\}$	$\{x \in \mathbf{R}\}$	$\{x \in \mathbf{R}\}$
Range	$\{f(x) \in \mathbf{R}\}$	$\{f(x) \in \mathbf{R} \mid$ $f(x) \geq 0\}$	$\{f(x) \in \mathbf{R} \mid$ $f(x) \neq 0\}$	$\{f(x) \in \mathbf{R} \mid$ $f(x) \geq 0\}$	$\{f(x) \in \mathbf{R} \mid$ $f(x) \geq 0\}$	$\{f(x) \in \mathbf{R} \mid$ $f(x) > 0\}$	$\{f(x) \in \mathbf{R} \mid$ $-1 \leq f(x) \leq 1\}$
Intervals of Increase	$(-\infty, \infty)$	$(0, \infty)$	None	$(0, \infty)$	$(0, \infty)$	$(-\infty, \infty)$	$[90(4k + 1),$ $90(4k + 3)]$ $K \in \mathbf{Z}$
Intervals of Decrease	None	$(-\infty, 0)$	$(-\infty, 0)(0, \infty)$	$(-\infty, 0)$	None	None	$[90(4k + 3),$ $90(4k + 1)]$ $K \in \mathbf{Z}$
Location of Discontinuities and Asymptotes	None	None	$y = 0$ $x = 0$	None	None	$y = 0$	None
Zeros	(0, 0)	(0, 0)	None	(0, 0)	(0, 0)	None	$180k$ $K \in \mathbf{Z}$
Y-Intercepts	(0, 0)	(0, 0)	None	(0, 0)	(0, 0)	(0, 1)	(0, 0)
Symmetry	Odd	Even	Odd	Even	Neither	Neither	Odd
End Behaviours	$x \to \infty, y \to \infty$ $x \to -\infty, y \to -\infty$	$x \to \infty, y \to \infty$ $x \to -\infty, y \to \infty$	$x \to \infty, y \to 0$ $x \to -\infty, y \to 0$	$x \to \infty, y \to \infty$ $x \to -\infty, y \to \infty$	$x \to \infty, y \to \infty$	$x \to \infty, y \to \infty$ $x \to -\infty, y \to 0$	Oscillating

13. It is important to name parent functions in order to classify a wide range of functions according to similar behaviour and characteristics.

14.

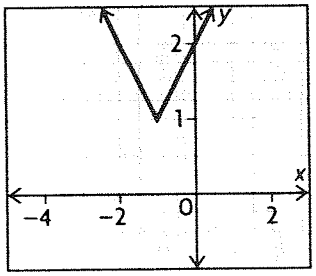


D = $\{x \in \mathbf{R}\}$, R = $\{f(x) \in \mathbf{R}\}$; interval of increase = $(-\infty, \infty)$, no interval of decrease, no discontinuities, x- and y-intercept at (0, 0), odd, $x \rightarrow \infty$, $y \rightarrow \infty$, and $x \rightarrow -\infty$, $y \rightarrow -\infty$. It is very similar to $f(x) = x$. It does not, however, have a constant slope.

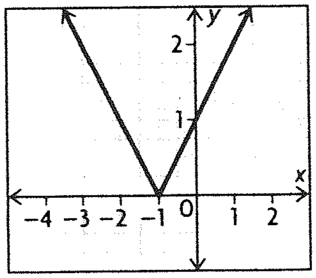
15. No, cos x is a horizontal translation of sin x.

16. The graph can have 0, 1, or 2 zeros.

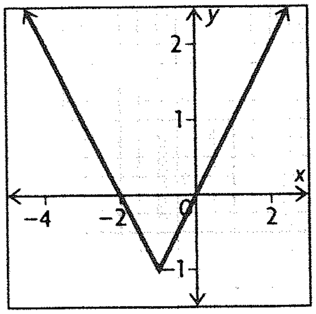
0 zeros:



1 zero:



2 zeros:



Mid-Chapter Review, p. 28

1. a) This is a function because every value in the domain goes to only one value in the range; D = $\{0, 3, 15, 27\}$, R = $\{2, 3, 4\}$

b) This is a function because every value in the domain goes to only one value in the range; D = $\{x \in \mathbf{R}\}$, R = $\{y \in \mathbf{R}\}$

c) This is not a function. It fails the vertical line test; D = $\{x \in \mathbf{R} | -5 \leq x \leq 5\}$, R = $\{y \in \mathbf{R} | -5 \leq y \leq 5\}$

d) This is not a function because 2, in the domain, goes to both 6 and 7 in the range; D = $\{1, 2, 10\}$, R = $\{-1, 3, 6, 7\}$

2. a) Yes. Every element in the domain gets sent to exactly one element in the range.

b) D = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

c) R = $\{10, 20, 25, 30, 35, 40, 45, 50\}$

3. a) D = $\{x \in \mathbf{R}\}$, R = $\{f(x) \in \mathbf{R}\}$; function

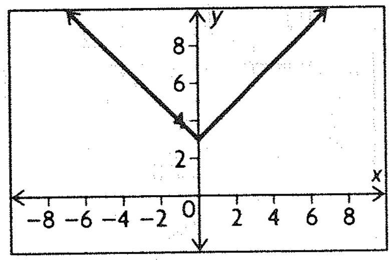
b) D = $\{x \in \mathbf{R} | -3 \leq x \leq 3\}$, R = $\{y \in \mathbf{R} | -3 \leq y \leq 3\}$; not a function

c) D = $\{x \in \mathbf{R} | x \leq 5\}$, R = $\{y \in \mathbf{R} | y \geq 0\}$; function

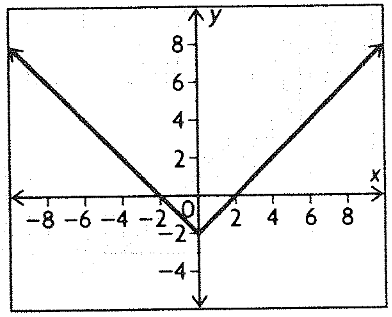
d) D = $\{x \in \mathbf{R}\}$, R = $\{y \in \mathbf{R} | y \geq -2\}$; function

4. $|-3| = 3$, $-|3| = -3$, $|5| = 5$, $|-4| = 4$, $|0| = 0$
$-|3| < |0| < |-3| < |-4| < |5|$

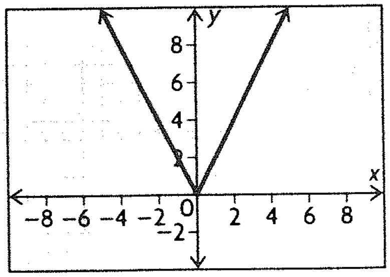
5. a)



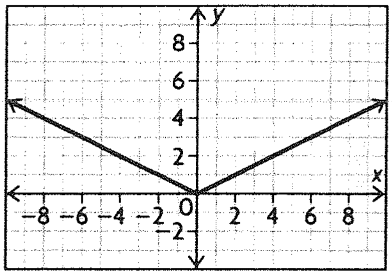
b)



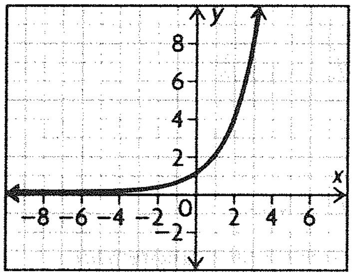
c)



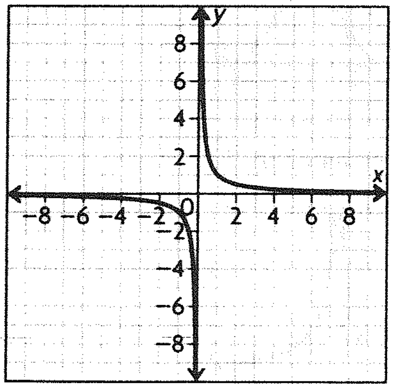
d)



6. a) The graph of $f(x) = 2^x$ is not symmetric about the y-axis nor the origin, and, therefore, is neither even nor odd. Looking at the graph we notice that $x \to \infty$ and $y \to \infty$.



b) $(-\infty, 0)$ and $(0, \infty)$ are both intervals of decrease for the function $f(x) = \frac{1}{x}$.



c) The function $f(x) = \sqrt{x}$ must have a domain greater than or equal to 0 because the square root of a negative number is undefined.

7. a) $f(x) = |2x|$

$f(-x) = |2(-x)| = |2x| = f(x)$

Since $f(x) = f(-x)$, the function is even.

b) $f(x) = (-x)^2$

$f(-x) = (-(-x))^2 = x^2 = (-x)^2 = f(x)$

Since $f(x) = f(-x)$, the function is even.

c) $f(x) = x + 4$

$f(-x) = (-x) + 4 = -x + 4$

Since $f(x) \neq f(-x)$ and $f(x) \neq -f(x)$, the function is neither odd nor even.

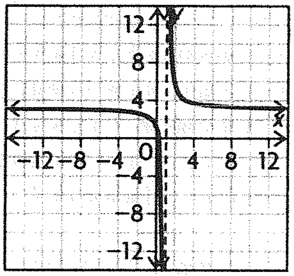
d) $f(x) = 4x^5 + 3x^3 - 1$

$f(-x) = 4(-x)^5 + 3(-x)^3 - 1$

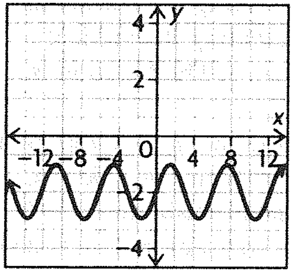
$= -4x^5 - 3x^3 - 1$

Since $f(x) \neq f(-x)$ and $f(x) \neq -f(x)$, the function is neither odd nor even.

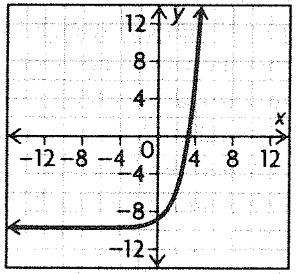
8. a) This is $f(x) = \frac{1}{x}$ translated right 1 and up 3; discontinuous



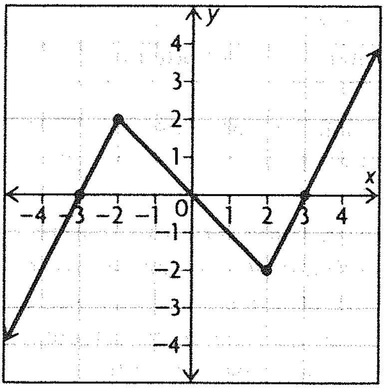
b) This is $f(x) = \sin x$ translated down 2; continuous



c) This is $f(x) = 2^x$ translated down 10; continuous



9.



1.4 Sketching Graphs of Functions, pp. 35–37

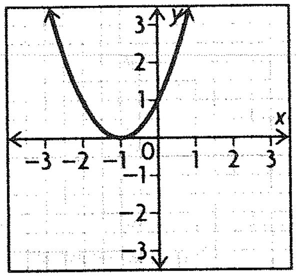
1. a) translation 1 unit down
b) horizontal compression by a factor of $\frac{1}{2}$, translation 1 unit right
c) reflection over the x-axis, translation 2 units up, translation 3 units right
d) reflection over the x-axis, vertical stretch by a factor of 2, horizontal compression by a factor of $\frac{1}{4}$
e) reflection over the x-axis, translation 3 units down, reflection over the y-axis, translation 2 units left
f) vertical compression by a factor of $\frac{1}{2}$, translation 6 units up, horizontal stretch by a factor of 4, translation 5 units right

2. a) Representing the reflection in the x-axis: $a = -1$, representing the horizontal stretch by a factor of 2: $k = \frac{1}{2}$, representing the horizontal translation: $d = 0$, representing the vertical translation 3 units up: $c = 3$. The function is $y = -\sin\left(\frac{1}{2}x\right) + 3$.
b) Representing the amplitude: $a = 3$, representing the horizontal stretch by a factor of 2: $k = \frac{1}{2}$, representing the horizontal translation: $d = 0$, representing the vertical translation 3 units down: $c = -2$. The function is $y = 3\sin\left(\frac{1}{2}x\right) - 2$.

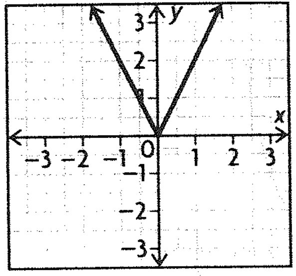
3. Consider the transformations of $f(x)$: horizontal compression by a factor of $\frac{1}{2}$, vertical stretch by a factor of 2, reflection across the x-axis, horizontal translation 5 units left, and vertical translation 4 units down. These transformations take $(2, 3)$ to $(1, 3)$, $(1, 6)$, $(1, -6)$, $(-4, -6)$, and finally to $(-4, -10)$.

4. a) Each y-coordinate gets multiplied by 2. $(2, 6)$, $(4, 14)$, $(-2, 10)$, $(-4, 12)$
b) Each x-coordinate gets increased by 3. $(5, 3)$, $(7, 7)$, $(1, 5)$, $(-1, 6)$
c) Each y-coordinate gets increased by 2. $(2, 5)$, $(4, 9)$, $(-2, 7)$, $(-4, 8)$
d) Each x-coordinate gets decreased by 1, and each y-coordinate gets decreased by 3. $(1, 0)$, $(3, 4)$, $(-3, 2)$, $(-5, 3)$
e) The points are reflected across the y-axis, so for x-coordinates that differ in sign switch the y-coordinates. $(2, 5)$, $(4, 6)$, $(-2, 3)$, $(-4, 7)$
f) The x-coordinates are reduced by a factor of $\frac{1}{2}$, and the y-coordinates are decreased by 1. $(1, 2)$, $(2, 6)$, $(-1, 4)$, $(-2, 5)$

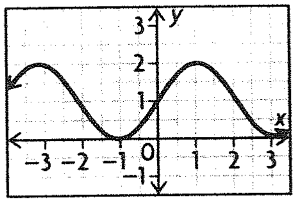
5. a) $f(x) = x^2$, translated left 1



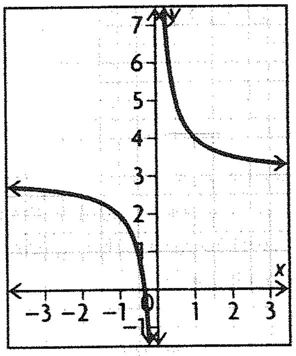
b) $f(x) = |x|$, vertical stretch by 2



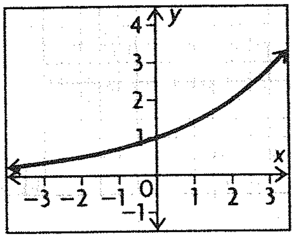
c) $f(x) = \sin(x)$, horizontal compression of $\frac{1}{3}$ translation up 1



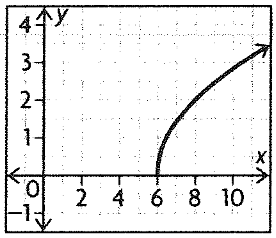
d) $f(x) = \frac{1}{x}$, translation up 3



e) $f(x) = 2^x$, horizontal stretch by 2

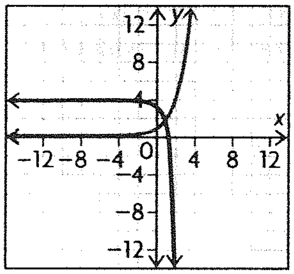


f) $f(x) = \sqrt{x}$, horizontal compression by $\frac{1}{2}$, translation right 6



6. a) $D = \{x \in \mathbf{R}\}$, $R = \{f(x) \in \mathbf{R} | f(x) \geq 0\}$
b) $D = \{x \in \mathbf{R}\}$, $R = \{f(x) \in \mathbf{R} | f(x) \geq 0\}$
c) $D = \{x \in \mathbf{R}\}$, $R = \{f(x) \in \mathbf{R} | 0 \leq f(x) \leq 2\}$
d) $D = \{x \in \mathbf{R} | x \neq 0\}$, $R = \{f(x) \in \mathbf{R} | f(x) \neq 3\}$
e) $D = \{x \in \mathbf{R}\}$, $R = \{f(x) \in \mathbf{R} | f(x) > 0\}$
f) $D = \{x \in \mathbf{R} | x \geq 6\}$, $R = \{f(x) \in \mathbf{R} | f(x) \geq 0\}$

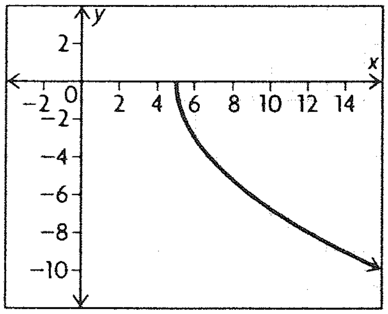
7. a)



b) The domain remains unchanged at $D = \{x \in \mathbf{R}\}$. The range must now be less than 4: $R = \{f(x) \in \mathbf{R} | f(x) < 4\}$. It changes from increasing on $(-\infty, \infty)$ to decreasing on $(-\infty, \infty)$. The end behaviour becomes as $x \rightarrow -\infty$, $y \rightarrow 4$, and as $x \rightarrow \infty$, $y \rightarrow -\infty$.

c) $g(x) = -2(2^{3(x-1)} + 4)$

8. $y = -3\sqrt{x - 5}$;



9. a) $(1, 8) \rightarrow (1 + 2, 8 \times 3) = (3, 24)$

b) $(1, 8) \rightarrow \left(\frac{1}{2}(1) - 1, 8 - 4\right) = (-0.5, 4)$

c) $(1, 8) \rightarrow \left(\frac{1}{-1}, 8(2) - 7\right) = (-1, 9)$

d) $(1, 8) \rightarrow \left(\frac{1}{4}(1) - 1, 8 \times -1\right) = (-0.75, -8)$

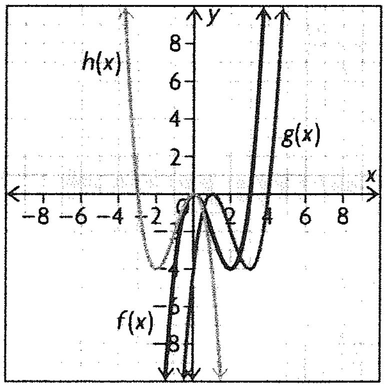
e) $(1, 8) \rightarrow \left(\frac{1}{-1}, 8 \times -1\right) = (-1, -8)$

f) $(1, 8) \rightarrow \left(\frac{1}{0.5}(1) - 3, 0.5(8) + 3\right) = (-1, 7)$

10. a) $g(x) = \sqrt{x - 2}$
$D = \{x \in \mathbf{R} | x \geq 2\}$, $R = \{g(x) \in \mathbf{R} | g(x) \geq 0\}$
b) $h(x) = 2\sqrt{x - 1} + 4$
$D = \{x \in \mathbf{R} | x \geq 1\}$, $R = \{h(x) \in \mathbf{R} | h(x) \geq 4\}$
c) $k(x) = \sqrt{-x} + 1$
$D = \{x \in \mathbf{R} | x \leq 0\}$, $R = \{k(x) \in \mathbf{R} | k(x) \geq 1\}$
d) $j(x) = 3\sqrt{2(x - 5)} - 3$
$D = \{x \in \mathbf{R} | x \geq 5\}$, $R = \{j(x) \in \mathbf{R} | j(x) \geq -3\}$

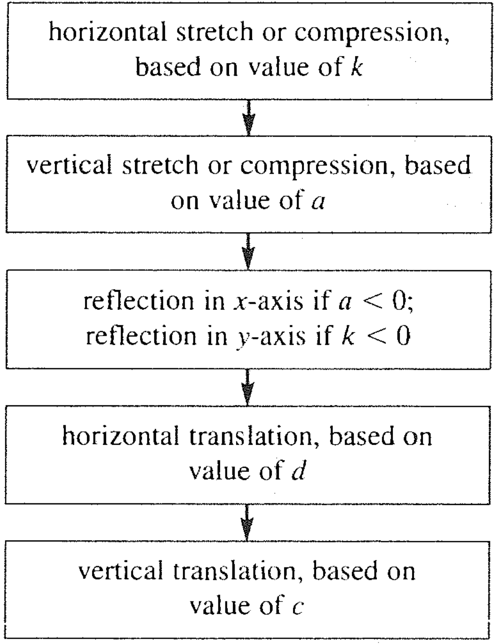
11. $y = 5(x^2 - 3)$ is the same as $y = 5x^2 - 15$, not $y = 5x^2 - 3$.

12.



13. a) a vertical stretch by a factor of 4
b) a horizontal compression by a factor of $\frac{1}{2}$
c) $(2x)^2 = 2^2x^2 = 4x^2$

14. Answers may vary. For example:

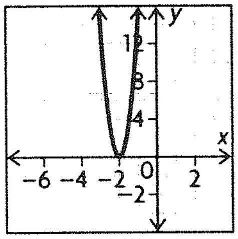


15. The new y-coordinate was produced by translating down 4 after a stretch by a factor of 2. To go backwards, we must translate up 4, which takes the 6 to 10, and then compress by a factor of $\frac{1}{2}$, which takes 10 to 5. The new x-coordinate was produced by translating left 1 unit. To go backwards, we translate right 1 unit, so 3 becomes 4. The original point is $(4, 5)$.

16. a) horizontal compression by a factor of $\frac{1}{3}$, translation 2 units to the left

b) Because they are equivalent expressions:
$3(x + 2) = 3x + 6$

c)



1.5 Inverse Relations, pp. 43–45

1. a) $(5, 2)$

b) $(-6, -5)$

c) $(-8, 4)$

d) $f(1) = 2 \rightarrow (1, 2)$
So, $(2, 1)$ is on the inverse.

e) $g(-3) = 0 \rightarrow (-3, 0)$
So, $(0, -3)$ is on the inverse.

f) $h(0) = 7 \rightarrow (0, 7)$
So, $(7, 0)$ is on the inverse.

2. The domain and the range of the original functions are switched for the inverses.

a) $D = \{x \in \mathbf{R}\}$, $R = \{y \in \mathbf{R}\}$

b) $D = \{x \in \mathbf{R}\}$, $R = \{y \in \mathbf{R} \mid y \geq 2\}$

c) $D = \{x \in \mathbf{R} \mid x < 2\}$, $R = \{y \in \mathbf{R} \mid y \geq -5\}$

d) $D = \{x \in \mathbf{R} \mid -5 < x < 10\}$, $R = \{y \in \mathbf{R} \mid y < -2\}$

3. Function A: $y = \frac{1}{2}x - 2$
The inverse of function A is:

$$x = \frac{1}{2}y - 2$$

$$x + 2 = \frac{1}{2}y$$

$$2x + 4 = y$$

Functions A and D match.
Function B: $y = x^2 + 2$ for $x \geq 0$
The inverse of function E is:
$x = y^2 + 2$
$x - 2 = y^2$
$\sqrt{x - 2} = y$, where $x \geq 2$
Functions B and F match.
Function C: $y = (x + 3)^2$ where $x \geq -3$
The inverse of function F is:
$x = (y + 3)^2$
$\sqrt{x} = y + 3$
$\sqrt{x} - 3 = y$
Functions C and E match.

4. a) $(4, 129)$

b) $(129, 4)$

c) $D = \{x \in \mathbf{R}\}$, $R = \{y \in \mathbf{R}\}$

d) $D = \{x \in \mathbf{R}\}$, $R = \{y \in \mathbf{R}\}$

e) Yes; it passes the vertical line test.

5. a) $(4, 248)$

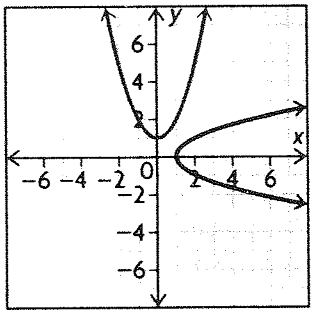
b) $(248, 4)$

c) $D = \{x \in \mathbf{R}\}$, $R = \{y \in \mathbf{R} \mid y \geq -8\}$

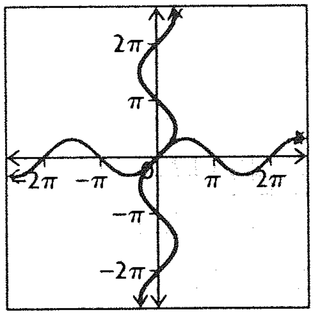
d) $D = \{x \in \mathbf{R} \mid x \geq -8\}$, $R = \{y \in \mathbf{R}\}$

e) No; $(248, 4)$ and $(248, -4)$ are both on the inverse relation.

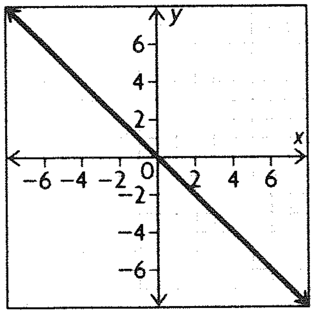
6. a) Not a function



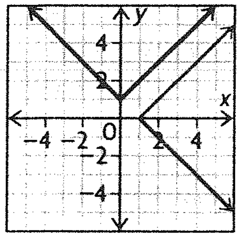
b) Not a function



c) Function



d) Not a function



7. a) $F = \frac{9}{5}C + 32$

$$F - 32 = \frac{9}{5}C$$

$$\frac{5}{9}(F - 32) = C$$

$C = \frac{5}{9}(F - 32)$; this allows you to convert from Fahrenheit to Celsius.

b) $F = \frac{9}{5}C + 32$

$$F = \frac{9}{5}(20) + 32 = 36 + 32 = 68$$

$$C = \frac{5}{9}(F - 32)$$

$$C = \frac{5}{9}(68 - 32) = \frac{5}{9}(36) = 20$$

20 °C = 68 °F

8. a) $A = \pi r^2$

$$\frac{A}{\pi} = r^2$$

$$\sqrt{\frac{A}{\pi}} = r$$

$r = \sqrt{\frac{A}{\pi}}$; this can be used to determine the radius of a circle when its area is known.

b) $A = \pi r^2 = \pi(5)^2 = 25\pi$

$$r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{25\pi}{\pi}} = \sqrt{25} = 5$$

$A = 25\pi$ cm², $r = 5$ cm

9. $y = kx^3 - 1$

$$x = ky^3 - 1$$

$$x + 1 = ky^3$$

$$\sqrt[3]{\frac{x + 1}{k}} = y$$

$$f^{-1}(x) = \sqrt[3]{\frac{x + 1}{k}}$$

$$f^{-1}(15) = \sqrt[3]{\frac{x + 1}{k}} = 2$$

$$\sqrt[3]{\frac{15 + 1}{k}} = 2$$

$$\frac{16}{k} = 2^3$$

$$16 = 8k$$

$$k = 2$$

10. $h(x) = 2x + 7$

$h^{-1}(x)$:

$$x = 2y + 7$$

$$\frac{x - 7}{2} = y$$

$$h^{-1}(x) = \frac{x - 7}{2}$$

a) $h(3) = 2(3) + 7 = 13$

b) $h(9) = 2(9) + 7 = 25$

c) $\frac{h(9) - h(3)}{9 - 3} = \frac{25 - 13}{6} = 2$

d) $h^{-1}(3) = \frac{3 - 7}{2} = \frac{-4}{2} = -2$

e) $h^{-1}(9) = \frac{9 - 7}{2} = \frac{2}{2} = 1$

f) $\frac{h^{-1}(9) - h^{-1}(3)}{9 - 3} = \frac{1 - (-2)}{6} = \frac{3}{6} = \frac{1}{2}$

11. No; several students could have the same grade point average.

12. a) $f(x) = 3x + 4$

$$x = 3y + 4$$

$$x - 4 = 3y$$

$$\frac{x - 4}{3} = y$$

$$f^{-1}(x) = \frac{1}{3}(x - 4)$$

b) $h(x) = -x$

$$x = -y$$

$$-x = y$$

$$h^{-1}(x) = -x$$

c) $g(x) = x^3 - 1$

$$x = y^3 - 1$$

$$x + 1 = y^3$$

$$\sqrt[3]{x + 1} = y$$

$$g^{-1}(x) = \sqrt[3]{x + 1}$$

d) $m(x) = -2(x + 5)$

$$x = -2(y + 5)$$

$$\frac{x}{-2} = y + 5$$

$$-\frac{x}{2} - 5 = y$$

$$m^{-1}(x) = -\frac{x}{2} - 5$$

13. a) $g(x) = 4(x - 3)^2 + 1$

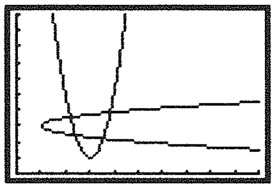
$x = 4(y - 3)^2 + 1$

b) $\frac{x - 1}{4} = (y - 3)^2$

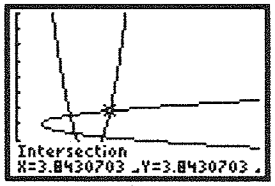
$\pm\sqrt{\frac{x - 1}{4}} + 3 = y$

$y = \pm\sqrt{\frac{x - 1}{4}} + 3$

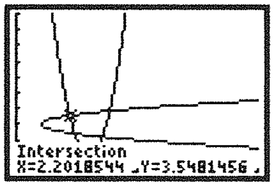
c)



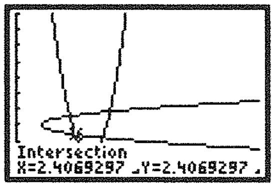
d) The points of intersection are approximately (2.20, 3.55), (2.40, 2.40), (3.55, 2.20), and (3.84, 3.84).



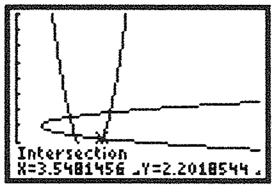
```
Intersection
X=3.8430703 Y=3.8430703
```



```
Intersection
X=2.2018544 Y=3.5481456
```



```
Intersection
X=2.4069297 Y=2.4069297
```



```
Intersection
X=3.5481456 Y=2.2018544
```

e) $x \geq 3$ because a negative square root is undefined.

f) $g(2) = 5$, but $g^{-1}(5) = 2$ or 4; the inverse is not a function if this is the domain of g.

14. For $y = -\sqrt{x + 2}$, $\mathrm{D} = \{x \in \mathbf{R} | x \geq -2\}$ and $\mathrm{R} = \{y \in \mathbf{R} | y \leq 0\}$. For $y = x^2 - 2$, $\mathrm{D} = \{x \in \mathbf{R}\}$ and $\mathrm{R} = \{y \in \mathbf{R} | y \geq -2\}$. The student would be correct if the domain of $y = x^2 - 2$ is restricted to $\mathrm{D} = \{x \in \mathbf{R} | x \leq 0\}$.

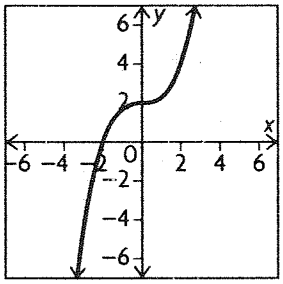
15. Yes; the inverse of $y = \sqrt{x + 2}$ is $y = x^2 - 2$ so long as the domain of this second function is restricted to $\mathrm{D} = \{x \in \mathbf{R} | x \geq 0\}$.

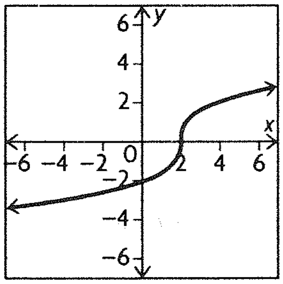
16. John is correct.

Algebraic: $y = \frac{x^3}{4} + 2$; $y - 2 = \frac{x^3}{4}$; $4(y - 2) = x^3$; $x = \sqrt[3]{4(y - 2)}$.

Numeric: Let $x = 4$. $y = \frac{4^3}{4} + 2 = \frac{64}{4} + 2$ $= 16 + 2 = 18$; $x = \sqrt[3]{4(y - 2)} = \sqrt[3]{4(18 - 2)}$ $= \sqrt[3]{4(16)} = \sqrt[3]{64} = 4$.

Graphical:





The graphs are reflections over the line $y = x$.

17. $f(x) = k - x$ works for all $k \in \mathbf{R}$.

$y = k - x$

Switch variables and solve for y: $x = k - y$

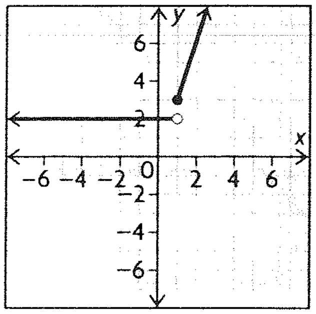
$y = k - x$

So the function is its own inverse.

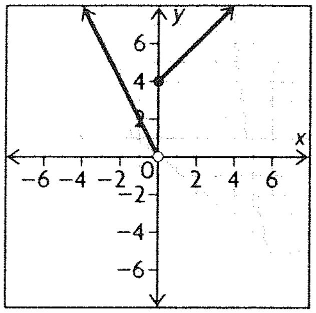
18. If a horizontal line hits the function in two locations, that means there are two points with equal y-values and different x-values. When the function is reflected over the line $y = x$ to find the inverse relation, those two points become points with equal x-values and different y-values, thus violating the definition of a function.

1.6 Piecewise Functions, pp. 51–53

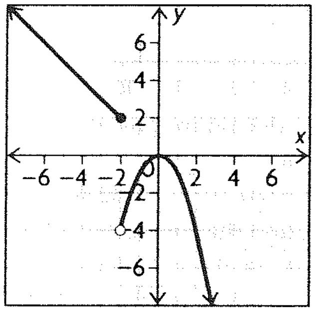
1. a)



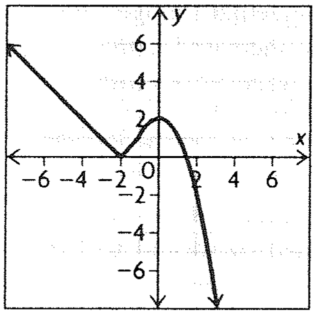
b)



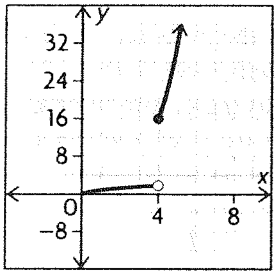
c)



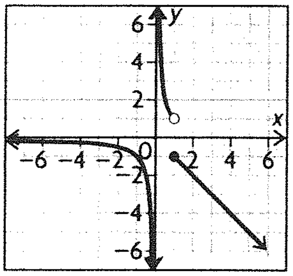
d)



e)



f)



2. a) Discontinuous at $x = 1$

b) Discontinuous at $x = 0$

c) Discontinuous at $x = -2$

d) Continuous

e) Discontinuous at $x = 4$

f) Discontinuous at $x = 1$ and $x = 0$

3. a) The function changes at $x = 1$. When $x \leq 1$, the function is a parabola represented by the equation $y = x^2 - 2$. When $x > 1$, it is a line represented by the equation $y = x + 1$.

$$f(x) = \begin{cases} x^2 - 2, \text{ if } x \leq 1 \\ x + 1, \text{ if } x > 1 \end{cases}$$

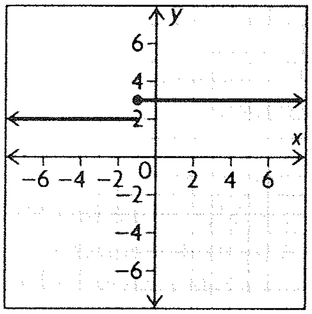
b) The function changes at $x = 1$. When $x < 1$, the function is an absolute value function represented by the equation $y = |x|$. When $x \geq 1$, it is a radical function represented by the equation $y = \sqrt{x}$.

$$f(x) = \begin{cases} |x|, \text{ if } x < 1 \\ \sqrt{x}, \text{ if } x \geq 1 \end{cases}$$

4. a) D = $\{x \in \mathbf{R}\}$; the function is discontinuous at $x = 1$.

b) D = $\{x \in \mathbf{R}\}$; the function is continuous.

5. a)

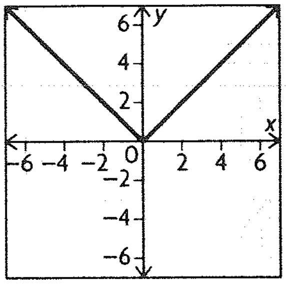


The function is discontinuous at $x = -1$.

D = $\{x \in \mathbf{R}\}$

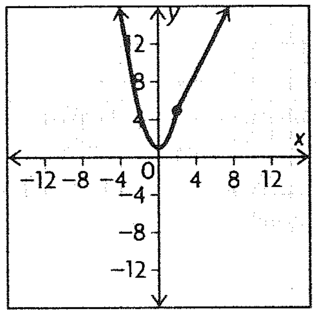
R = $\{2, 3\}$

b)



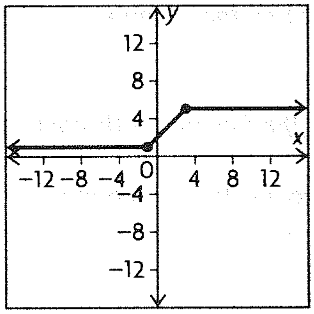
The function is continuous.
$D = \{x \in \mathbf{R}\}$
$R = \{f(x) \in \mathbf{R} | f(x) \geq 0\}$

c)



The function is continuous.
$D = \{x \in \mathbf{R}\}$
$R = \{f(x) \in \mathbf{R} | f(x) \geq 1\}$

d)



The function is continuous.
$D = \{x \in \mathbf{R}\}$
$R = \{f(x) \in \mathbf{R} | 1 \leq f(x) \leq 5\}$

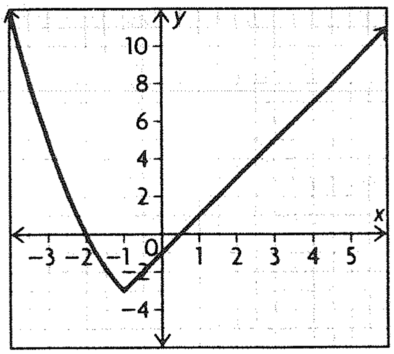
6. There is a flat fee of $15 for the first 500 minutes which is represented by the top equation. Over 500 minutes results in a rate represented by the bottom equation.

$$f(x) = \begin{cases} 15, \text{ if } 0 \leq x \leq 500 \\ 15 + 0.02x, \text{ if } x \geq 500 \end{cases}$$

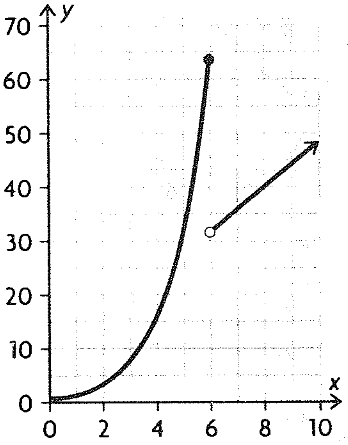
7.

$$f(x) = \begin{cases} 0.35x, \text{ if } 0 \leq x \leq 100\,000 \\ 0.45x - 10\,000, \text{ if } 100\,000 < x \leq 500\,000 \\ 0.55x - 60\,000, \text{ if } x > 500\,000 \end{cases}$$

8. In order for the function to be continuous the two pieces must have the same value for $x = -1$.
$1 - k = -2 - 1$, or $k = 4$.

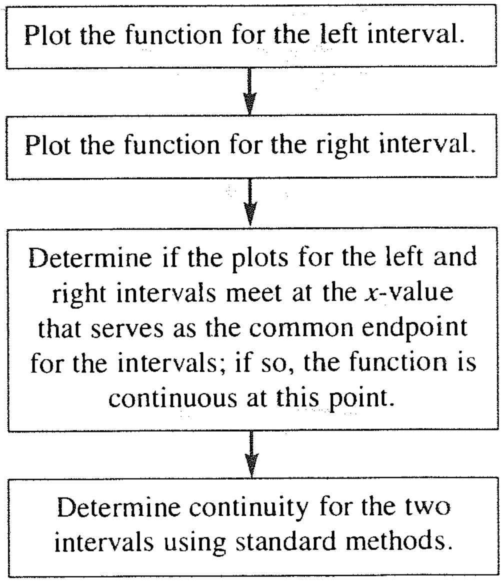


9. a)



b) The function is discontinuous at $x = 6$.
c) $2^x - (4x + 8)$ at $x = 6$
$2^6 - (4(6) + 8) = 64 - 32 = 32$ fish
d) Using the function that represents the time after the spill, $4x + 8 = 64$; $4x = 56$; $x = 14$
e) Answers may vary. For example: three possible events are environmental changes, introduction of a new predator, and increased fishing.

10. Answers may vary. For example:



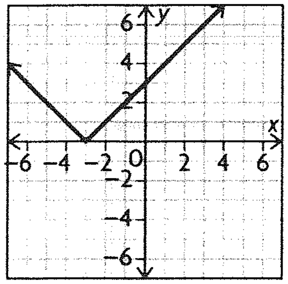
Plot the function for the left interval.

Plot the function for the right interval.

Determine if the plots for the left and right intervals meet at the x-value that serves as the common endpoint for the intervals; if so, the function is continuous at this point.

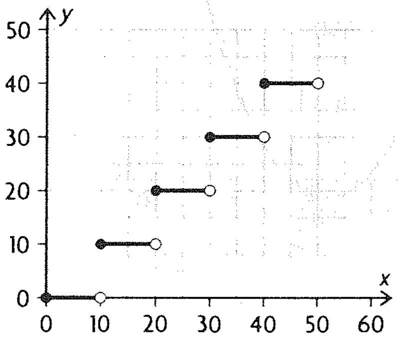
Determine continuity for the two intervals using standard methods.

11. $f(x) = |x + 3| = \begin{cases} x + 3, \text{if } x \geq -3 \\ -x - 3, \text{if } x < -3 \end{cases}$



12. The function is discontinuous at $p = 0$ and $p = 15$; continuous at $0 < p < 15$ and $p > 15$.

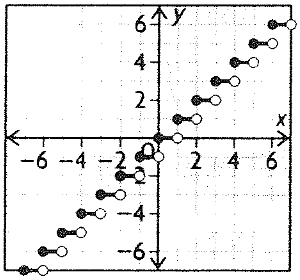
13. $f(x) = \begin{cases} 0, \text{ if } \ 0 \leq x < 10 \\ 10, \text{if } 10 \leq x < 20 \\ 20, \text{if } 20 \leq x < 30 \\ 30, \text{if } 30 \leq x < 40 \\ 40, \text{if } 40 \leq x < 50 \end{cases}$



It is often referred to as a step function because the graph looks like steps.

14. To make the first two pieces continuous, $5(-1) = -1 + k$, so $k = -4$. But if $k = -4$, the graph is discontinuous at $x = 3$.

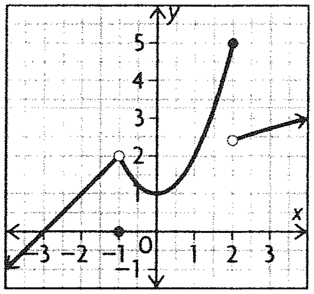
15.



16. Answers may vary. For example:

a) $f(x) = \begin{cases} x + 3, \text{if } x < -1 \\ x^2 + 1, \text{if } -1 \leq x \leq 2 \\ \sqrt{x} + 1, \text{if } x > 2 \end{cases}$

b)



c) The function is not continuous. The last two pieces do not have the same value for $x = 2$.

d) $f(x) = \begin{cases} x + 3, \text{if } x < -1 \\ x^2 + 1, \text{if } -1 \leq x \leq 1 \\ \sqrt{x} + 1, \text{if } x > 1 \end{cases}$

1.7 Exploring Operations with Functions, pp. 56–57

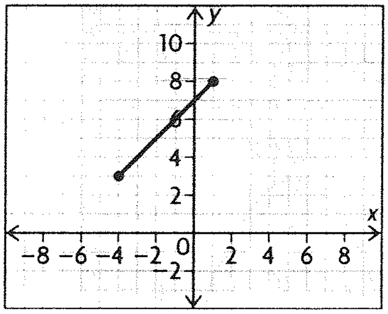
1. a) Add y-coordinates for the same x-coordinates of f and g.
$f + g = \{(-4, 6), (-2, 5), (1, 5), (4, 10)\}$

b) Subtract the y-coordinate of g from the y-coordinate of f for the same x-coordinates of f and g.
$f - g = \{(-4, 2), (-2, 3), (1, 1), (4, 2)\}$

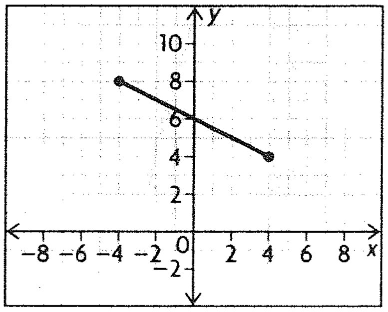
c) Subtract the y-coordinate of f from the y-coordinate of g for the same x-coordinates of f and g.
$g - f = \{(-4, -2), (-2, -3), (1, -1), (4, -2)\}$

d) Multiply y-coordinates for the same x-coordinates of f and g.
$fg = \{(-4, 8), (-2, 4), (1, 6), (4, 24)\}$

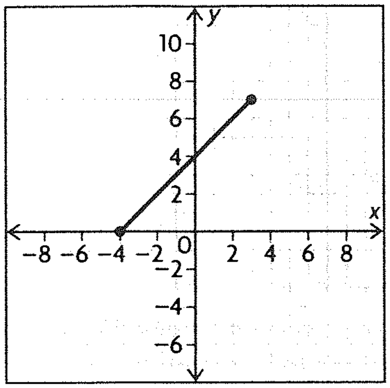
2. a)



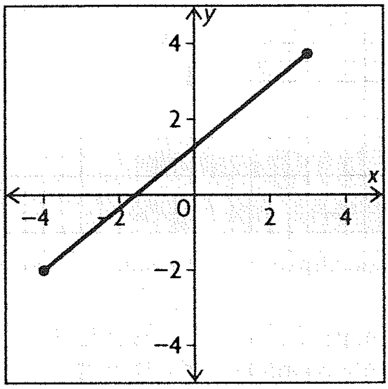
b)



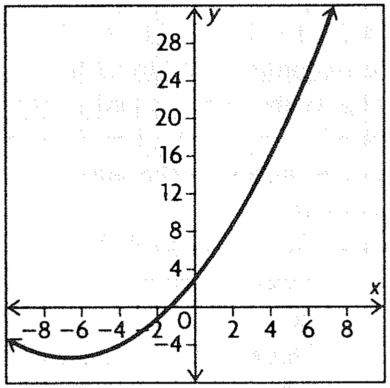
3. a)



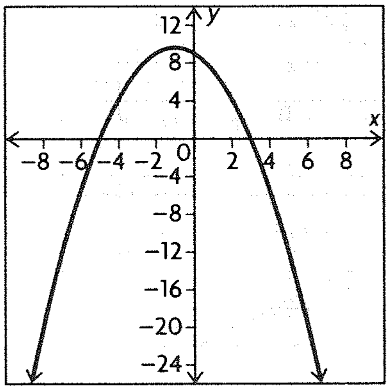
b)



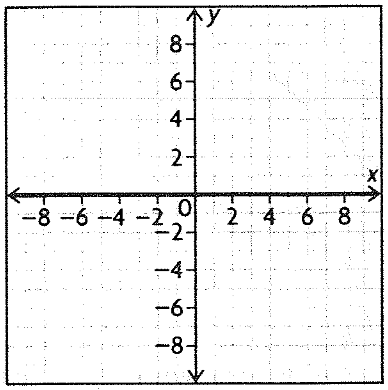
4. a)



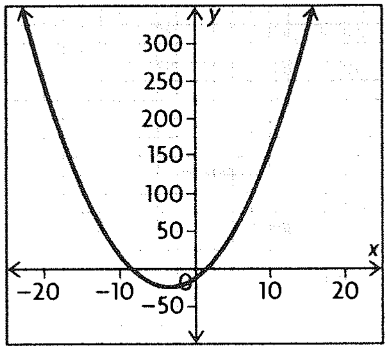
b)



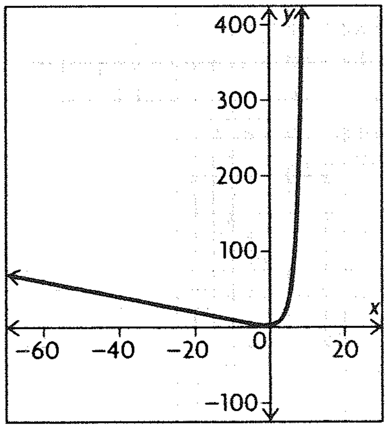
5. a) $h(x) = f(x) + g(x)$
$= x^2 + (-x^2)$
$= 0$



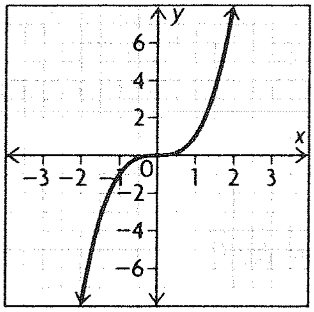
b) $p(x) = m(x) - n(x)$
$= x^2 - (-7x + 12)$
$= x^2 + 7x - 12$



c) $r(x) = s(x) + t(x)$
$= |x| + 2^x$

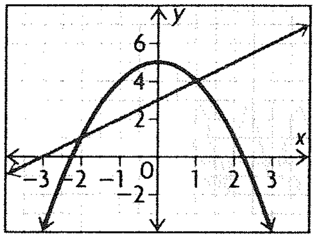


d) $a(x) = b(x) \times c(x)$
$= x \times x^2$
$= x^3$



6. a)–b) Answers may vary. For example, properties of the original graphs such as intercepts and sign at various values of the independent variable figure prominently in the shape of the new function.

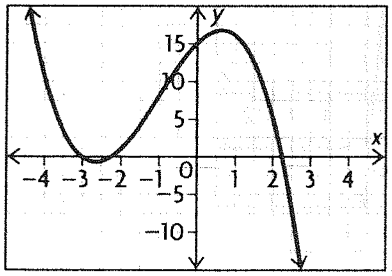
7. a)



b)

x	$f(x)$	$g(x)$	$h(x) = f(x) \times g(x)$
−3	0	−4	0
−2	1	1	1
−1	2	4	8
0	3	5	15
1	4	4	16
2	5	1	5
3	6	−4	−24

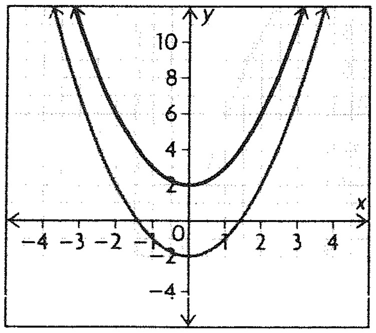
c)



d) $h(x) = (x + 3)(-x^2 + 5)$
$= -x^3 - 3x^2 + 5x + 15$; degree is 3

e) $D = \{x \in \mathbf{R}\}$; this is the same as the domain of both f and g.

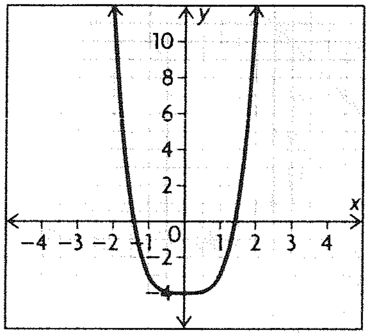
8. a)



b)

x	$f(x)$	$g(x)$	$h(x) = f(x) \times g(x)$
−3	11	7	77
−2	6	2	12
−1	3	−1	−3
0	2	−2	−4
1	3	−1	−3
2	6	2	12
3	11	7	77

c)



d) $h(x) = (x^2 + 2)(x^2 - 2) = x^4 - 4$; degree is 4

e) $D = \{x \in \mathbf{R}\}$

Chapter Review, pp. 60–61

1. a) This is a function; $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R}\}$

b) This is a function; $D = \{x \in \mathbf{R}\}$
$R = \{y \in \mathbf{R} | y \le 3\}$

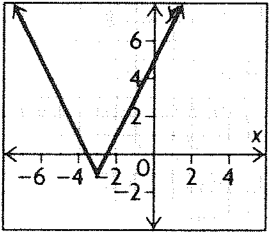
c) This is not a function; $D = \{x \in \mathbf{R} | -1 \le x \le 1\}$;
$R = \{y \in \mathbf{R}\}$

d) This is a function; $D = \{x \in \mathbf{R} | x > 0\}$
$R = \{y \in \mathbf{R}\}$

2. a) $C(t) = 30 + 0.02t$

b) $D = \{t \in \mathbf{R} | t \ge 0\}$; $R = \{C(t) \in \mathbf{R} | C(t) \ge 30\}$

3. $D = \{x \in \mathbf{R}\}$; $R = \{f(x) \in \mathbf{R} | f(x) \ge 1\}$



4. The number line has open circles at 2 and −2.
$|x| < 2$

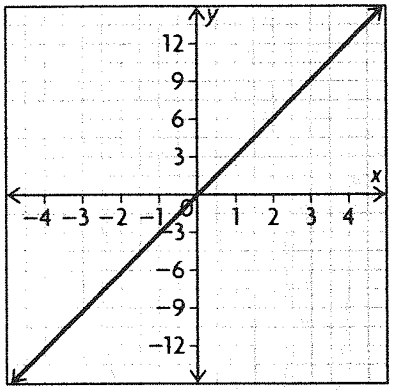
5. a) Both functions have a domain of all real numbers, but the ranges differ.

b) Both functions are odd but have different domains.

c) Both functions have the same domain and range, but x^2 is smooth and $|x|$ has a sharp corner at $(0, 0)$.

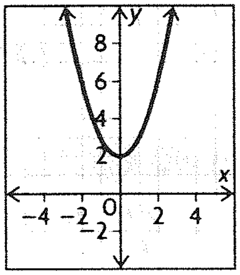
d) Both functions are increasing on the entire real line, but 2^x has a horizontal asymptote while x does not.

6. a)



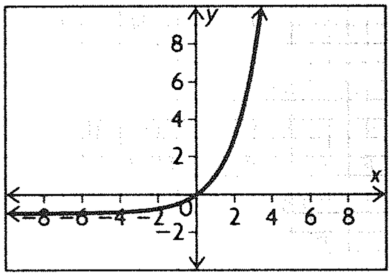
Increasing on $(-\infty, \infty)$; odd; $D = \{x \in \mathbf{R}\}$; $R = \{f(x) \in \mathbf{R}\}$

b)



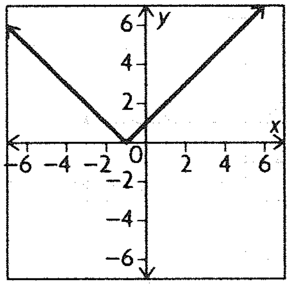
Decreasing on $(-\infty, 0)$; increasing on $(0, \infty)$; even; $D = \{x \in \mathbf{R}\}$; $R = \{f(x) \in \mathbf{R} | f(x) \geq 2\}$

c)

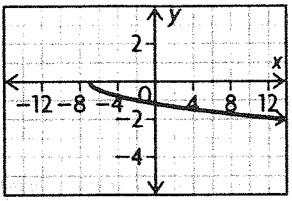


Increasing on $(-\infty, \infty)$; neither even nor odd; $D = \{x \in \mathbf{R}\}$; $R = \{f(x) \in \mathbf{R} | f(x) > -1\}$

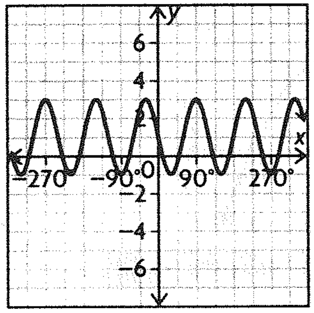
7. a) Parent: $y = |x|$; translated left 1



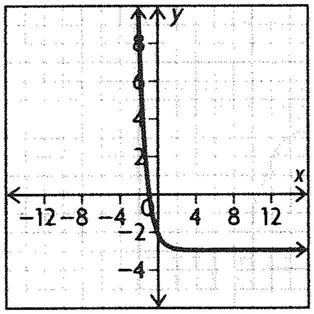
b) Parent: $y = \sqrt{x}$; compressed vertically by a factor of 0.25, reflected across the x-axis, compressed horizontally by a factor of $\frac{1}{3}$, and translated left 7



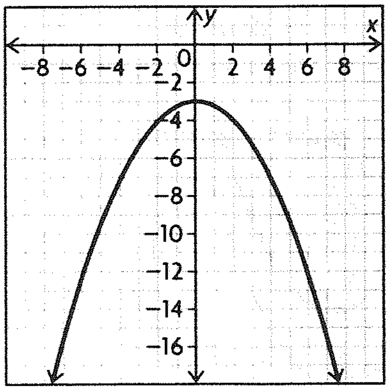
c) Parent: $y = \sin x$; reflected across the x-axis, expanded vertically by a factor of 2, compressed horizontally by a factor of $\frac{1}{3}$, translated up by 1



d) Parent: $y = 2^x$; reflected across the y-axis, compressed horizontally by a factor of $\frac{1}{2}$, and translated down by 3.



8. $y = -\left(\frac{1}{2}x\right)^2 - 3$;



9. a) $(2, 1) \rightarrow \left(\frac{2}{-1}, -1 + 2\right) = (-2, 1)$

b) $(2, 1) \rightarrow \left(-\frac{1}{2}(2) - 9, 1 - 7\right) = (-10, -6)$

c) $(2, 1) \rightarrow (2 + 2, 1 + 2) = (4, 3)$

d) $(2, 1) \rightarrow \left(\frac{1}{5}(2) + 3, 0.3(1)\right) = \left(\frac{17}{5}, 0.3\right)$

e) $(2, 1) \rightarrow (-2 + 1, -1 + 1) = (-1, 0)$

f) $(2, 1) \rightarrow \left(\frac{1}{2}(2) + 8, -1 \times 1\right) = (9, -1)$

10. a) $(1, 2) \rightarrow (2, 1)$

b) $(-1, -9) \rightarrow (-9, -1)$

c) $(0, 7) \rightarrow (7, 0)$

d) $f(5) = 7 \rightarrow (5, 7)$

So, $(7, 5)$ is on the inverse.

e) $g(0) = -3 \rightarrow (0, -3)$

So, $(-3, 0)$ is on the inverse.

f) $h(1) = 10 \rightarrow (1, 10)$

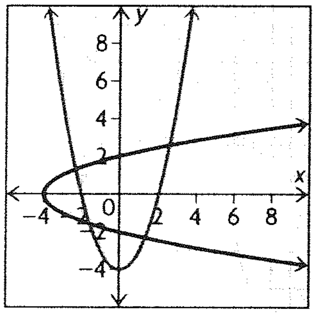
So, $(10, 1)$ is on the inverse.

11. The domain and the range of the original functions are switched for the inverses.

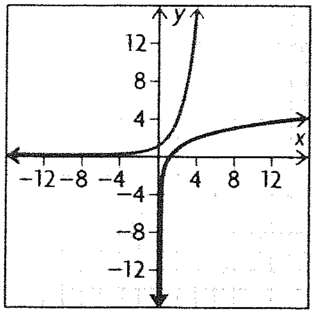
a) $D = \{x \in \mathbf{R} \mid -2 < x < 2\}$, $R = \{y \in \mathbf{R}\}$

b) $D = \{x \in \mathbf{R} \mid x < 12\}$, $R = \{y \in \mathbf{R} \mid y \geq 7\}$

12. a) The inverse relation is not a function.



b) The inverse relation is a function.



13. a) $f(x) = 2x + 1$

$x = 2y + 1$

$x - 1 = 2y$

$\frac{x - 1}{2} = y$

$f^{-1}(x) = \frac{x - 1}{2}$

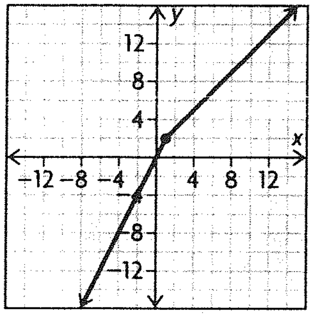
b) $g(x) = x^3$

$x = y^3$

$\sqrt[3]{x} = y$

$g^{-1}(x) = \sqrt[3]{x}$

14.



The function is continuous; $D = \{x \in \mathbf{R}\}$, $R = \{y \in \mathbf{R}\}$

15. $f(x) = \begin{cases} 3x - 1, \text{ if } x \leq 2 \\ -x, \text{ if } x > 2 \end{cases}$;

The function is discontinuous at $x = 2$.

16. In order for $f(x)$ to be continuous at $x = 1$, the two pieces must have the same value when $x = 1$. When $x = 1$, $x^2 + 1 = 2$, and $3x = 3$. The two pieces are not equal when $x = 1$, so the function is not continuous at $x = 1$.

17. a) For any number of minutes up to 200, the cost is \$30. For any number above 200 minutes, the charge is \$30 plus \$0.03 per minute above 200.

$30 + 0.03(x - 200) = 30 + 0.03x - 6$

$= 24 + 0.03x$

$f(x) = \begin{cases} 30, \text{ if } x \leq 200 \\ 24 + 0.03x, \text{ if } x > 200 \end{cases}$

b) $24 + 0.03(350) = \$34.50$

c) $180 < 200$, so the cost is \$30.

18. a) For x-coordinates that f and g have in common, add the corresponding y-coordinates.

$f + g = \{(1, 7), (4, 15)\}$

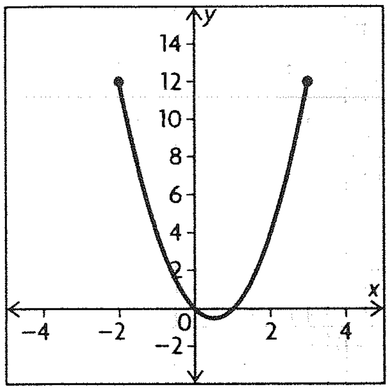
b) For x-coordinates that f and g have in common, subtract the corresponding y-coordinates.

$f - g = \{(1, -1), (4, -1)\}$

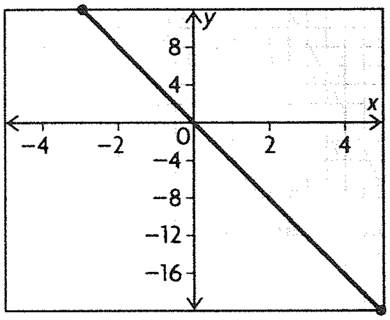
c) For x-coordinates that f and g have in common, multiply the corresponding y-coordinates.

$fg = \{(1, 12), (4, 56)\}$

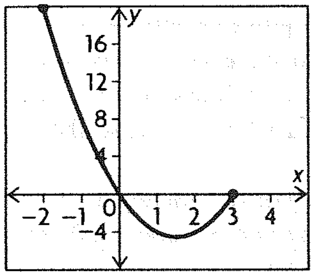
19. a)



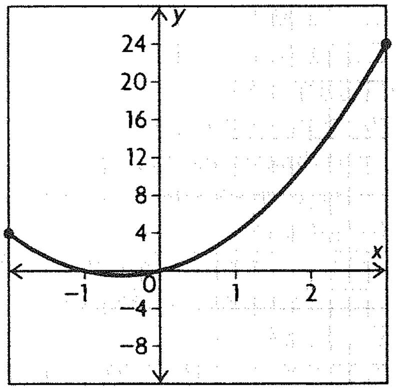
b)



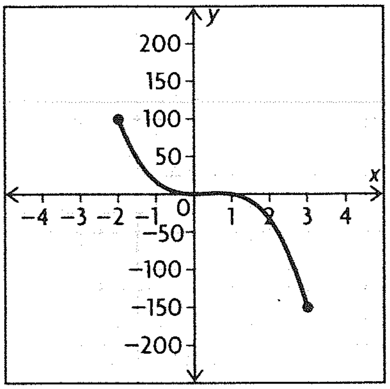
c) $f + g = 2x^2 - 2x + (-4x)$
$= 2x^2 - 6x, -2 \leq x \leq 3$



d) $f - g = 2x^2 - 2x - (-4x)$
$= 2x^2 + 2x, -2 \leq x \leq 3$



e) $fg = (2x^2 - 2x)(-4x)$
$= -8x^3 + 8x^2, -2 \leq x \leq 3$



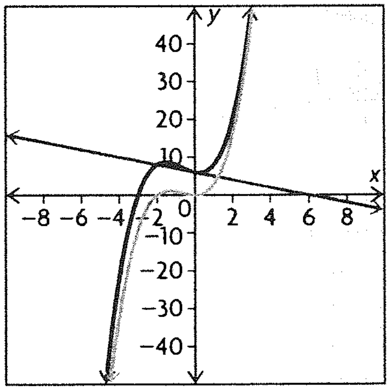
20. $f(x) = x^2 + 2x, g(x) = x + 1$
A $f(x) + g(x) = x^2 + 2x + x + 1$
$= x^2 + 3x + 1$
B $f(x) - g(x) = x^2 + 2x - (x + 1)$
$= x^2 + x - 1$
C $g(x) - f(x) = x + 1 - (x^2 + 2x)$
$= -x^2 - x + 1$
D $f(x) \times g(x) = (x^2 + 2x)(x + 1)$
$= x^3 + 3x^2 + 2x$

a) D
b) C
c) A
d) B

21. a)

x	−3	−2	−1	0	1	2
f(x)	−9	0	1	0	3	16
g(x)	9	8	7	6	5	4
(f + g)(x)	0	8	8	6	8	20

b)–c)



d) $(f + g)(x) = x^3 + 2x^2 + (-x + 6)$
$= x^3 + 2x^2 - x + 6$

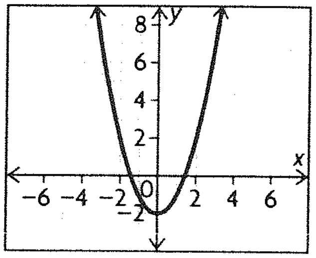
e) Answers may vary. For example, (0, 0) belongs to f, (0, 6) belongs to g, and (0, 6) belongs to $f + g$. Also, (1, 3) belongs to f, (1, 5) belongs to g, and (1, 8) belongs to $f + g$.

Chapter Self-Test, p. 62

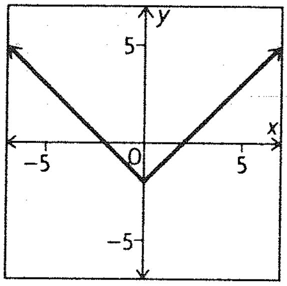
1. a) Yes. It passes the vertical line test.
b) $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} | y \geq 0\}$

2. a) $f(x) = x^2$ or $f(x) = |x|$

b)



or

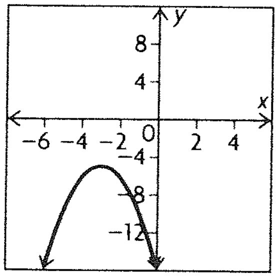


c) The graph was translated 2 units down.

3. $f(-x) = |3(-x)| + (-x)^2 = |3x| + x^2 = f(x)$

4. 2^x has a horizontal asymptote while x^2 does not. The range of 2^x is $\{y \in \mathbf{R} | y > 0\}$ while the range of x^2 is $\{y \in \mathbf{R} | y \geq 0\}$. 2^x is increasing on the whole real line and x^2 has an interval of decrease and an interval of increase.

5. reflection over the x-axis, translation down 5 units, translation left 3 units



6. horizontal stretch by a factor of 2, translation 1 unit up; $f(x) = |\frac{1}{2}x| + 1$

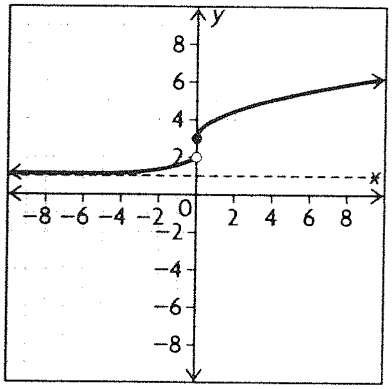
7. a) $(3, 5) \rightarrow (-3, -1, 5(3) + 2) = (-4, 17)$
b) $(3, 5) \rightarrow (5, 3)$

8.
$$f(x) = -2(x + 1)$$
$$x = -2(y + 1)$$
$$-\frac{x}{2} = y + 1$$
$$-\frac{x}{2} - 1 = y$$
$$f^{-1}(x) = -\frac{x}{2} - 1$$

9. a) $0.12(125000) - 6000 = \$9000$

b) $f(x) = \begin{cases} 0.05x, \text{ if } x \leq 50\,000 \\ 0.12x - 6\,000, \text{ if } x > 50\,000 \end{cases}$

10. a)



b) $f(x)$ is discontinuous at $x = 0$ because the two pieces do not have the same value when $x = 0$. When $x = 0$, $2^x + 1 = 2$ and $\sqrt{x} + 3 = 3$.
c) intervals of increase: $(-\infty, 0)$, $(0, \infty)$; no intervals of decrease
d) $D = \{x \in \mathbf{R}\}$, $R = \{y \in \mathbf{R} | 0 < y < 2 \text{ or } y \geq 3\}$

CHAPTER 2
Functions: Understanding Rates of Change

Getting Started, p. 66

1. The slope between two points can be found by dividing the change in y by the change in x, $\frac{\Delta y}{\Delta x}$.

a) $\frac{7-3}{5-2} = \frac{4}{3}$

b) $\frac{5-(-1)}{-4-3} = -\frac{6}{7}$

2. a) $-1 - 1 = -2$

$-5 - (-1) = -4$

$-13 - (-5) = -8$

$-29 - (-13) = -16$

$-61 - (-29) = -32$

Each successive first difference is 2 times the previous first difference. The function is exponential.

b) First differences

$11 - 0 = 11$

$28 - 11 = 17$

$51 - 28 = 23$

$80 - 51 = 29$

$115 - 80 = 35$

Second differences

$17 - 11 = 6$

$23 - 17 = 6$

$29 - 23 = 6$

$35 - 29 = 6$

The second differences are constant so the function is quadratic.

3. a) $0 = 2x^2 - x - 6$

$0 = (2x + 3)(x - 2)$

$0 = 2x + 3$ and $0 = x - 2$

$0 - 3 = 2x + 3 - 3$

$-3 = 2x$

$-\frac{3}{2} = x$

$0 + 2 = x - 2 + 2$

$2 = x$

The zeros are $-\frac{3}{2}$ and 2.

b) $0 = 2^x - 1$

$0 + 1 = 2^x - 1 + 1$

$1 = 2^x$

Any non-zero number raised to the exponent of 0 is 1, so $x = 0$.

c) $0 = \sin(x - 45°), 0° \le x \le 360°$

$\sin(0°)$, $\sin(180°)$, and $\sin(360°) = 0$.

$0° = x - 45°$, $180° = x - 45°$, and $360° = x - 45°$

$0° + 45° = x - 45° + 45°$

$45° = x$

$180° + 45° = x - 45° + 45°$

$225° = x$

$360° + 45° = x - 45° + 45°$

$405° = x$

Because $0° \le x \le 360°$, 405° cannot be a zero. The zeros are 45° and 225°.

d) $0 = 2\cos(x)$

$0 = \cos(x)$

For $-360° \le x \le 0°$, $\cos(-90°) = 0$ and $\cos(-270°) = 0$.

The zeros are $-90°$ and $-270°$.

4. a) $f(x)$ is compressed vertically by a factor of $\frac{1}{2}$.

b) $f(x)$ is stretched vertically by a factor of 2 and translated right 4 units.

c) $f(x)$ is stretched vertically by a factor of 3, reflected in the x-axis, and translated up 7 units.

d) $f(x)$ is stretched vertically by a factor of 5, translated right 3 units, and translated down 2 units.

5. a) \$1000 is P. 8% or 0.08 is i. $1 + i$ is 1.08. n is t. $A = P(1 + i)^n$ becomes $A = 1000(1.08)^t$

b) t is 3, $A = 1000(1.08)^3$ or \$1259.71

c) No, since the interest is compounded each year you earn more interest than the previous year. The interest earns interest.

6. a) $y = \sin x$ is a maximum at 90° so $15°t = 90°$ or $t = 6$.

$h(6) = 8 + 7\sin(15° \times 6)$. $h(6) = 15$ m. $y = \sin x$ is a minimum at 270° so $15°t = 270°$ or $t = 18$.

$h(18) = 8 + 7\sin(15° \times 18)$. $h(18) = 1$ m.

b) The period of $y = \sin x$ is 360°.

$15°t = 360°$ or $t = 24$ s.

c) $t = 30$. $h(30) = 8 + 7\sin(15° \times 30)$.

$h(30) = 15$ m.

7.

Rates of Change

Linear relations	**Nonlinear relations**
constant; same as slope of line; positive for lines that slope up from left to right; negative for lines that slope down from left to right; 0 for horizontal lines.	variable; can be positive, negative, or 0 for different parts of the same relation

Lesson 2.1 Determining Average Rate of Change, pp. 76–78

1. The average rate of change is equal to the change in y divided by the change in x.

a) $g(4) = 4(4)^2 - 5(4) + 1$
$= 64 - 20 + 1$
$= 45$

$g(2) = 4(2)^2 - 5(2) + 1$
$= 16 - 10 + 1$
$= 7$

Average rate of change $= \frac{45 - 7}{4 - 2} = 19$

b) $g(3) = 4(3)^2 - 5(3) + 1$
$= 36 - 15 + 1$
$= 22$

$g(2) = 7$

Average rate of change $= \frac{22 - 7}{3 - 2} = 15$

c) $g(2.5) = 4(2.5)^2 - 5(2.5) + 1$
$= 25 - 12.5 + 1$
$= 13.5$

$g(2) = 7$

Average rate of change $= \frac{13.5 - 7}{2.5 - 2} = 13$

d) $g(2.25) = 4(2.25)^2 - 5(2.25) + 1$
$= 20.25 - 11.25 + 1$
$= 10$

$g(2) = 7$

Average rate of change $= \frac{10 - 7}{2.25 - 2} = 12$

e) $g(2.1) = 4(2.1)^2 - 5(2.1) + 1$
$= 17.64 - 10.5 + 1$
$= 8.14$

$g(2) = 7$

Average rate of change $= \frac{8.14 - 7}{2.1 - 2} = 11.4$

f) $g(2.01) = 4(2.01)^2 - 5(2.01) + 1$
$= 16.1604 - 10.05 + 1$
$= 7.1104$

$g(2) = 7$

Average rate of change $= \frac{7.1104 - 7}{2.01 - 2} = 11.04$

2. a) i) According to the table, the height at $t = 2$ is 42.00 m and the height at $t = 1$ is 27.00 m.

$\frac{42 - 27}{2 - 1} = 15$ m/s

ii) According to the table, the height at $t = 4$ is 42.00 and $t = 3$ is 47.00 m.

$\frac{42 - 47}{4 - 3} = -5$ m/s

b) The flare is gaining height at 15 m/s and then loses height at 5 m/s.

3. $f(x)$ is always increasing at a constant rate. $g(x)$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, so the rate of change is not constant.

4. a) 1st half hour: $\frac{176 - 0}{0.5 - 0} = 352$ people/h

2nd half hour: $\frac{245 - 176}{1.0 - 0.5} = 138$ people/h

3rd half hour: $\frac{388 - 245}{1.5 - 1.0} = 286$ people/h

4th half hour: $\frac{402 - 388}{2.0 - 1.5} = 28$ people/h

5th half hour: $\frac{432 - 402}{2.5 - 2.0} = 60$ people/h

6th half hour: $\frac{415 - 432}{3.0 - 2.5} = -34$ people/h

b) the rate of growth of the crowd at the rally

c) A positive rate of growth indicates that people were arriving at the rally. A negative rate of growth indicates that people were leaving the rally.

5. a) Day 1: $\frac{203 - 0}{1 - 0} = 203$ km/day

Day 2: $\frac{396 - 203}{2 - 1} = 193$ km/day

Day 3: $\frac{561 - 396}{3 - 2} = 165$ km/day

Day 4: $\frac{739.5 - 561}{4 - 3} = 178.5$ km/day

Day 5: $\frac{958 - 739.5}{5 - 4} = 218.5$ km/day

Day 6: $\frac{1104 - 958}{6 - 5} = 146$ km/day

b) No; some days the distance travelled was greater than others.

6. The function is $f(x) = 4x$. To find the average rate of change find $\frac{\Delta f(x)}{\Delta x}$. The rate of change from $x = 2$ to $x = 6$ is:

$$\frac{f(6) - f(2)}{6 - 2}$$
$$= \frac{4(6) - 4(2)}{6 - 2}$$
$$= \frac{24 - 8}{6 - 2}$$
$$= \frac{16}{4} = 4$$

The rate of change from 2 to 26 is:

$$\frac{f(26) - f(2)}{26 - 2}$$
$$= \frac{4(26) - 4(2)}{26 - 2}$$
$$= \frac{104 - 8}{26 - 2}$$
$$= \frac{96}{24} = 4$$

The average rate of change is always 4 because the function is linear, with a slope of 4.

7. For any amount of time up to and including 250 minutes, the monthly charge is $39, therefore the rate of change is 0 for that interval. After 250 minutes the rate of change is a constant 10 cents per minute. The rate is not constant.

8. a) Find the ordered pairs for the intervals given.

Interval i): (20, 20) and (0, 5)
Interval ii): (40, 80) and (20, 20)
Interval iii): (60, 320) and (40, 80)
Interval iv): (60, 320) and (0, 5)

Use this information to find the change in population over the change in time.

i) $\frac{20 - 5}{20 - 0} = \frac{15}{20} = \frac{3}{4}$ or 750 people per year

ii) $\frac{80 - 20}{40 - 20} = \frac{60}{20} = 3$ or 3000 people per year

iii) $\frac{320 - 80}{60 - 40} = \frac{240}{20} = 12$ or 12 000 people per year

iv) $\frac{320 - 5}{60 - 0} = \frac{315}{60} = 5.25$ or 5250 people per year

b) No; the rate of growth increases as the time increases.

c) Assume that the growth continues to follow this pattern and that the population will be 5 120 000 people in 2050.

9. The function is $h(t) = 18t - 0.8t^2$. The average rate of change is $\frac{\Delta h(t)}{\Delta t}$ for the interval $10 \leq t \leq 15$.

$$\frac{\Delta h(t)}{\Delta t} = \frac{h(15) - h(10)}{15 - 10}$$
$$= \frac{18(15) - 0.8(15)^2 - (18(10) - 0.8(10)^2)}{15 - 10}$$
$$= \frac{90 - 100}{15 - 10}$$
$$= \frac{-10}{5}$$
$$= -2 \text{ m/s}$$

10. a) The function is
$P(s) = -0.30s^2 + 3.5s + 11.5$
The average rate of change is $\frac{\Delta P(s)}{\Delta s}$.

i) $\frac{P(2) - P(1)}{2 - 1}$

$$P(2) = -0.3(2)^2 + 3.5(2) + 11.15$$
$$= 16.95$$
$$P(1) = -0.3(1)^2 + 3.5(1) + 11.15$$
$$= 14.35$$
$$\frac{P(2) - P(1)}{2 - 1} = \frac{16.95 - 14.35}{2 - 1}$$
$$= 2.6$$

$2.60 per sweatshirt

ii) $P(3) = -0.3(3)^2 + 3.5(3) + 11.15$

$$= 18.95$$
$$P(2) = 16.95$$
$$\frac{P(3) - P(2)}{3 - 2} = \frac{18.95 - 16.95}{3 - 2}$$
$$= 2$$

$2.00 per sweatshirt

iii) $P(4) = -0.3(4)^2 + 3.5(4) + 11.15$

$$= 20.35$$
$$P(3) = 18.95$$
$$\frac{P(4) - P(3)}{4 - 3} = \frac{20.35 - 18.95}{4 - 3}$$
$$= 1.4$$

$1.40 per sweatshirt

iv) $P(5) = -0.3(5)^2 + 3.5(5) + 11.15$

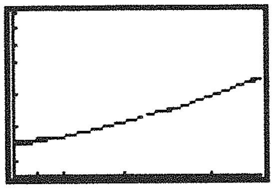
$$= 21.15$$
$$\frac{P(5) - P(4)}{5 - 4} = \frac{21.15 - 20.35}{5 - 4}$$
$$= 0.8$$

$0.80 per sweatshirt

b) The rate of change is still positive, but it is decreasing. This means that the profit is still increasing, but at a decreasing rate.

c) No; after 6000 sweatshirts are sold, the rate of change becomes negative. This means that the profit begins to decrease after 6000 sweatshirts are sold.

11. a)



b) If we were to find the average rate of change of an interval that is farther in the future, such as 2025–2050 instead of 2010–2015, the average rate of change would be greater. The graph indicates that the change in population increases as time increases. The graph is getting steeper as the values of t increase.

c) The function is $P(t) = 50t^2 + 1000t + 20\,000$.

The average rate of change is $\frac{\Delta P(t)}{\Delta t}$.

i)
$$\begin{aligned} P(10) &= 50(10)^2 + 1000(10) + 20\,000 \\ &= 35\,000 \\ P(0) &= 50(0)^2 + 1000(0) + 20\,000 \\ &= 20\,000 \\ \frac{P(10) - P(0)}{10 - 0} &= \frac{35\,000 - 20\,000}{10 - 0} \\ &= 1500 \text{ people per year} \end{aligned}$$

ii)
$$\begin{aligned} P(12) &= 50(12)^2 + 1000(12) + 20\,000 \\ &= 39\,200 \\ P(2) &= 50(2)^2 + 1000(2) + 20\,000 \\ &= 22\,200 \\ \frac{P(12) - P(2)}{12 - 2} &= \frac{39\,200 - 22\,200}{12 - 2} \\ &= 1700 \text{ people per year} \end{aligned}$$

iii)
$$\begin{aligned} P(15) &= 50(15)^2 + 1000(15) + 20\,000 \\ &= 46\,250 \\ P(5) &= 50(5)^2 + 1000(5) + 20\,000 \\ &= 26\,250 \\ \frac{P(15) - P(5)}{15 - 5} &= \frac{46\,250 - 26\,250}{15 - 5} \\ &= 2000 \text{ people per year} \end{aligned}$$

iv)
$$\begin{aligned} P(20) &= 50(20)^2 + 1000(20) + 20\,000 \\ &= 60\,000 \\ P(10) &= 35\,000 \\ \frac{P(20) - P(10)}{20 - 10} &= \frac{60\,000 - 35\,000}{20 - 10} \\ &= 2500 \text{ people per year} \end{aligned}$$

d) The prediction was correct.

12. Answers may vary. For example:

a) Someone might calculate the average increase in the price of gasoline over time. One might calculate the average decrease in the price of computers over time.

b) An average rate of change would be useful when there are several different rates of change over a specific interval.

c) The average rate of change is found by taking the change in y for the specified interval and dividing it by the change in x over that same interval.

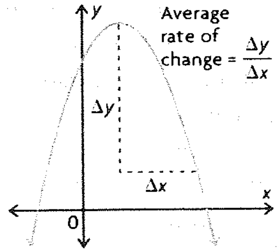
13. The car's starting value is \$23 500. After 8 years the car is only worth \$8750.

The average rate of change in the value of the car is

$$\begin{aligned} \frac{8750 - 23\,500}{8 - 0} &= \frac{-14\,750}{8} \\ &= -1843.75. \end{aligned}$$

The value of the car decreases, on average, by \$1843.75 per year. As a percent of the car's original value, this is $\frac{1843.75}{23\,500} \times 100$, or 7.8% decrease, or -7.8%

14. Answers may vary. For example:

AVERAGE RATE OF CHANGE		
Definition in your own words	**Personal example**	**Visual representation**
the change in one quantity divided by the change in a related quantity	I record the number of miles I run each week versus the week number. Then, I can calculate the average rate of change in the distance I run over the course of weeks.	

15. Calculate the fuel economy for several values of x.

x	$F(x) = -0.005x^2 + 0.8x + 12$
10	19.5
20	26.0
30	31.5
40	36.0
50	39.5
60	42.0
70	43.5
80	44.0
90	43.5
100	42.0
110	39.5

The fuel economy increases as x increases to 80 and then decreases. The speed that gives the best fuel economy is 80 km/h.

Lesson 2.2 Estimating Instantaneous Rates of Change from Tables of Values and Equations, pp. 85–88

1. a) The function is $f(x) = 5x^2 - 7$.

The average rate of change is $\frac{\Delta f(x)}{\Delta x}$.

$$f(2) = 13, f(1) = -2, \frac{\Delta f(x)}{\Delta x} = 15$$

$$f(2) = 13, f(1.5) = 4.25, \frac{\Delta f(x)}{\Delta x} = 17.5$$

$$f(2) = 13, f(1.9) = 11.05, \frac{\Delta f(x)}{\Delta x} = 19.5$$

$$f(2) = 13, f(1.99) = 12.8, \frac{\Delta f(x)}{\Delta x} = 19.95$$

$$f(3) = 38, f(2) = 13, \frac{\Delta f(x)}{\Delta x} = 25$$

$$f(2.5) = 24.25, f(2) = 13, \frac{\Delta f(x)}{\Delta x} = 22.5$$

$$f(2.1) = 15.05, f(2) = 13, \frac{\Delta f(x)}{\Delta x} = 20.5$$

$$f(2.01) = 13.2, f(2) = 13, \frac{\Delta f(x)}{\Delta x} = 20.05$$

Preceding Interval	$\Delta f(x)$	Δx	Average Rate of Change, $\frac{\Delta f(x)}{\Delta x}$
$1 \le x \le 2$	$13 - (-2) = 15$	$2 - 1 = 1$	15
$1.5 \le x \le 2$	8.75	0.5	17.5
$1.9 \le x \le 2$	1.95	0.1	19.5
$1.99 \le x \le 2$	0.1995	0.01	19.95

Following Interval	$\Delta f(x)$	Δx	Average Rate of Change, $\frac{\Delta f(x)}{\Delta x}$
$2 \le x \le 3$	$38 - 13 = 25$	$3 - 2 = 1$	25
$2 \le x \le 2.5$	11.25	0.5	22.5
$2 \le x \le 2.1$	2.05	0.1	20.5
$2 \le x \le 2.01$	0.2005	0.01	20.05

b) As the values of x get closer together on both sides of 2, the average rate of change gets closer to 20.

2. a) Find the average rate of change for intervals that approach 2.0 from both sides.

$$\frac{30.9 - 20.6}{2.0 - 1} = 10.3$$

$$\frac{30.9 - 26.98}{2.0 - 1.5} = 7.84$$

$$\frac{31.4 - 30.9}{3.0 - 2.0} = 0.5$$

$$\frac{32.38 - 30.9}{2.5 - 2.0} = 2.96$$

$$\frac{7.84 + 2.96}{2} = 5.4$$

$$\frac{10.3 + 0.5}{2} = 5.4$$

The instantaneous rate of change appears to be approaching 5.4.

b) Find the average rate of change for intervals that approach 2.0.

$$\frac{31.4 - 20.6}{3.0 - 1.0} = 5.4$$

$$\frac{32.38 - 26.98}{2.5 - 1.5} = 5.4$$

The instantaneous rate of change is approximately 5.4.

c) Answers may vary. For example: I prefer the centred interval method. Fewer calculations are required, and it takes into account points on each side of the given point in each calculation.

3. a) The population at 2.5 months is $P(2.5)$.

$$P(2.5) = 100 + 30(2.5) + 4(2.5)^2$$
$$= 200$$

b)

$$P(0) = 100 + 30(0) + 4(0)^2$$
$$= 100$$

$$\frac{200 - 100}{2.5 - 0} = 40 \text{ raccoons per month}$$

c) Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a + h) - f(a)}{h}$, where h is a very small value.

$$f(2.51) = 100 + 30(2.51) + 4(2.51)^2$$
$$= 200.5004$$

$$\frac{200.5004 - 200}{0.01} = 50.04 \text{ or } 50 \text{ raccoons per month}$$

d) Part **a)** asks for the value of $P(t)$ at 2.5; part **b)** asks for the average rate of change over a certain interval; part **c)** ask for the instantaneous rate of change at 2.5—they are all different values.

4. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.

a) $f(-1.99) = 6(-1.99)^2 - 4$
$= 19.7606$
$f(-2) = 6(-2)^2 - 4$
$= 20$

$\frac{20 - 19.76}{-2 - (-1.99)} = -23.94$ or -24

b) $f(0.01) = 6(0.01)^2 - 4$
$= -3.9994$
$f(0) = 6(0)^2 - 4$
$= -4$

$\frac{-3.9994 - (-4)}{0.01} = 0.06$ or 0

c) $f(4.01) = 6(4.01)^2 - 4$
$= 92.4806$
$f(4) = 6(4)^2 - 4$
$= 92$

$\frac{92.48 - 92}{0.01} = 48.06$ or 48

d) $f(8.01) = 6(8.01)^2 - 4$
$= 380.9606$
$f(8) = 6(8)^2 - 4$
$= 380$

$\frac{380.96 - 380}{0.01} = 96.06$ or 96

5. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.

The function is
$h(x) = -5x^2 + 3x + 65$.
$f(3.01) = -5(3.01)^2 + 3(3.01) + 65$
$= 28.7295$
$f(3) = -5(3)^2 + 3(3) + 65$
$= 29$

$\frac{28.7295 - 29}{0.01} = -27.05$ m/s or -27 m/s

6. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.

The function is $H(t) = 125\,000(1.06)^t$.
$H(8.01) = 125\,000(1.06)^{8.01}$
$= 199\,347.13$
$H(8) = 125\,000(1.06)^8$
$= 199\,231.01$

$\frac{199\,347.13 - 199\,231.01}{0.01} = \$11\,612$ per year or about $11 610 per year

7. a) The function is $P(t) = -1.5t^2 + 36t + 6$.

The average rate of change is $\frac{\Delta y}{\Delta x}$.
$P(24) = -1.5(24)^2 + 36(24) + 6$
$= 6$
$P(0) = -1.5(0)^2 + 36(0) + 6$
$= 6$

$\frac{6 - 6}{24 - 0} = 0$ people/year

b) Answers may vary. For example: Yes, it makes sense. It means that the populations in 2000 and 2024 are the same, so their average rate of change is 0.

c) $P(12) = -1.5(12)^2 + 36(12) + 6$
$= 222$
$P(0) = 6$

$\frac{222 - 6}{12 - 0} = 18$ thousand/year

$P(24) = 6$
$P(12) = 222$

$\frac{6 - 222}{24 - 12} = -18$ thousand/year

The average rate of change during the first 12 years was 18 000 per year. During the second 12 years it was − 18 000 per year. The population during year 0 is 6000 and during year 24 is 6000.

d) Because the average rate of change is the same on each side of 12, we know that the instantaneous rate of change would be 0 at 12.

8. Use the difference quotient to find the instantaneous rate of change.

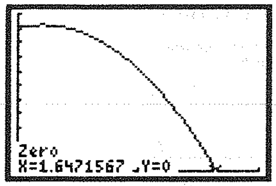
$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.

The function is $V(t) = 18\,999(0.93)^t$.
$f(5.01) = 18\,999(0.93)^{5.01}$
$= 13\,207.79$
$f(5) = 18\,999(0.93)^5$
$= 13\,217.38$

$\frac{13\,207.79 - 13\,217.38}{0.01} = -959$

When the car turns five, it loses about $960/year.

9. a) The diver will hit the water when $h(t) = 0$.
$10 + 2t - 4.9t^2 = 0$
Use a graphing calculator to determine the value of t for which the equation is true.



The diver enters the water at about $t = 1.65$ s.

b) Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a + h) - f(a)}{h}$, where h is a very small value.

$$h(1.66) = 10 + 2(1.66) - 4.9(1.66)^2$$
$$= -0.182\,44$$
$$h(1.65) = 10 + 2(1.65) - 4.9(1.65)^2$$
$$= -0.040\,25$$
$$\frac{-0.182\,44 + 0.040\,25}{0.01} = -14.219$$

The diver is travelling at a rate of about 14 m/s.

10. Use the centered interval method to estimate the instantaneous rate of change at $r = 5$. Find values of $V(r)$ on either side of 5.

$$V(5.1) = \frac{4}{3}\pi(5.1)^3 = 176.868\pi$$
$$V(4.9) = \frac{4}{3}\pi(4.9)^3 = 156.865\pi$$
$$\frac{176.868\pi - 156.865\pi}{5.1 - 4.9} = 314.21 \text{ or } 100\pi \text{ cm}^3\text{/cm}$$

Now, use the difference quotient to find the instantaneous rate of change.

$$V(5.01) = \frac{4}{3}\pi(5.01)^3 = 167.669\pi$$
$$V(5) = \frac{4}{3}\pi(5)^3 = 166.667\pi$$
$$\frac{167.669\pi - 166.667\pi}{0.01} = 314.63 \text{ or } 100\pi \text{ cm}^3\text{/cm}$$

11. David simply needs to keep track of the total distance that he's travelled and the amount of time that it has taken him to travel that distance. Dividing the distance travelled by the time required to travel that distance will give him his average speed.

12. a) Use a centered interval to find the instantaneous rate of change. $\frac{305 - 350}{5 - 3} = -22.5$ °F/min

b) Answers may vary. For example: A quadratic model for the oven temperature versus time is $y = -1.96x^2 - 9.82x + 400.71$. Using this model, the instantaneous rate of change at $x = 4$ is about -25.5 °F/min.

c) Answers may vary. For example, the first rate is using a larger interval to estimate the instantaneous rate.

d) Answers may vary. For example, the second estimate is better as it uses a much smaller interval to estimate the instantaneous rate.

13. Answers may vary. For example:

Method of Estimating Instantaneous Rate of Change	**Advantage**	**Disadvantage**
series of preceding intervals and following intervals	accounts for differences in the way that change occurs on either side of the given point	must do two sets of calculations
series of centred intervals	accounts for points on either side of the given interval in the same calculation	to get a precise answer, numbers involved will need to have several decimal places
difference quotient	more precise	calculations can be tedious or messy

14. a) The formula for finding the area of a circle is $A = \pi r^2$, where r is the radius. The average rate of change is $\frac{\Delta A}{\Delta r}$.

$$A = \pi(100)^2$$
$$= 10\,000\pi$$
$$A = \pi(0)^2$$
$$= 0$$
$$\frac{10\,000\pi - 0}{100 - 0} = 100\pi$$

The average rate of change is 100π cm²/cm.

b) Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a + h) - f(a)}{h}$, where h is a very small value.

$$A = \pi(120.01)^2$$
$$= \pi(14\,402.4001)$$
$$A = \pi(120)^2$$
$$= 14\,400\pi$$
$$\frac{14\,402.4001\pi - 14\,400\pi}{0.01} \doteq 754.01 \text{ cm}^2\text{/cm or}$$

240π cm²/cm

15. The formula for the surface area of a cube given the length of a side is $V = 6s^2$, where s is the side length of the cube. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a + h) - f(a)}{h}$, where h is a very small value.

$$SA = 6(3.01)^2$$
$$= 54.3606$$

$$\begin{aligned} SA &= 6(3)^2 \\ &= 54 \end{aligned}$$

$$\frac{54.3606 - 54}{0.01} = 36.06 \text{ cm}^2/\text{cm}$$

The instantaneous rate of change is about 36 cm^2/cm.

16. The formula for finding the surface area of a sphere is $SA = 4\pi r^2$. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h) - f(a)}{h}$, where h is a very small value.

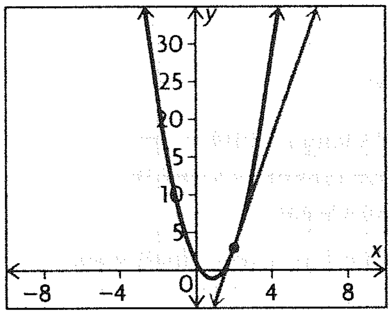
$$\begin{aligned} SA &= 4\pi(20.01)^2 \\ &= 1601.6004\pi \\ SA &= 4\pi(20)^2 \\ &= 1600\pi \end{aligned}$$

$$\frac{1601.6004\pi - 1600\pi}{0.01} \doteq 502.78 \text{ cm}^2/\text{cm}$$

The instantaneous rate of change is about 502.78 cm^2/cm or 160π cm^2/cm.

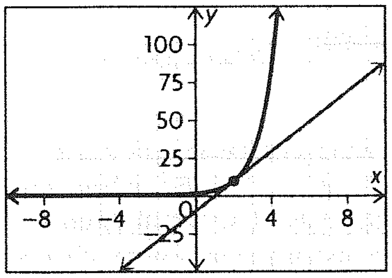
Lesson 2.3 Exploring Instantaneous Rates of Change Using Graphs, pp. 91–92

1. a) Answers may vary. For example:



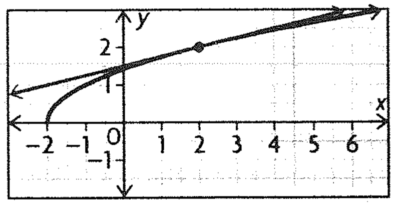
The slope is about 7.

b) Answers may vary. For example:



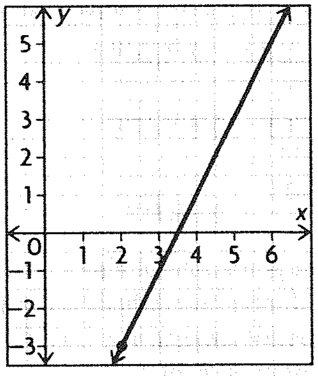
The slope is about 10.

c) Answers may vary. For example:



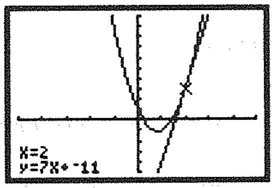
The slope is about 0.25.

d) Answers may vary. For example:



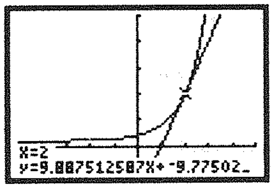
Because the graph is linear, the slope is the same everywhere. The slope is 2.

2. a)



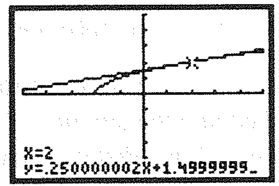
```
X=2
Y=7X+⁻11
```

b)



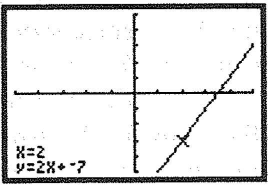
```
X=2
Y=9.8875125807X+⁻9.77502_
```

c)



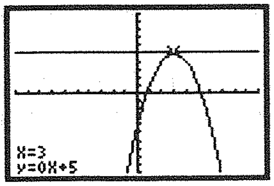
```
X=2
Y=.25000000002X+1.4999999_
```

d)



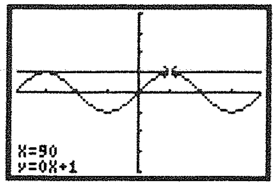
```
X=2
Y=2X+⁻7
```

3. a) Set A:

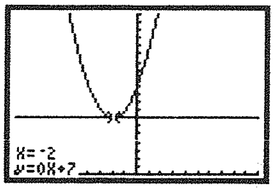


```
X=3
Y=0X+5
```

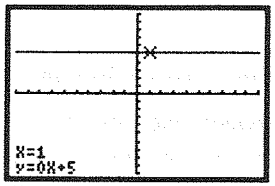
The slope of the tangent at $x = 3$ is 0.



The slope of the tangent at $x = 90°$ is 0.

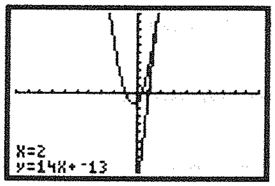


The slope of the tangent at $x = -2$ is 0.

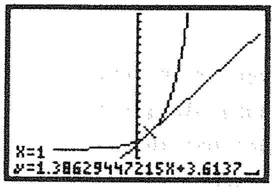


The slope of the tangent at $x = 1$ is 0.

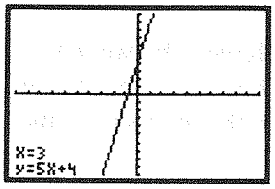
Set B:



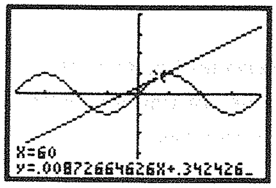
The slope of the tangent at $x = 2$ is 14.



The slope of the tangent at $x = 1$ is about 1.4.

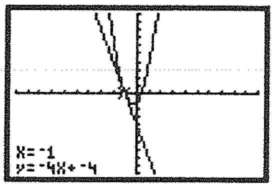


The slope of the tangent at $x = 3$ is 5.

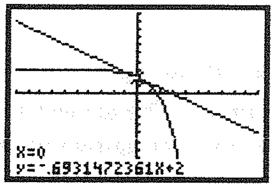


The slope of the tangent at $x = 60°$ is about 0.009.

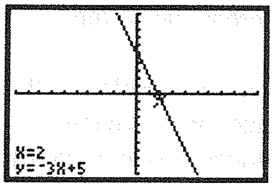
Set C:



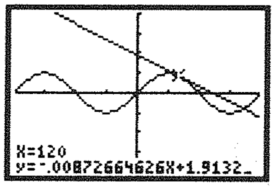
The slope of the tangent at $x = -1$ is -4.



The slope of the tangent at $x = 0$ is about -0.69.



The slope of the tangent at $x = 2$ is -3.



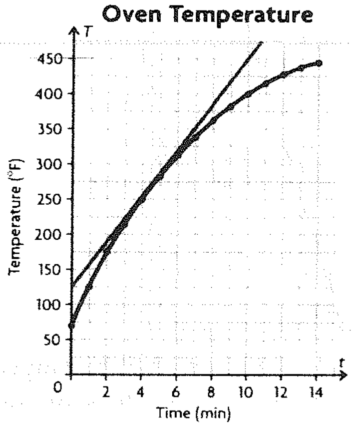
The slope of the tangent at $x = 120°$ is about -0.009.

b) Set: A: All slopes are zero.

Set: B: All slopes are positive.

Set: C: All slopes are negative.

4. a) and **b)**



c) The y-intercept of the tangent line appears to be 125 °F. Find the slope between the points (0, 125) and (5, 280).

$$\frac{280 - 125}{5 - 0} = 31$$

The slope is 31.

d) Use the data points (6, 310) and (4, 250).

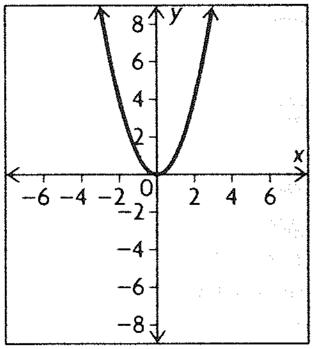
$$\frac{310 - 250}{6 - 4} = 30$$

The rate of change is about 30 °F/min at $x = 5$.

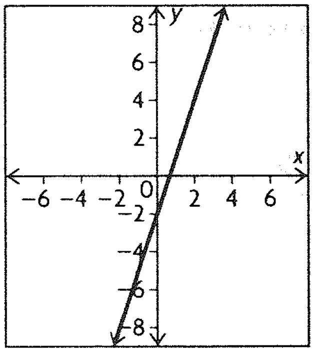
e) Answers may vary. For example: The answer in part d) is the slope of the line connecting two points on either side of $x = 5$. The answer in part c) is the slope of the line tangent to the function at point $x = 5$. The two lines are different and so their slopes will be different.

5. Answers may vary. For example, similarity: the calculation; difference: average rate of change is over an interval while instantaneous rate of change is at a point.

6. a)



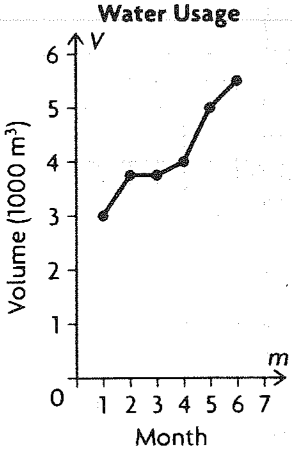
b)



c) From the graph, it appears that the tangent to the curve at (1.5, 2.25) would have the same slope as the secant line drawn.

Mid-Chapter Review, p. 95

1. a)



b) Rate of change is $\frac{\Delta f(x)}{\Delta x}$. Since we are looking for the amount of change between each month, Δx will always be 1 month. Therefore, we just need to find the difference in volume between each month.

m_1: $3.75 - 3.00 = 0.75$ 1000 m^3/month or 750 m^3/month

m_2: $3.75 - 3.75 = 0.00$ 1000 m^3/month or 0 m^3/month

m_3: $4.0 - 3.75 = 0.25$ 1000 m^3/month or 250 m^3/month

m_4: $5.10 - 4.00 = 1.10$ 1000 m^3/month or 1100 m^3/month

m_5: $5.50 - 5.10 = 0.40$ 1000 m^3/month or 400 m^3/month

c) Examine each of the answers from the previous exercises. The greatest amount is the greatest amount of change between two months.

$1.10 > 0.75 > 0.40 > 0.25 > 0.00$

The greatest amount of change occurred during $m4$, between April and May.

d) The change in y is the difference between the volume of water used in each month. The change in x is the difference between the numbers of the months.

$$\frac{5.50 - 3.75}{5 - 2} = 0.580 \times 1000 \text{ m}^3\text{/month or}$$

580 m^3/month

2. a) The equation models exponential growth. This means that the average rate of change between consecutive years will always increase.

b) Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a + h) - f(a)}{h}$, where h is a very small value.

$$\begin{aligned} f(10.01) &= 1.2(1.05)^{10.01} \\ &= 1.955\,627\,473 \\ f(10) &= 1.2(1.05)^{10} \\ &= 1.954\,673\,552 \end{aligned}$$

$$\frac{1.955\,627\,473 - 1.954\,673\,552}{0.01} = 0.095\,39$$

$0.095\,39 \times 10\,000 \doteq 950$ people per year

3. a) The average change for a specific interval is $\frac{\Delta h(t)}{\Delta t}$. The function is $h(t) = -5t^2 + 20t + 1$.

$$\begin{aligned} h(2) &= -5(2)^2 + 20(2) + 1 \\ &= 21 \\ h(0) &= -5(0)^2 + 20(0) + 1 \\ &= 1 \end{aligned}$$

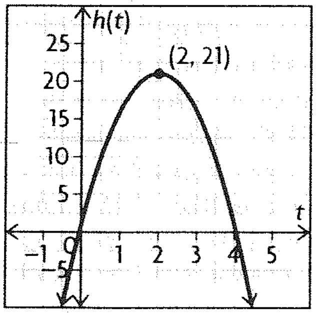
$$\frac{21 - 1}{2 - 0} = 10 \text{ m/s}$$

$$\begin{aligned} h(4) &= -5(4)^2 + 20(4) + 1 \\ &= 1 \\ h(2) &= 21 \end{aligned}$$

$$\frac{1 - 21}{4 - 2} = -10 \text{ m/s}$$

b) $t = 2$; Answers may vary. For example: The graph has its vertex at (2, 21). It appears that a tangent line at this point would be horizontal.

$$\frac{(f(2.01) - f(1.99))}{0.02} \doteq 0$$



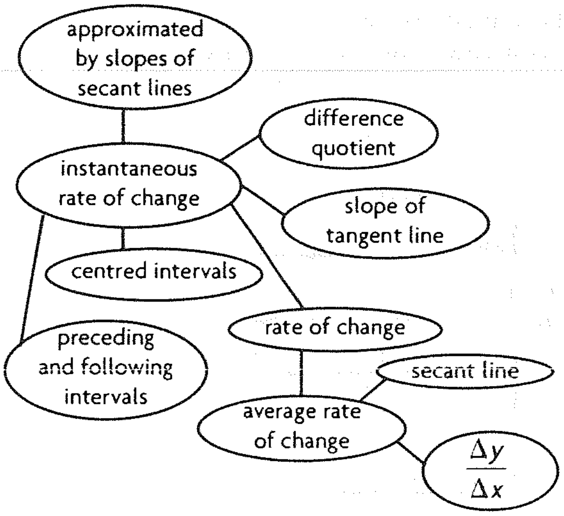
4. Use a centred interval.

$$\begin{aligned} d(20.01) &= 0.01(20.01)^2 + 0.5(20.01) \\ &= 14.009\,001 \\ d(19.99) &= 0.01(19.99)^2 + 0.5(19.99) \\ &= 13.991\,001 \end{aligned}$$

$$\frac{d(20.01) - d(19.99)}{20.01 - 19.99} = 0.9.$$

So the instantaneous rate of change in the glacier's position after 20 days is about 0.9 m/day.

5. Answers may vary. For example:

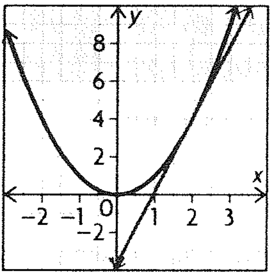


6. Answers may vary. For example: Find the value of y for different values of x on both sides of $x = 2$. Put this information in a table.

Points	**Slope of Secant**
(2, 9) and (1, 2)	7
(2, 9) and (1.5, 4.375)	9.25
(2, 9) and (1.9, 7.859)	11.41
(2, 9) and (2.1, 10.261)	12.61
(2, 9) and (2.5, 16.625)	15.25
(2, 9) and (3, 28)	19

The slope of the tangent line at (2, 9) is about 12.

7. Examine the graph.



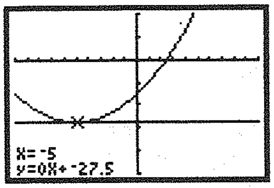
The tangent line appears to be passing through the points (1, 0) and (2, 4). Use this information to help determine the slope of the tangent line.

$$\begin{aligned} m &= \frac{\Delta f(x)}{\Delta x} \\ &= \frac{4 - 0}{2 - 1} \\ &= 4 \end{aligned}$$

The slope of the line is 4.

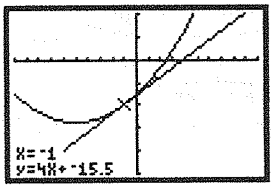
8. The instantaneous rate of change of the function whose graph is shown is 4 at $x = 2$.

9. a) Answers may vary. For example:



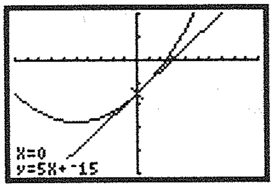
The slope is 0.

b)



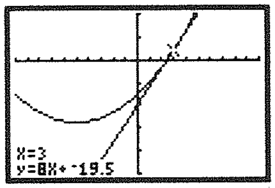
The slope is 4.

c)



The slope is 5.

d)



The slope is 8.

Lesson 2.4 Using Rates of Change to Create a Graphical Model, pp. 103–106

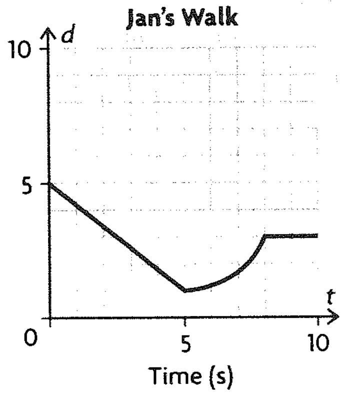
1. a) Graph a indicates that as time increases, distance also increases; C.

b) Graph b indicates that as time increases, distance decreases; A.

c) Graph c indicates that as time increases, the distance does not change; B.

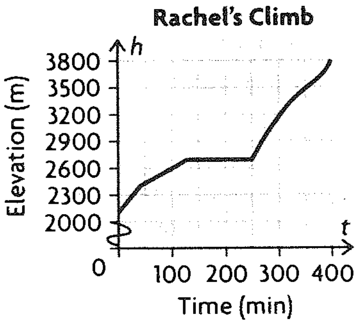
2. Graph a indicates that distance is increasing at a steady rate over time, meaning that the speed is constant. However, graph b indicates that distance is decreasing at a steady rate over time—this also indicates that the speed is constant. Graph c indicates that distance does not change so speed is 0, a constant. All 3 are constant speed.

3. Draw a graph of Jan's distance from the sensor over time. Jan is 5 m away from the sensor, which means that her initial position is (0, 5). She then walks 4 m towards the sensor for 5 seconds, which means that she will be standing 1 m away from the sensor. Her second position will be (5, 1). She then walks 3 metres away for 3 seconds, which means that she will be 4 m away from the sensor. Her third position will be (8, 4). Jan then stops and waits for 2 seconds, which means she stays 4 m away from the sensor for 2 seconds. Her fourth position will be (10, 4). Use this information to draw the graph.



4. a) Answers may vary. For example, draw a graph of Rachel's distance over time while climbing Mt. Fuji. Rachel begins the climb at Level 5 and so her initial position is (0, 2100). She walks for 40 minutes at a constant rate to move from Level 5 to Level 6, which means that her second position will be (40, 2400). It then takes he 90 minutes to move from Level 6 to Level 7, which means that her third position will be (130, 2700). Rachel then decides to rest for 2 hours, which means that her position does not change. So her fourth position is (250, 2700). After her break, it took Rachel 40 minutes to reach Level 8. Her fifth position is (290, 3100). It took Rachel 45 minutes to go from Level 8 to Level 9. Her next position is (335, 3400). After the walk from Level 9 to Level 10, Rachel reached the top. This position can be represented as (395, 3740).

Use this information to plot the graph.



b) Use the data points from the previous question to determine Rachel's average speed during each part of her journey.

$$\frac{2400 - 2100}{40 - 0} = 7.5 \text{ m/min}$$

$$\frac{2700 - 2400}{130 - 40} = 3.3 \text{ m/min}$$

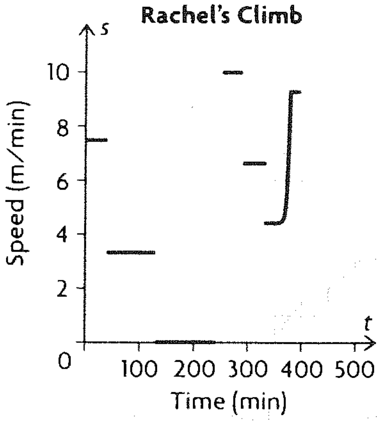
$$\frac{2700 - 2700}{250 - 130} = 0 \text{ m/min}$$

$$\frac{3100 - 2700}{290 - 250} = 10.0 \text{ m/min}$$

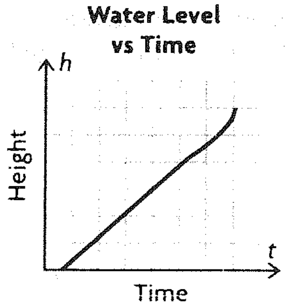
$$\frac{3400 - 3100}{335 - 290} = 6.7 \text{ m/min}$$

$$\frac{3740 - 3400}{395 - 335} = 5.7 \text{ m/min}$$

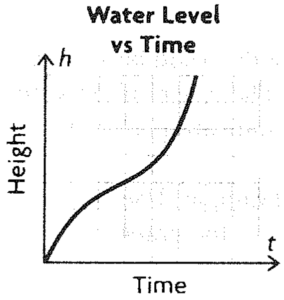
c) Answers may vary. For example, use Rachel's average rates to make a graph of her speed. During the first 40 minutes of her journey, her speed was 7.5 m/min. This can be represented by a straight line from (0, 7.5) to (40, 7.5). Rachel's speed during the next 90 minutes is 3.3 m/min. This speed can be represented by a straight line from (40, 3.3) to (130, 3.3). Rachel then rested for 2 hours. This can be represented with a straight line from (130, 0) to (250, 0). Rachel travelled at a rate of 10.0 m/min for the next 40 minutes. This speed can be represented by a straight line from (250, 10.0) to (290, 10.0). Then she travelled at a rate of 6.7 m/min for 45 minutes, 4.4 m/min for 45 minutes, and 9.3 m/min for 15 minutes. The speeds for these parts or her walk can be represented by the following segments: (290, 6.7) to (335, 6.7), (335, 4.4) to (380, 4.4), and (380, 9.3) to (395, 9.3).



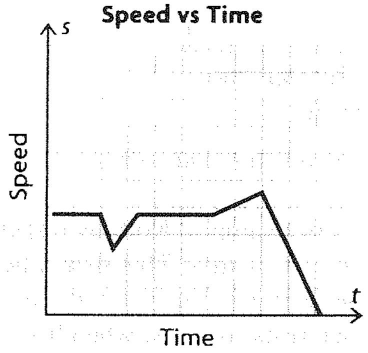
5. a) Answers may vary. For example, the 2 L plastic pop bottle has a uniform shape for the most part. Therefore, as long as the rate of water flowing into the bottle remains constant, the rate at which the height is changing will also remain constant.



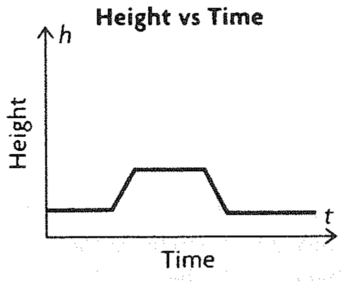
b) Answers may vary. For example, the circumference of the vase changes for any given height on the vase. Therefore, the rate of change of the height of the water flowing into the vase will vary over time—faster at the very bottom of the vase, slower in the middle and then faster again at the top.



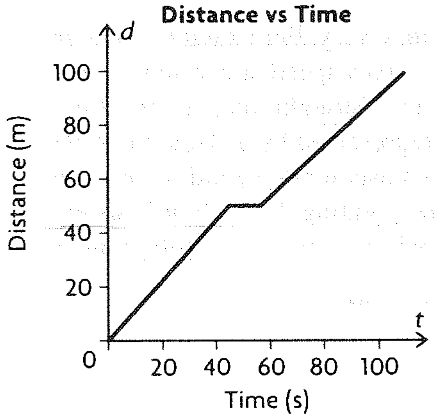
6. a) Answers may vary. For example, on a graph that represents John's speed, a constant speed would be represented by a straight line, any increase in rate would be represented by a slanted line pointing up, and any decrease in rate would be represented by a slanted line pointing down. John's speed over his bike ride could be represented following graph.



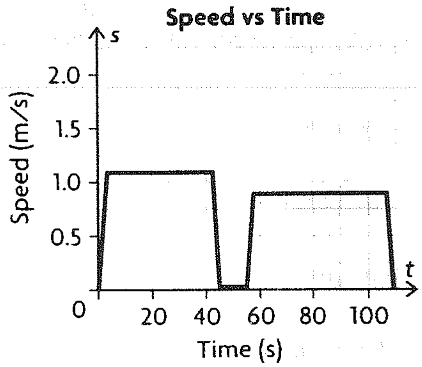
b) Answers may vary. For example, the first part of John's bicycle ride is along a flat road. His height over this time would be constant. As he travels up the hill, his height would increase. At the top of the hill, his height would again be constant. As he goes down the hill, his height would decrease. As he climbs the second hill his height would again increase. The graph of his height over time would look something like this.



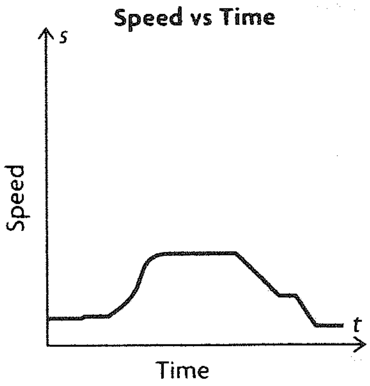
7. a) Kommy travels 50 m in 45 seconds. This means that his speed would be $\frac{50}{45} = 1.11$ m/s.
b) During the second part of his swim he travelled 50 m in 55 s. This means that his speed would be $\frac{50}{55} = 0.91$ m/s.
c) The graph of the first length would be steeper, indicating a quicker speed. The graph of the second length would be less steep, indicating a slower speed.
d) Answers may vary. For example: Use the information for part c) to draw the graph of Kommy's distance over time.



e) At $t = 50$, Kommy is resting, and so his speed would be 0.
f) Answers may vary. For example: Kommy's speed for the first 45 seconds is 1.11 m/s. This would be represented by the line segment from (0, 1.11) to (45, 1.11). Kommy then rests for 10 s, when his speed would be 0. This would be represented by a line segment from (45, 0) to (55, 0). Kommy's speed during the second half of his swim is 0.91 m/s. This would be represented by a line segment from (55, 0.91) to (110, 0.91).



8. a) A – if the rate at which a speed is increasing increases, this would be represented by an upward curve.
b) C – if the rate at which a speed is decreasing decreases over time, this would be represented by a curve that drops sharply at first and then drops more gradually.
c) D – if the rate at which a speed is decreasing increases, this would be represented by a downward curve.
d) B – if the rate at which a speed is increasing decreases, this would be represented by a curve that rises sharply at first and then rises more gradually.
9. Answers may vary. For example: Because the jockey is changing the horse's speed at a non-constant rate—at first slowly and then more quickly—the lines will have an upward curve when the horse is accelerating and a downward curve when decelerating. The horse's speed during the first part of the warm up is constant, which would be represented by a straight line. She then increases the horse's speed to a canter and keeps this rate for a while. Draw a graph of this information with speed over time.



10. a) Graph i) shows that distance is decreasing and then increasing. The first graph shows a person standing 5 m away from the motion sensor then moving to 2 m away. The person then moves back to 5 m away from the motion sensor. The person is always moving at a constant rate. Graph ii) shows a person's initial position being 6 m away from the motion sensor. This person then moves 2 m closer to the sensor over 2 seconds. Then, he or she rests for a second and then moves 2 m closer to the sensor over 2 more seconds. Finally, this person moves 2 m away from the sensor over 1 second to end up at a final position of about 4 m away from the sensor. The person is always moving at a constant rate.

b) For each graph, determine the (t, d) point for each position.

Graph A

(0, 5); (3, 2); (6, 5)

Graph B

(0, 6); (2, 4); (3, 4); (5, 2); (6, 3.5)

Use these points to find the various speeds.

Graph A

$$\frac{2-5}{3-0} = -1, \text{ so the speed is } 1 \text{ m/s}$$

$$\frac{5-2}{6-3} = 1, \text{ so the speed is } 1 \text{ m/s}$$

Graph B

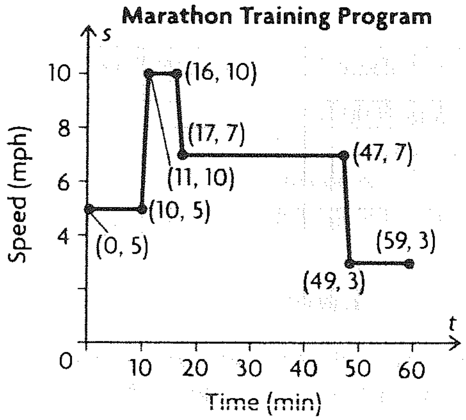
$$\frac{4-6}{2-0} = -2, \text{ so the speed is } 2 \text{ m/s}$$

$$\frac{4-4}{3-2} = 0, \text{ so the speed is } 0 \text{ m/s}$$

$$\frac{2-4}{5-3} = -1, \text{ so the speed is } 1 \text{ m/s}$$

$$\frac{3.5-2}{6-5} = 1.5, \text{ so the speed is } 1.5 \text{ m/s}$$

11. a) Answers may vary. For example: Draw a graph of the runner's speed over time. The runner's positions on the graph will be represented by the following points: (0, 5), (10, 5), (11, 10), (16, 10), (17, 7), (47, 7), (49, 3), (59, 3). Plot the points on a graph. Because the runner accelerates and decelerates at a constant rate, the lines will always be straight.



b) Use the data points on either side of $t = 10.5$ to estimate the instantaneous rate of change at that point. The points are (10, 5), (11, 10).

$$\frac{10-5}{11-10} = 5 \text{ mi/h/min}$$

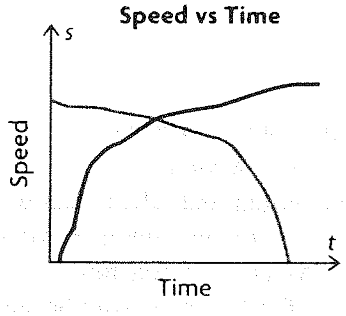
c) The runner's speed at minute 11 is 10 miles per hour. The runner's speed at minute 49 is 3 miles per hour.

$$\frac{3-10}{49-11} = \frac{-7}{38} = -0.1842 \text{ miles per hour per minute}$$

d) The answer to part c) is an average rate of change over a long period, but the runner does not slow down at a constant rate during this period.

12. Answers may vary. For example: Walk from (0, 0) to (5, 5) and stop for 5 s. Then run to (15, 30). Continue walking to (25, 5) and end at (25, 0). What is the maximum speed and the minimum speed on any interval? Create the speed time graph from these data.

13. Answers may vary. For example: Graphing both women's speeds on the same graph would mean that there are two lines on the graph. The first woman is decelerating; this means that her line would have a downward direction. Because she is decelerating slowly first and then more quickly, the line would also have a downward curve. The second woman is accelerating; this means that her line will have an upward direction. Because she is accelerating quickly at first and then more slowly, the graph would have a sharp upward curve. The line on the graph would look something like this:



14. If the original graph showed an increase in rate, it would mean that the distance travelled during each successive unit of time would be greater—meaning a graph that curves upward. If the original graph showed a straight, horizontal line, then it would mean that the distance travelled during each successive unit of time would be greater—meaning a steady increasing straight line on the second graph. If the original graph showed a decrease in rate, it would mean that the distance travelled during each successive unit of time would be less—meaning a line that curves down.

Lesson 2.5 Solving Problems Involving Rates of Change, pp. 111–113

1. Answers may vary. For example: Verify that the most economical production level occurs when 1500 items are produced by examining the rate of change at $x = 1500$. Because x is in thousands, use $a = 1.5$. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h) - f(a)}{h}$, where h is a very small value.

$$\begin{aligned} C(1.5) &= 0.3(1.5)^2 - 0.9(1.5) + 1.675 \\ &= 1 \\ C(1.501) &= 0.3(1.501)^2 - 0.9(1.501) + 1.675 \\ &= 1.000\,000\,3 \end{aligned}$$

$$\frac{1.000\,000\,3 - 1}{0.01} = 0.000\,03$$

When 1500 items are produced, the instantaneous rate of change is zero. Therefore, the most economical production level occurs when 1500 items are produced.

2. The function is $P(t) = -20\cos(300°t) + 100$. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h) - f(a)}{h}$, where h is a very small value.

$$\begin{aligned} P(3) &= -20\cos(300° \times 3) + 100 \\ &= 120 \\ P(3.01) &= -20\cos(300° \times 3.001) + 100 \\ &= 119.999\,73 \end{aligned}$$

$$\frac{119.999\,73 - 120}{0.001} = -0.27 \text{ or } 0$$

The blood pressure is dropping at a rate of 0 millimetres of mercury per second.

3. a) If $(a, f(a))$ is a maximum, then the points to the left of, and very close to the maximum, have a positive rate of change. As $x(a)$ approaches $(a, f(a))$ from the left, $y(f(a))$ is increasing because $(a, f(a))$ is a maximum.

b) If $(a, f(a))$ is a maximum, then the points to the right of, and very close to the maximum, have negative rate of change. As $x(a)$ moves away from $(a, f(a))$ to the right, $y(f(a))$ is decreasing because $(a, f(a))$ is a maximum.

4. a) If $(a, f(a))$ is a minimum, then the points to the left of, and very close to the maximum, have negative rate of change. As $x(a)$ moves toward $(a, f(a))$ from the left, $y(f(a))$ is decreasing because $(a, f(a))$ is a minimum.

b) If $(a, f(a))$ is a minimum, then the points to the right of, and very close to the maximum, have a positive rate of change. As $x(a)$ moves away from $(a, f(a))$ towards the right, $y(f(a))$ is increasing because $(a, f(a))$ is a maximum.

5. a) The leading coefficient is positive, and so the value given will be a minimum. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h) - f(a)}{h}$, where h is a very small value.

Find $f(-6)$ and $f(-5.99)$. The function is $f(x) = 0.5x^2 + 6x + 7.5$.

$$\begin{aligned} f(-6) &= 0.5(-6)^2 + 6(-6) + 7.5 \\ &= -10.5 \\ f(-5.99) &= 0.5(-5.99)^2 + 6(-5.99) + 7.5 \\ &= -10.499\,995 \end{aligned}$$

$$\frac{-10.5 - (-10.499\,995)}{0.01} = -0.0005 \text{ or } 0$$

The slope is very small, pretty close to zero, and so it can be assumed that $(-6, -10.5)$ is the minimum.

b) The leading coefficient is negative, and so the value given will be a maximum. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h) - f(a)}{h}$, where h is a very small value.

Find $f(0.5)$ and $f(0.501)$. The function is $f(x) = -6x^2 + 6x + 9$.

$$\begin{aligned} f(0.501) &= -6(0.501)^2 + 6(0.501) + 9 \\ &= 10.499\,994 \\ f(0.5) &= -6(0.5)^2 + 6(0.5) + 9 \\ &= 10.5 \end{aligned}$$

$$\frac{10.499\,994 - 10.5}{0.01} = -0.000\,6 \text{ or } 0$$

The number is very close to zero, and so we can assume that the point has an instantaneous rate of change of zero and is a maximum.

c) The function is $f(x) = 5\sin(x)$. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h) - f(a)}{h}$, where h is a very small value.

Find $f(90°)$ and $f(90.01°)$.

$$\begin{aligned} f(90.01°) &= 5\sin(90.01°) \\ &= 4.999\,999 \\ f(90°) &= 5\sin(90°) \\ &= 5 \end{aligned}$$

$$\frac{4.999\,999 - 5}{0.01} = -0.0001 \text{ or } 0$$

The number is very close to zero, and so we can assume that the instantaneous rate of change at the point is zero, and so the point is a maximum.

d) The function is $f(x) = -4.5 \cos (2x)$. Use the difference quotient to find the instantaneous rate of change.

$\dfrac{f(a+h) - f(a)}{h}$, where h is a very small value.

Find $f(0)$ and $f(0.01)$.

$$\begin{aligned} f(0.01) &= -4.5 \cos (2 \times 0.01)^\circ \\ &= -4.499\,999 \\ f(0) &= -4.5 \cos (2 \times 0)^\circ \\ &= -4.5 \end{aligned}$$

$$\frac{-4.499\,999 - (-4.5)}{0.01} = 0.0001 \text{ or } 0$$

The number is very close to zero, and so we can assume that the instantaneous rate of change at the point is zero, and so the point is a maximum.

6. Examine the instantaneous rates of change on either side of the point in question. If the point to the left of the point in question is negative, then the point is a minimum. If the point to the left of the point in question is positive, then the point is a maximum. If the point to the right of the point in question is positive, then the point is a minimum. If the point to the right of the point in question is negative, then the point is a maximum. Use the difference quotient to find the instantaneous rate of change.

$\dfrac{f(a+h) - f(a)}{h}$, where h is a very small value.

a) $f(x) = x^2 - 4x + 5$; $(2, 1)$

Examine $x = 1$, which is to the left of $(2, 1)$.

$$\begin{aligned} f(1.01) &= (1.01)^2 - 4(1.01) + 5 \\ &= 1.97 \\ f(1) &= (1)^2 - 4(1) + 5 \\ &= 2 \end{aligned}$$

$$\frac{1.97 - 2}{0.01} = -3$$

The instantaneous rate of change of $(1, 2)$ is negative and so $(2, 1)$ is a minimum.

b) $f(x) = -x^2 - 12x + 5.75$; $(-6, 41.75)$

Examine $x = -5$, which is to the right of $(-6, 41.75)$.

$$\begin{aligned} f(-4.99) &= -(-4.99)^2 - 12(-4.99) + 5.75 \\ &= 40.7299 \\ f(-5) &= -(-5)^2 - 12(-5) + 5.75 \\ &= 40.75 \end{aligned}$$

$$\frac{40.7299 - 40.75}{0.01} = -2.01$$

The instantaneous rate of change of $(-5, 40.75)$ is -2.01, and so $(-6, 41.75)$ is a maximum.

c) $f(x) = x^2 - 9x$; $(4.5, -20.25)$

Examine $x = 5$, which is to the right of $(4.5, -20.25)$.

$$\begin{aligned} f(5.01) &= (5.01)^2 - 9(5.01) \\ &= -19.899 \\ f(5) &= (5)^2 - 9(5) \\ &= -20 \end{aligned}$$

$$\frac{-19.899 - (-20)}{0.01} = 10.1$$

The instantaneous rate of change at $(5, -20)$ is positive and so $(4.5, -20.25)$ is a minimum.

d) $f(x) = 3 \cos x$; $(0^\circ, 3)$

Examine $x = -1^\circ$, which is to the left of $(0^\circ, 3)$.

$$\begin{aligned} f(-0.99^\circ) &= 3 \cos (-0.99^\circ) \\ &= 2.999\,55 \\ f(-1^\circ) &= 3 \cos (-1^\circ) \\ &= 2.999\,54 \end{aligned}$$

$$\frac{2.999\,55 - 2.999\,54}{0.01} = 0.001$$

The instantaneous rate of change at $(-1^\circ, 2.99)$ is positive, and so $(0^\circ, 3)$ is a maximum.

e) $f(x) = x^3 - 3x$; $(-1, 2)$

Examine $x = 0$, which is to the right of $(-1, 2)$.

$$\begin{aligned} f(0.01) &= (0.01)^3 - 3(0.01) \\ &= -0.029\,999 \\ f(0) &= (0)^3 - 3(0) \\ &= 0 \end{aligned}$$

$$\frac{-0.029\,999 - 0}{0.01} = -2.9999$$

The instantaneous rate of change at $(0, 0)$ is -2.9999, and so $(-1, 2)$ is a maximum.

f) $f(x) = -x^3 + 12x - 1$; $(2, 15)$

Examine $x = 1$, which is to the left of $(2, 15)$.

$$\begin{aligned} f(1.01) &= -(1.01)^3 + 12(1.01) - 1 \\ &= 10.0897 \\ f(1) &= -(1)^3 + 12(1) - 1 \\ &= 10 \end{aligned}$$

$$\frac{10.0897 - 10}{0.01} = 8.97$$

The instantaneous rate of change at $(1, 10)$ is 8.97, and so $(2, 15)$ is a maximum.

7. Use a table to inspect several values of $h(t)$.

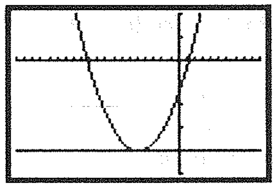
t	$h(t)$
0	10 000
1	10 074
2	10 116
3	10 126
4	10 104
5	10 050

The height is definitely decreasing after $t = 3$, but for this data the exact maximum cannot be determined. Examine other values of t to help determine the maximum.

t	$h(t)$
2.75	10 126.50
3.25	10 123.50

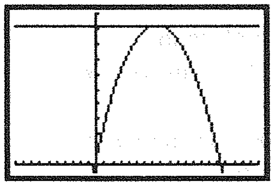
The maximum appears to be pretty close to 2.75. The slopes of tangents for values of t less than about 2.75 would be positive, while slopes of tangents for values of t greater than about 2.75 would be negative.

8. a) i)



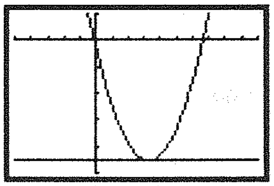
The minimum is at approximately $x = -5$.

ii)



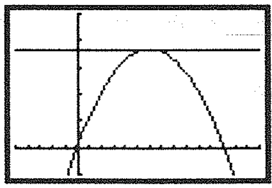
The maximum is at $x = 7.5$.

iii)



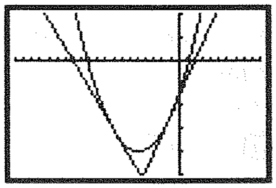
The minimum is at approximately $x = 3.25$.

iv)

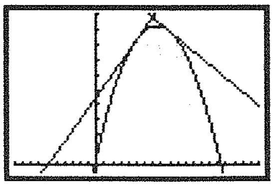


The maximum is at $x = 6$.

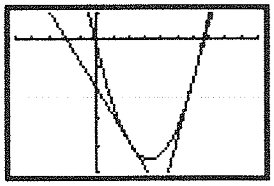
b) i)



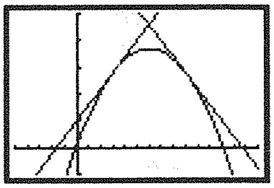
ii)



iii)

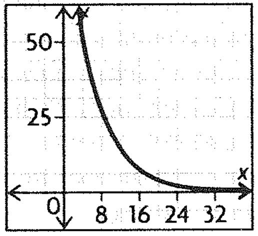


iv)



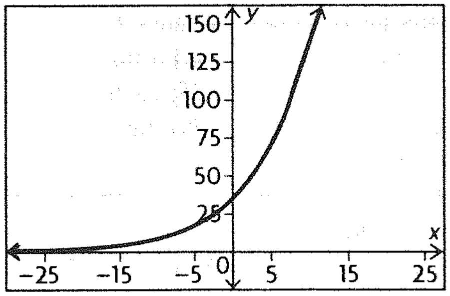
c) Answers may vary. For example, if the sign of the slope of the tangent changed from positive to negative, there was a maximum. If the sign of the slope of the tangent changed from negative to positive, there was a minimum.

9. a) i) Examine the graph of the equation.



The maximum for the interval $0 \le t \le 5$ appears to be at $x = 0$ or (0, 100). The minimum appears to occur at $t = 5$ or (5, 44.4). This cannot be verified with the difference quotient because the graph will always be decreasing. This means that the instantaneous rate of change for any point on the graph will always be negative and never be zero.

ii) Examine the graph of the function.



b) The minimum appears to be at $x = 0$ or (0, 35) and the maximum at $x = 10$ or (10, 141.6). This cannot be verified with the difference quotient because the graph will always be increasing. This means that the instantaneous rate of change for any point on the graph will always be positive and never be zero.

10. Answers may vary. For example, examine points on either side of $t = 0.5$ s to make sure that the diver's height is increasing before the point and decreasing afterwards.

$$\begin{aligned} h(0.49) &= -5(0.49)^2 + 5(0.49) + 10 \\ &= 11.2495 \\ h(0.5) &= -5(0.5)^2 + 5(0.5) + 10 \\ &= 11.25 \end{aligned}$$

$$\frac{11.25 - 11.2495}{0.01} = 0.05$$

The slope to the right of the point is positive.

$$\begin{aligned} h(0.51) &= -5(0.51)^2 + 5(0.51) + 10 \\ &= 11.2495 \\ h(0.5) &= 11.25 \end{aligned}$$

$$\frac{11.2495 - 11.25}{0.01} = -0.05$$

The function is increasing up to 0.5 s and decreasing after 0.5 s—the point is a maximum.

11. Answers may vary. For example, yes, this observation is correct. The slope of the tangent at 1.5 s is 0.

The slopes of the tangents between 1 s and 1.5 s are negative, and the slopes of the tangent lines between 1.5 s and 2 s are positive. So, the minimum of the function occurs at 1.5 s.

12. Answers may vary. For example, estimate the slope of the tangent line to the curve when $x = 5$ by writing an equation for the slope of an secant line on the graph if $R(x)$. If the slope of the tangent is 0, this will confirm there may be a maximum at $x = 5$. If the slopes of tangent lines to the left are positive and the slopes of tangent lines to the right are negative, this will confirm that a maximum occurs at $x = 5$.

13. Answers may vary. For example, because $\sin 90°$ gives a maximum value of 1, I know that a maximum occurs when $(k(x - d)) = 90°$. Solving this equation for x will tell me what types of x-values will give a maxim. For example, when $k = 2$ and $d = 3$,

$$\begin{aligned} (2(x - 3°)) &= 90° \\ (x - 3°) &= 45° \\ x &= 48° \end{aligned}$$

14. Myra is plotting (instantaneous) velocity versus time. The rates of change Myra calculates represent acceleration. When Myra's graph is increasing, the car is accelerating. When Myra's graph is decreasing, the car is decelerating. When Myra's graph is constant, the velocity of the car is constant; the car is neither accelerating nor decelerating.

15. Choose a method and determine the instantaneous rates of change for the points given. Use tables to examine the relationship between x and the instantaneous rate of change at x.

x	Rate of Change
−2	−4
−1	−2
2	4
3	6

The instantaneous rate of change appears to be 2 times the x-coordinate or $2x$. Now use a table to examine the relationship between the points given and their instantaneous rates of change for the function $f(x) = x^3$.

x	Rate of Change
−2	12
−1	3
2	12
3	27

The instantaneous rate of change appears to be 3 times the square of the x-coordinate or $3x^2$.

Chapter Review, pp. 116–117

1. a) Examine the rate of change between each interval. If the rate of change is the same for each interval, then the data follows a linear relation.

$$\frac{297.50 - 437.50}{17 - 25} = 17.5$$

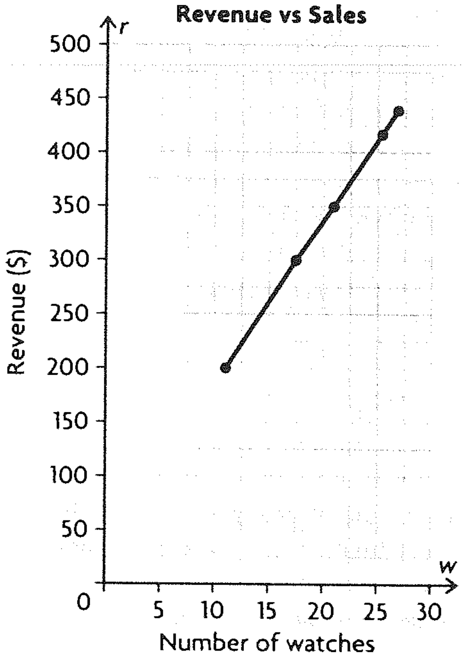
$$\frac{350.00 - 297.50}{20 - 17} = 17.5$$

$$\frac{210.00 - 350.00}{12 - 20} = 17.5$$

$$\frac{420.00 - 210.00}{24 - 12} = 17.5$$

The slope between each interval is the same, and so the relation is linear.

b) Answers may vary. For example:



The graph appears to be linear, and so it would appear that my hypothesis is correct.

c) The average rate of change from $w = 20$ to $w = 25$.

$$\frac{437.50 - 350.00}{25 - 20} = \$17.50 \text{ per watch}$$

d) The cost of one watch is \$17.50; this is the slope of the line on the graph.

2. a) Calculate the average rate of change for the interval [0, 4]. The second point is (4, 7); the first is (0, 1).

$$\frac{7 - 1}{4 - 0} = 1.5 \text{ m/s}$$

b) Calculate the average rate of change for the interval [4, 8]. The second point is (8, 1). The first point is (4, 7).

$$\frac{1 - 7}{8 - 4} = -1.5 \text{ m/s}$$

c) The time intervals have the same length. The amount of change is the same, but with opposite signs for the two intervals. So the rates of change are the same for the two intervals, but with opposite signs.

3. a) The company spends \$2500 per month in expenses—this can be represented by $2500m$. The initial expenses were 10 000. The whole equation is $E = 2500m + 10\,000$.

b) Find the expenses for $m = 6$ and $m = 3$.

$$2500(6) + 10\,000 = 25\,000$$
$$2500(3) + 10\,000 = 17\,500$$
$$\frac{25\,000 - 17\,500}{6 - 3} = 2500$$

The average rate of change is \$2500 per month.

c) No, the equation that represents this situation is linear, and the rate of change over time for a linear equation is constant.

4. a) Answers may vary. For example: Because the unit of the equation is years, do not choose $3 \le t \le 4$ and $4 \le t \le 5$. A better choice would be $3.75 \le t \le 4.0$ and $4.0 \le t \le 4.25$.

b) Answers may vary. For example, the equation is $V(t) = 2500(1.15)^t$. Find $V(4.0)$ and $V(4.25)$.

$$\begin{aligned} V(4.0) &= 2500(1.15)^{4.0} \\ &= 4372.515\,625 \\ V(4.25) &= 2500(1.15)^{4.25} \\ &= 4527.993\,869 \end{aligned}$$

$$\frac{4527.993\,869 - 4372.515\,625}{4.25 - 4.0} = 621.912\,976$$

$$V(3.75) = 2500(1.15)^{3.75} = 4222.376\,055$$
$$V(4.0) = 4372.515\,625$$

$$\frac{4372.515\,625 - 4222.376\,055}{4.0 - 3.75} = 600.558\,280$$

$$\frac{600.558\,280 + 621.912\,976}{2} = 611.24$$

5. a) Answers may vary. For example, squeezing the interval.

b) Squeezing the interval will be a good method. Use the interval $11.99 \le t \le 12.01$. The equation is $y = 2 \sin(120°t)$.

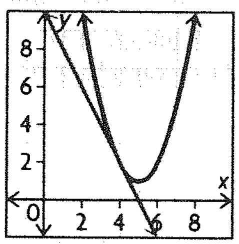
$$2 \sin(120°(11.99)) = -0.0419$$
$$2 \sin(120°(12.01)) = 0.0419$$
$$\frac{0.0419 - (-0.0419)}{12.01 - 11.99} = 4.19 \text{ cm/s}$$

Now use the interval $11.999 \le t \le 12.001$.

$$2 \sin(120°(11.999)) = -0.004\,19$$
$$2 \sin(120°(12.001)) = 0.004\,19$$
$$\frac{0.004\,19 - (-0.004\,19)}{12.001 - 11.999} = 4.19 \text{ cm/s}$$

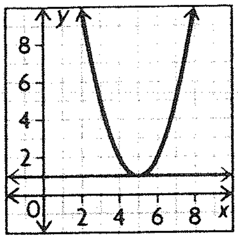
6. For each point, draw a line tangent to the graph at the point given.

a)



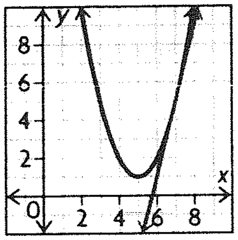
The slope of the line appears to be −2.

b)



The slope of the line appears to be 0.

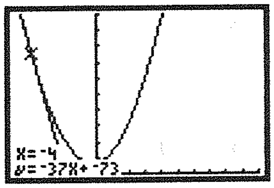
c)



The slope of the line appears to be 4.

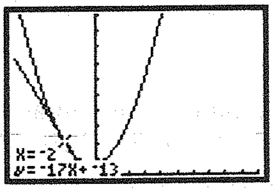
7. Graph the original equation. Find the corresponding y for each value of x given. Use this information to draw a tangent line to the original graph with a graphing calculator.

a)



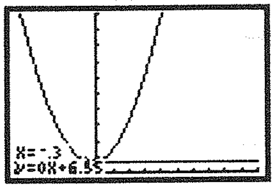
The slope of the line, and therefore the instantaneous rate of change at $x = -4$, is -37.

b)



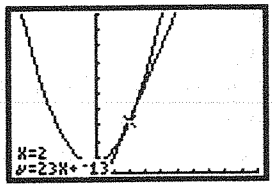
The slope of the line, and therefore the instantaneous rate of change at $x = -2$, is -17.

c)



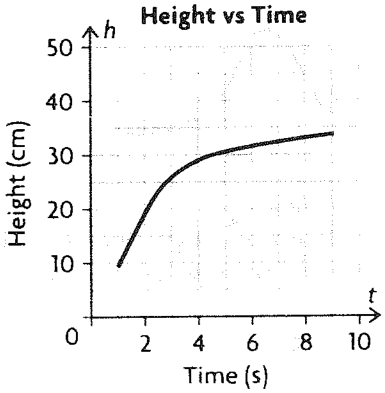
The slope of the line, and therefore the instantaneous rate of change at $x = -0.3$, is 0.

d)

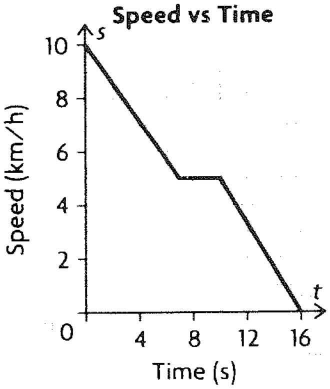


The slope of the line, and therefore the instantaneous rate of change at $x = 2$, is 23.

8.



9. a) Answers may vary. For example:



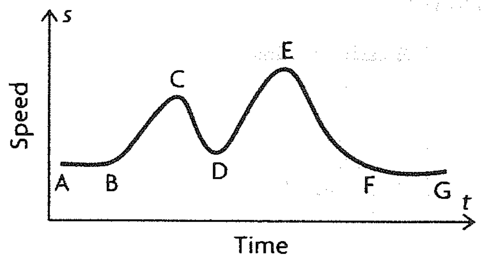
b) Find the average rate of change in the bicycle rider's speed on the interval $0 \le t \le 7$. The speed at $t = 0$ was 10 km/h. The speed at $t = 7$ was 5 km/h. The average rate of change in speed is $\frac{5 - 10}{7 - 0} = -\frac{5}{7}$ km/h/s.

c) From $(7, 5)$ to $\left(12, \frac{10}{3}\right)$, the average rate of change of speed is $-\frac{1}{3}$ km/h/s.

d) The speed is decreasing at a constant rate from $t = 10$ to $t = 16$. So find the average rate of change on any interval between those two numbers and it will be the same as the instantaneous rate of change at $t = 12$.

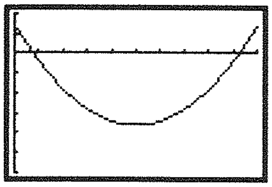
$$\frac{0 - 5}{16 - 10} = -\frac{5}{6} \text{ km/h/s}$$

10. The roller coaster moves at a slow steady speed between A and B. At B it begins to accelerate as it moves down to C. Going uphill from C to D it decelerates. At D it starts to move down and accelerates to E, where the speed starts to decrease until, where it maintains a slower speed to G, the end of the track.



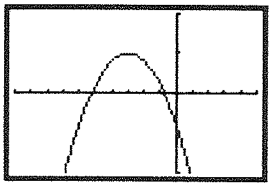
11. Graph each function using a graphing calculator to determine whether the point given is a maximum or a minimum.

a)



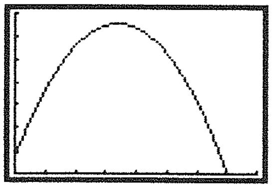
The graph shows that $(5, -18)$ is a minimum.

b)



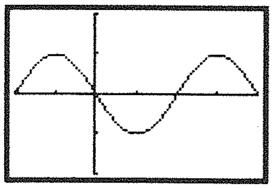
The graph shows that $(-3, 5)$ is a maximum.

c)



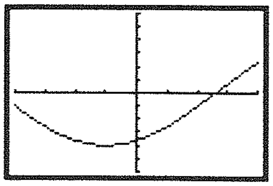
The graph shows that $(17, 653)$ is a maximum.

d)



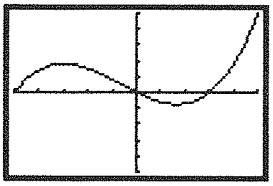
The graph shows that $(45°, -1)$ is a minimum.

e)



The graph shows that $(-25°, -4)$ is a minimum.

f)



The graph shows that $\left(-3, \frac{9}{5}\right)$ is a maximum.

12. a)

i) $f(x) = x^2 - 30x$

$$\begin{aligned} f(2+h) &= (2+h)^2 - 30(2+h) \\ &= 2^2 + 2(2)h + h^2 - 30(2) - 30h \\ &= -56 - 26h + h^2 \\ f(2) &= (2)^2 - 30(2) \\ &= -56 \end{aligned}$$

$$\frac{-56 - 26h + h^2 - (-56)}{2 + h - 2} = h - 26$$

The slope is $m = h - 26$.

ii) $g(x) = -4x^2 - 56x + 16; a = -1$

$$\begin{aligned} g(-1+h) &= -4(-1+h)^2 - 56(-1+h) + 16 \\ &= -4(1 - 2h + h^2) + 56 - 56h + 16 \\ &= -4 + 8h - 4h^2 + 56 - 56h + 16 \\ &= -4h^2 - 48h + 68 \\ g(-1) &= -4(-1)^2 - 56(-1) + 16 \\ &= -4 + 56 + 16 \\ &= 68 \end{aligned}$$

$$\frac{4h^2 - 48h + 68 - 68}{-1 + h - (-1)} = -4h - 48$$

b) For each of the points given, the value of h would be equal to zero. Substitute 0 in for h to find the instantaneous rate of change for each point.

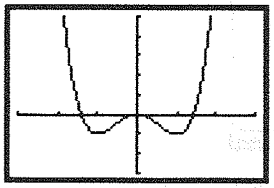
i) $m = 0 - 26 = -26$

ii) $m = -4(0) - 48 = -48$

13. a) To the left of a maximum, the instantaneous rates of change are positive. To the right, the instantaneous rates of change are negative.

b) To the left of a minimum, the instantaneous rates of change are negative. To the right, the instantaneous rates of change are positive.

14. a)



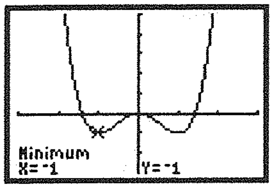
b) minimum: $x = -1, x = 1$

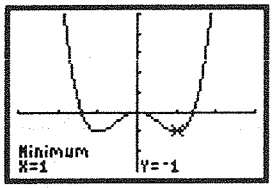
maximum: $x = 0$

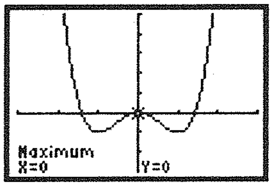
c) The slopes of tangent lines for points to the left of a minimum will be negative, while the slopes of tangent lines for points to the right of a minimum will be positive.

The slopes of tangent lines for points to the left of a maximum will be positive, while the slopes of tangent lines for points to the right of a minimum will be negative.

d)

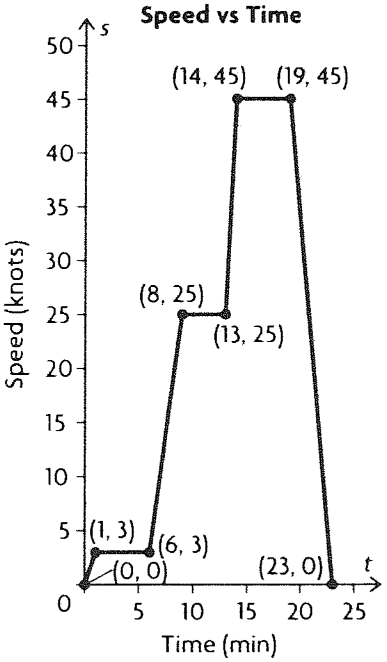






Chapter Self-Test, p. 118

1. a)



b) At $t = 8$ s the speed is approximately 25 knots. At $t = 6$ the speed is approximately 3 knots.

$$\frac{25-3}{8-6} = 11 \text{ kn/min}$$

At $t = 13$ the boats speed is 25 knots.

$$\frac{25-25}{13-8} = 0 \text{ kn/min}$$

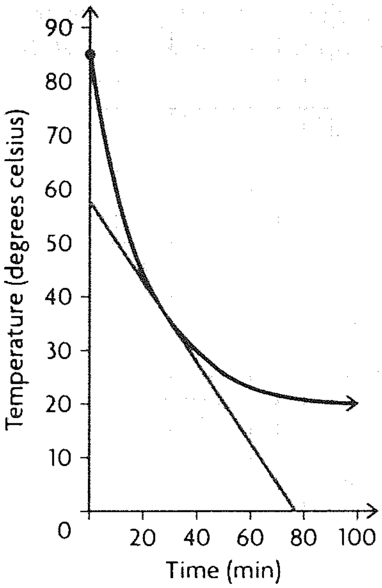
The two different average rates of change indicate that the boat was increasing its speed from $t = 6$ to $t = 8$ at a rate of 11 kn/min and moving at a constant speed from $t = 8$ to $t = 13$.

c) Because the rate of change is constant over the interval, the instantaneous rate of change at $t = 7$ would be the same as it was over the interval, $6 \leq t \leq 8$, 11 kn/min.

2. a) The slope of the secant line between (5, 70) and (50, 25) would be $\frac{25-70}{50-5} = -1$.

b) The hot cocoa is cooling by 1 °C/min on average.

c) Examine the graph to and draw a line tangent to the graph at the point (30, 35).



The slope of the tangent line is -0.75.

d) The hot cocoa is cooling by 0.75 °C/min after 30 min.

e) The rate of decrease decreases over the interval, until it is nearly 0 and constant.

3. a) Calculate both $P(10)$ and $P(8)$.

$$P(10) = -5(10)^2 + 400(10) - 2550$$
$$= 950$$
$$P(8) = -5(8)^2 + 400(8) - 2550$$
$$= 330$$
$$\frac{950-330}{10-8} = 310$$

The average rate of change is $310 per dollar spent.

b) Use the different quotient to estimate the instantaneous rate of change.

$$P(50.01) = -5(50.01)^2 + 400(50.01) - 2550$$
$$= 4948.9995$$

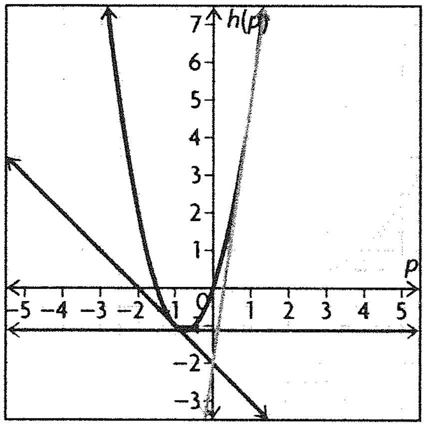
$$P(50) = -5(50)^2 + 400(50) - 2550$$
$$= 4950$$
$$\frac{4948.9995 - 4950}{0.01} = -100.05$$

The instantaneous rate of change is approximately $-\$100$ per dollar spent.

c) The positive sign for part a) means that the company is increasing its profit when it spends between \$8000 and \$10 000 on advertising. The negative sign means that the company's profit is decreasing when it spends \$50 000 on advertising.

4. Graph each function and approximate the tangent line at each of the given points. Estimate the instantaneous rate of change at each point given by determining the slope of the tangent line at the given point.

a)

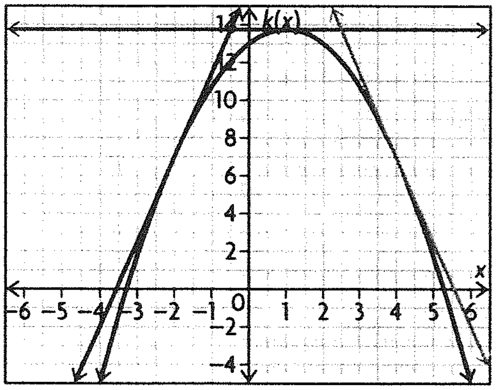


The instantaneous rate of change when $p = -1$ is -1.

The instantaneous rate of change when $p = -0.75$ is 0. The point is a minimum.

The instantaneous rate of change when $p = 1$ is 7.

b)



The instantaneous rate of change when $x = -2$ is 4.5.

The instantaneous rate of change when $x = 4$ is -4.5.

The instantaneous rate of change when $x = 1$ is 0. This point is a maximum.

CHAPTER 3

Polynomial Functions

Getting Started, p. 122

1. a) $2x^2(3x - 11)$
$6x^3 - 22x^2$
b) $(x - 4)(x + 6)$
$x^2 + 6x - 4x - 24$
$x^2 + 2x - 24$
c) $4x(2x - 5)(3x + 2)$
$(8x^2 - 20x)(3x + 2)$
$24x^3 + 16x^2 - 60x^2 - 40x$
$24x^3 - 44x^2 - 40x$
d) $(5x - 4)(x^2 + 7x - 8)$
$5x^3 + 35x^2 - 40x - 4x^2 - 28x + 32$
$5x^3 + 31x^2 - 68x + 32$
2. a) $x^2 + 3x - 28$
$(x + 7)(x - 4)$
b) $2x^2 - 18x + 28$
$2(x^2 - 9x + 14)$
$2(x - 2)(x - 7)$
3. a) $3x + 7 = x - 5$
$3x + 7 - x - 7 = x - 5 - x - 7$
$2x = -12$
$x = -6$
b) $(x + 3)(2x - 9) = 0$
$x + 3 = 0$ and $2x - 9 = 0$
$x = -3$ and $x = \frac{9}{2}$
$x = -3, 4.5$
c) $x^2 + 11x + 24 = 0$
$(x + 3)(x + 8) = 0$
$x + 3 = 0$ and $x + 8 = 0$
$x = -3 \qquad x = -8$
d) $6x^2 + 22x = 8$
$6x^2 + 22x - 8 = 0$
$3x^2 + 11x - 4 = 0$
$(3x - 1)(x + 4) = 0$
$3x - 1 = 0$ and $x + 4 = 0$
$x = \frac{1}{3} \qquad x = -4$

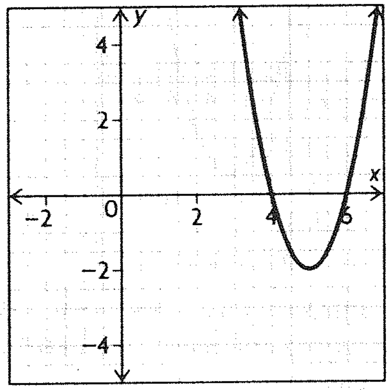
4. a) $y = \frac{1}{4}(x - 3)^2 + 9$

vertical compression by a factor of $\frac{1}{4}$, horizontal translation 3 units to the right; vertical translation 9 units up

b) $y = \left(\frac{1}{2}x\right)^2 - 7$

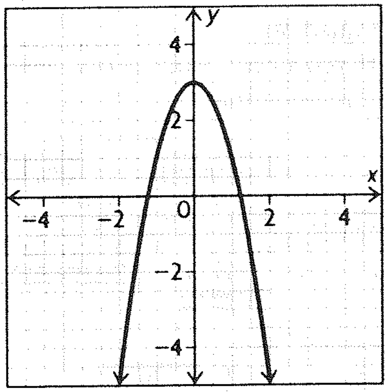
vertical compression by a factor of $\frac{1}{4}$, vertical translation 7 units down

5. a)



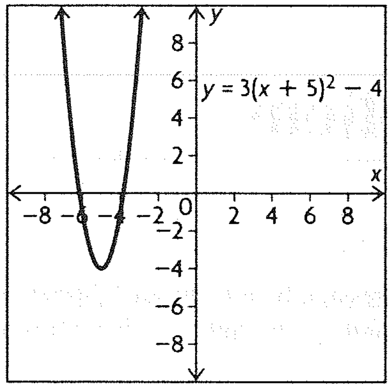
Working from the parent function $y = x^2$, the vertex is at $(5, -2)$. It opens upward. It is vertically stretched by a factor of 2. The equation is $y = 2(x - 5)^2 - 2$.

b)

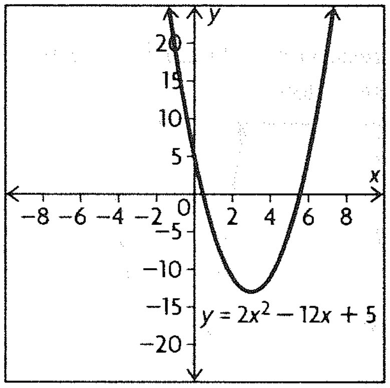


Working from the parent function $y = x^2$, the vertex is at $(0, 3)$. It opens downward. It is vertically stretched by a factor of 2. The equation is $y = -2x^2 + 3$.

6. a)



b)



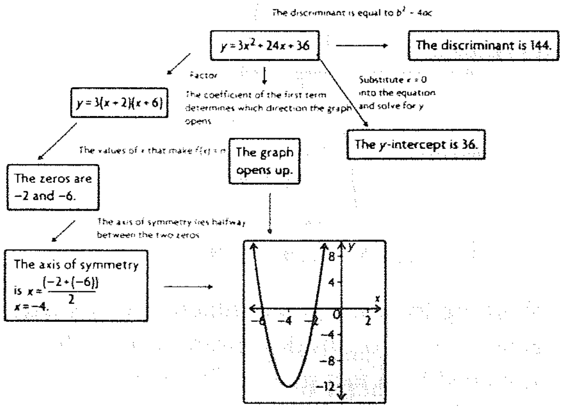
7. a) The first differences are −5.8, −5.6, −5.4, and −5.2. The second differences are all 0.2, so the function is quadratic.

b) The first differences are −6, −3, 5, and 6. The second differences are 3, 8, and 1. The function is not linear or quadratic.

c) The first differences are 4, 12, 36, and 108. The second differences are 8, 24, and 72. The function is not linear or quadratic.

d) The first differences are −0.5, −0.5, −0.5, and −0.5. The function is linear.

8.



3.1 Exploring Polynomial Functions, pp. 127–128

1. a) This represents a polynomial function because the domain is the set of all real numbers, the range does not have a lower bound, and the graph does not have horizontal or vertical asymptotes.

b) This represents a polynomial function because the domain is the set of all real numbers, the range is the set of all real numbers, and the graph does not have horizontal or vertical asymptotes.

c) This is not a polynomial function because it has a horizontal asymptote.

d) This represents a polynomial function because the domain is the set of all real numbers, the range does not have an upper bound, and the graph does not have horizontal or vertical asymptotes.

e) This is not a polynomial function because its domain is not all real numbers.

f) This is not a polynomial function because it is a periodic function.

2. a) polynomial; the exponents of the variables are all natural numbers

b) polynomial; the exponents of the variables are all natural numbers

c) polynomial; the exponents of the variables are all natural numbers

d) other; the variable is under a radical sign

e) other; the function contains another function in the denominator

f) polynomial; the exponents of the variables are all natural numbers

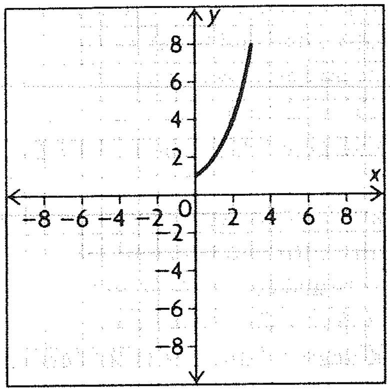
3. a) The first differences are all 25, so the function is linear.

b) The first differences are 15, 5, −5, and −15. The second differences are all −10, so the function is quadratic.

c) The first differences are all 25, so the function is linear.

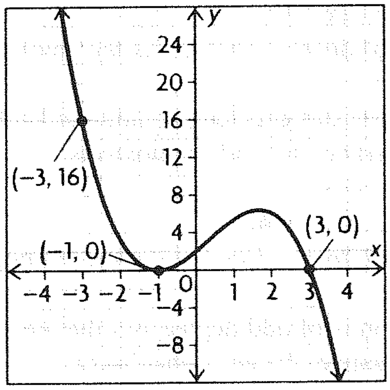
d) The first differences are 4, 28, 76, 148, 244, and 364. The second differences are 24, 48, 72, 96, and 120. The third differences are all 24, so the function is cubic.

4.



a) The graph looks like one half of a parabola, which is the graph of a quadratic equation.
b) There is a variable in the exponent.

5.



6. Answers may vary. For example: Any equation of the form $y = a(-\frac{4}{3}x^2 + \frac{8}{3}x + 4)$ will have the same zeros, but have a different y-intercept and a different value for $f(-3)$. Any equation of the form $y = x(-\frac{4}{3}x^2 + \frac{8}{3}x + 4)$ would have two of the same zeros, but a different value for $f(-3)$ and different positive/negative intervals.
7. $y = x + 5$, $y = x^2 + 5$, $y = x^3 + 5$, $y = x^4 + 5$ all have a y-intercept of 5.
8. Answers may vary. For example:

Definition A polynomial is an expression of the form $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ where $a_0, a_1, \ldots, a_n$ are real numbers and n is a whole number.	**Characteristics** The domain of the function is all real numbers, but the range can have restrictions; except for polynomial functions of degree zero (whose graphs are horizontal lines), the graphs of polynomials do not have horizontal or vertical asymptotes. The shape of the graph depends on its degree.
Polynomials	
Examples $x^2 + 4x + 6$	**Non-Examples** $\sqrt{x + 1}$

3.2 Characteristics of Polynomial Functions, pp. 136–138

1. a) $f(x) = -4x^4 + 3x^2 - 15x + 5$
The function is of degree 4 and −4 is the leading coefficient.
Since the function is of even degree and the leading coefficient is negative, the end behaviour is:
as $x \rightarrow +/-\infty$, $y \rightarrow -\infty$
b) $g(x) = 2x^5 - 4x^3 + 10x^2 - 13x + 8$
The function is of degree 5 and 2 is the leading coefficient.
Since the function is of odd degree and the leading coefficient is positive, the end behaviour is:
as $x \rightarrow -\infty$, $y \rightarrow -\infty$ and as $x \rightarrow \infty$, $y \rightarrow \infty$
c) $p(x) = 4 - 5x + 4x^2 - 3x^3$
$= -3x^3 + 4x^2 - 5x + 4$
The function is of degree 3 and −3 is the leading coefficient.
Since the function is of odd degree and the leading coefficient is negative, the end behaviour is:
as $x \rightarrow -\infty$, $y \rightarrow \infty$ and as $x \rightarrow \infty$, $y \rightarrow -\infty$
d) $h(x) = 2x(x - 5)(3x + 2)(4x - 3)$
$= (2x^2 - 10x)(12x^2 - x - 6)$
$= 24x^4 - 2x^3 - 12x^2 - 120x^3 + 10x^2 + 60x$
$= 24x^4 - 122x^3 - 2x^2 + 60x$
The function is of degree 4 and 24 is the leading coefficient.
Since the function is of even degree and the leading coefficient is positive, the end behaviour is:
as $x \rightarrow +/-\infty$, $y \rightarrow \infty$
2. $f(x) = -4x^4 + 3x^2 - 15x + 5$
a) Turning points
a) Since the function is of degree 4, it will have at least 1 turning point and at most 4 − 1 or 3 turning points.
b) Since the degree of the function is even, the minimum number of zeros is 0 and the maximum number of zeros is equal to the degree of the function, 4.
$g(x) = 2x^5 - 4x^3 + 10x^2 - 13x + 8$
c) Since the function is of degree 5, it will have at least 0 turning points and at most 5 − 1 or 4 turning points.
d) Since the degree of the function is odd, the minimum number of zeros is 1 and the maximum number of zeros is equal to the degree of the function, 5.
$p(x) = 4 - 5x + 4x^2 - 3x^3$

b) Zeros

a) Since the function is of degree 3, it will have at least 0 turning points and at most 3 − 1 or 2 turning points.

b) Since the degree of the function is odd, the minimum number of zeros is 1 and the maximum number of zeros is equal to the degree of the function, 3.

$h(x) = 24x^4 - 122x^3 - 2x^2 + 60$

c) Since the function is of degree 4, it will have at least 1 turning point and at most 4 − 1 or 3 turning points.

d) Since the degree of the function is even, the minimum number of zeros is 0 and the maximum number of zeros is equal to the degree of the function, 4.

3. i. a) There are 3 turning points which means the degree is 3 + 1 = 4. The degree is even.

b) The leading coefficient is negative because as $x \to +/-\infty$, $y \to -\infty$.

ii. a) There are 5 turning points which means the degree is 5 + 1 = 6. The degree is even.

b) The leading coefficient is negative because as $x \to +/-\infty$, $y \to -\infty$.

iii. a) There are 4 turning points which means the degree is 4 + 1 = 5. The degree is odd.

b) The leading coefficient is negative because as $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$.

iv. a) There is 1 turning point which means the degree is 1 + 1 = 2. The degree is even.

b) The leading coefficient is positive because as $x \to +/-\infty$, $y \to \infty$.

v. a) There are 2 turning points which means the degree is 2 + 1 = 3. The degree is odd.

b) The leading coefficient is negative because as $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$.

vi. a) There are 2 turning points which means the degree is 2 + 1 = 3. The degree is odd.

b) The leading coefficient is positive because as $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$.

4. a) $f(x) = 2x^2 - 3x + 5$

The function is of degree 2 and 2 is the leading coefficient.

Since the function is of even degree and the leading coefficient is positive, the end behaviour is:

as $x \to +/-\infty$, $y \to \infty$

b) $f(x) = -3x^3 + 2x^2 + 5x + 1$

The function is of degree 3 and −3 is the leading coefficient.

Since the function is of odd degree and the leading coefficient is negative, the end behaviour is:

as $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$

c) $f(x) = 5x^3 - 2x^2 - 2x + 6$

The function is of degree 5 and 2 is the leading coefficient.

Since the function is of odd degree and the leading coefficient is positive, the end behaviour is:

as $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$

d) $f(x) = -2x^4 + 5x^3 - 2x^2 + 3x - 1$

The function is of degree 4 and −4 is the leading coefficient.

Since the function is of even degree and the leading coefficient is negative, the end behaviour is:

as $x \to +/-\infty$, $y \to -\infty$

e) $f(x) = 0.5x^4 + 2x^2 - 6$

The function is of degree 4 and 0.5 is the leading coefficient.

Since the function is of even degree and the leading coefficient is positive, the end behaviour is:

as $x \to +/-\infty$, $y \to +\infty$.

f) $f(x) = -3x^5 + 2x^3 - 4x$

The function is of degree 5 and −3 is the leading coefficient.

Since the function is of odd degree and the leading coefficient is negative, the end behaviour is:

as $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$

5. a) D: the graph extends from quadrant III to quadrant I and the y-intercept is 2

b) A: the graph extends from quadrant III to quadrant IV

c) E: the graph extends from quadrant II to quadrant I and the y-intercept is −5

d) C: the graph extends from quadrant II to quadrant I and the y-intercept is 0

e) F: the graph extends from quadrant II to quadrant IV

f) B: the graph extends from quadrant III to quadrant I and the y-intercept is 1

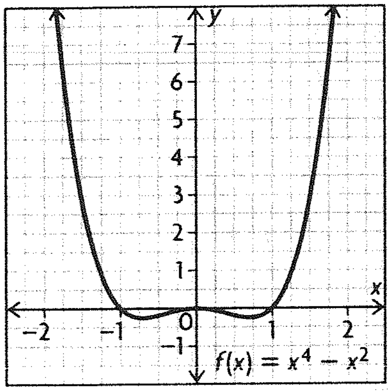
6. a) Answers may vary, but the function should be an odd degree with a positive leading coefficient. One example is $f(x) = 2x^3 + 5$.

b) Answers may vary, but the function should be an even degree with a positive leading coefficient. One example is $f(x) = 6x^2 + x - 4$.

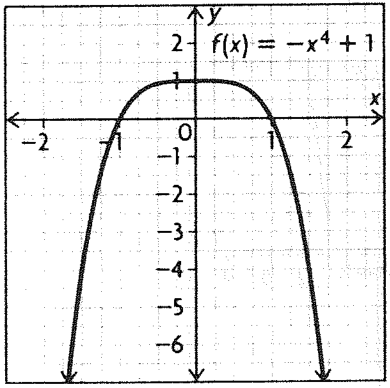
c) Answers may vary, but the function should be an even degree with a negative leading coefficient. One example is $f(x) = -x^4 - x^3 + 7$.

d) Answers may vary, but the function should be an odd degree with a negative leading coefficient. One example is $f(x) = -9x^5 + x^4 - x^3 - 2$.

7. a) Answers may vary. For example:

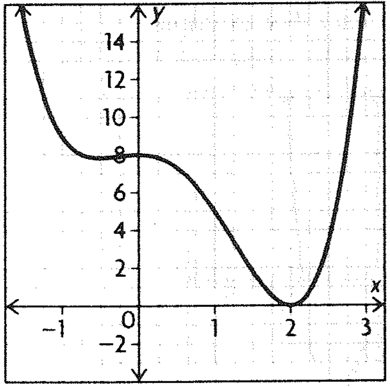


b) Answers may vary. For example:

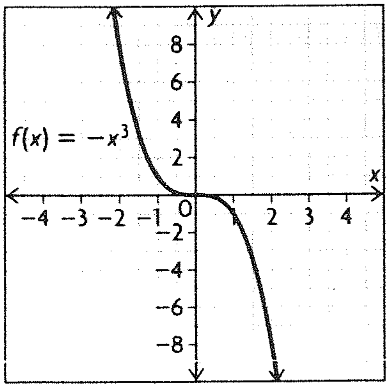


c) Answers may vary. For example:

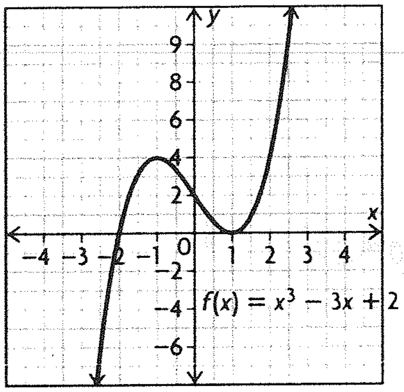
$f(x) = x^4 - 2x^3 - 2x^2 + 8$



d) Answers may vary. For example:

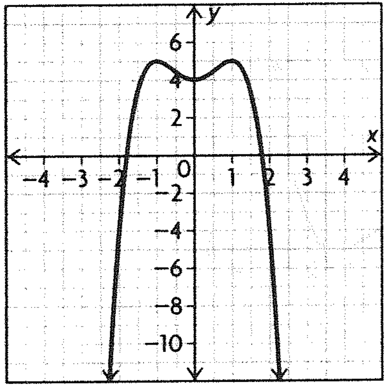


e) Answers may vary. For example:



f) Answers may vary. For example:

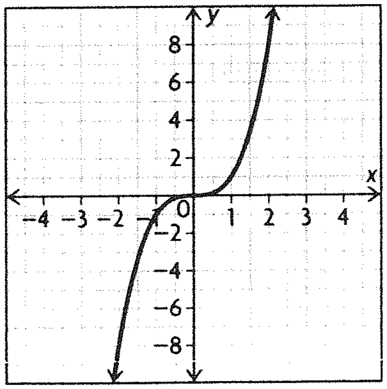
$f(x) = -x^4 + 2x^2 + 4$



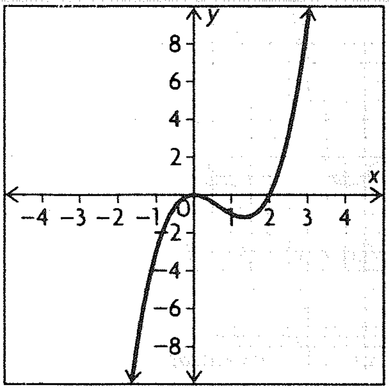
8. An odd-degree polynomial can have only local maximums and minimums because the y-value goes to $-\infty$ and ∞ at each end of the function. Whereas an even-degree polynomial can have absolute maximums and minimums because it will go to either $-\infty$ at both ends or ∞ at both ends of the function.

9. A polynomial of even degree cannot be symmetrical with respect to the origin, since the end behaviours must be the same. Therefore, $f(x)$ is of odd degree, meaning that the number of turning points is even.

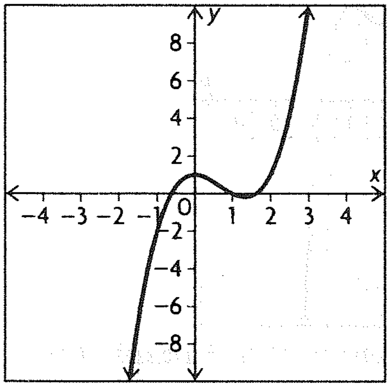
10. a) Answers may vary. For example:
$f(x) = x^3$



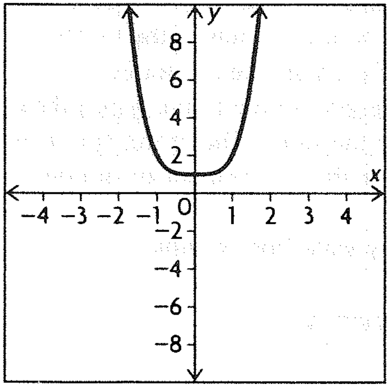
b) Answers may vary. For example:
$f(x) = x^3 - 2x^2$



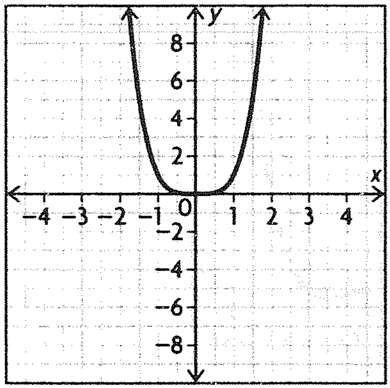
c) Answers may vary. For example:
$f(x) = x^3 - 2x^2 + 1$



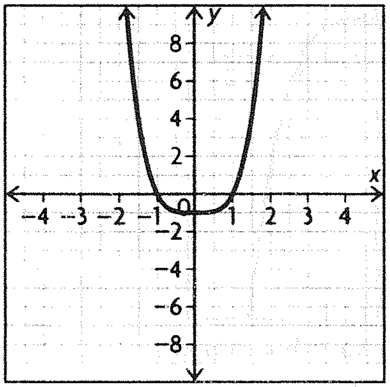
11. a) Answers may vary. For example:
$f(x) = x^4 + 1$



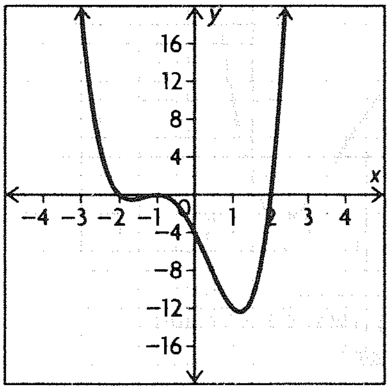
b) Answers may vary. For example: $f(x) = x^4$



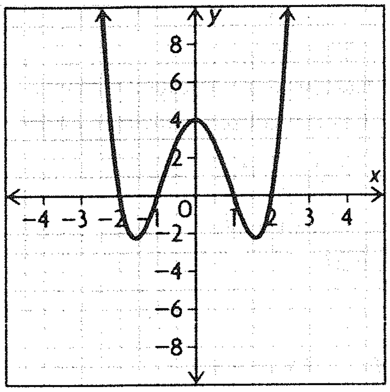
c) Answers may vary. For example:
$f(x) = x^4 - 1$



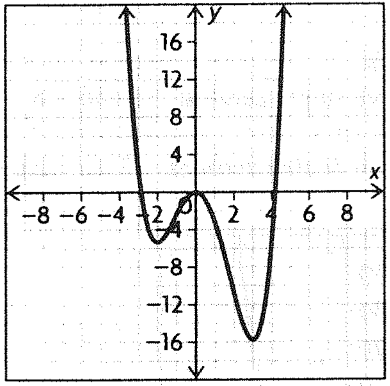
d) Answers may vary. For example:
$f(x) = x^4 + 2x^3 - 3x^2 - 8x - 4$



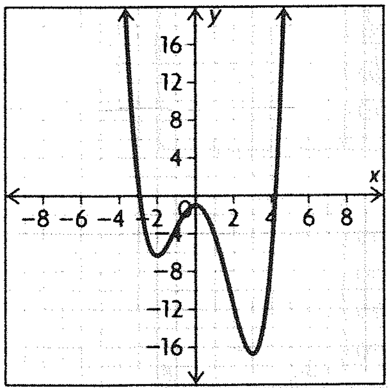
e) Answers may vary. For example:
$f(x) = x^4 - 5x^2 + 4$



12. a) Answers may vary. For example:
$f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2$



and $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2 - 1$



b) the zeros of the function and the leading coefficient of the function

13. a) In 1900, $x = 0$.
$y = -0.1(0)^4 + 0.5(0)^3 + 0.4(0)^2 + 10(0) + 7$
The population was 700.

b) Though the population will grow from the original number, it will end up decreasing because the leading coefficient is negative.

14. a) False; answers may vary; for example, $f(x) = x^2 + x$ is not an even function.
b) true
c) False; answers may vary; for example, $f(x) = x^2 + 1$ has no zeros.
d) False; answers may vary; for example, $f(x) = -x^2$ has end behaviour opposite the behaviour stated.

15. Answers may vary. For example: "What are the turning points of the function?", "What is the leading coefficient of the function?", and "What are the zeros of the function?" If the function has 0 turning points or an even number of turning points, then it must extend to the opposite side of the x-axis. If it has an odd number of turning points, it must extend to the same side of the x-axis. If the leading coefficient is known, it can be determined exactly which quadrants the function extends to/from and if the function has been vertically stretched. If the zeros are known, it can be determined if the function has been vertically translated up or down.

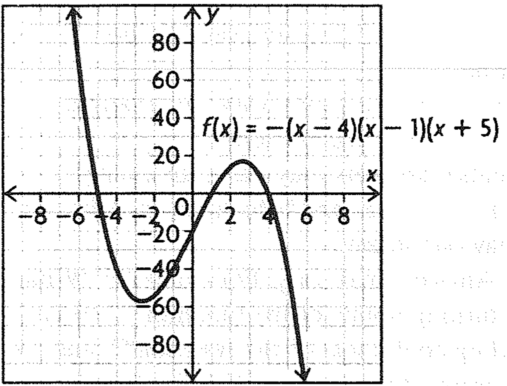
16. a) Since f is an even function, $f(x) = f(-x)$.
$ax^2 + bx + c = a(-x)^2 + b(-x) + c$
$ax^2 + bx + c = ax^2 - bx + c$
$2bx = 0$
So, $b = 0$.

b) Since g is an odd function, $-g(x) = g(-x)$ for all real x.
$-ax^3 - bx^2 - cx - d$
$= a(-x)^3 + b(-x)^2 + c(-x) + d$
$-ax^3 - bx^2 - cx - d = -ax^3 + bx^2 - cx + d$
$-bx^2 - d = bx^2 + d$
Since the coefficient of x^2 and the constant on each side must be equal, $-b = b$ and $-d = d$. So $b = 0$ and $d = 0$.

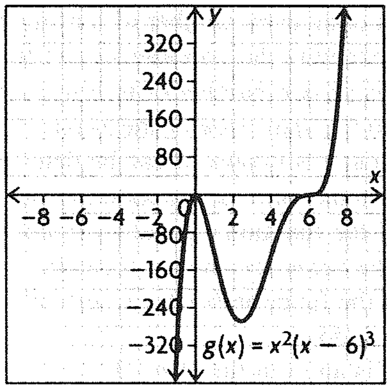
3.3 Characteristics of Polynomial Functions in Factored Form, pp. 146–148

1. a) C: the graph has zeros of −1 and 3, and it extends from quadrant III to quadrant I
b) A: the graph has zeros of −1 and 3, and it extends from quadrant II to quadrant I
c) B: the graph has zeros of −1 and 3, and it extends from quadrant II to quadrant IV
d) D: the graph has zeros of −1, 0, 3, and 5, and it extends from quadrant II to quadrant I

2. a)



b)



3. a) $f(x) = k(x + 1)(x - 4)$

$f(x) = 4(x + 1)(x - 4)$

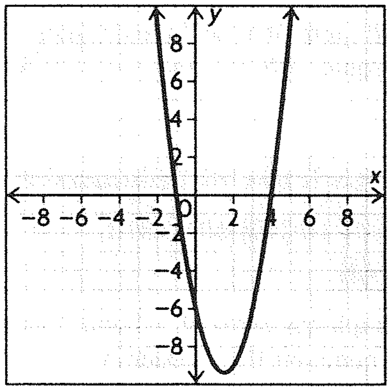
$f(x) = -2(x + 1)(x - 4)$

b) $9 = k(5 + 1)(5 - 4)$

$9 = k(6)(1)$

$\frac{3}{2} = k$

$f(x) = \frac{3}{2}(x + 1)(x - 4)$



4. a) The zeros of the function are located at −3, 2, and 5.

$y = a(x + 3)(x - 2)(x - 5)$

$8 = a(1 + 3)(1 - 2)(1 - 5)$

$8 = a(4)(-1)(-4)$

$8 = 16a$

$\frac{1}{2} = a$

$y = 0.5(x + 3)(x - 2)(x - 5)$

b) The zeros of the function are located at −1, 2, and 4. Since $x = -1$ is a turning point, this factor is squared.

$y = a(x + 1)^2(x - 2)(x - 4)$

$-12 = a(1 + 1)^2(1 - 2)(1 - 4)$

$-12 = a(2)^2(-1)(-3)$

$-12 = 12a$

$-1 = a$

$y = -(x + 1)^2(x - 2)(x - 4)$

5. Family 1: A, G, I

These functions have single zeros at 3 and −5.

Family 2: B, E

These functions have double zeros at 3 and single zeros at −5.

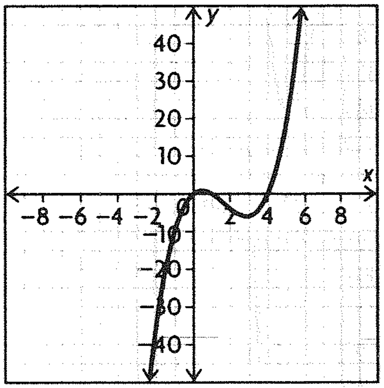
Family 3: C, F, H, K

These functions have single zeros at −6 and −8.

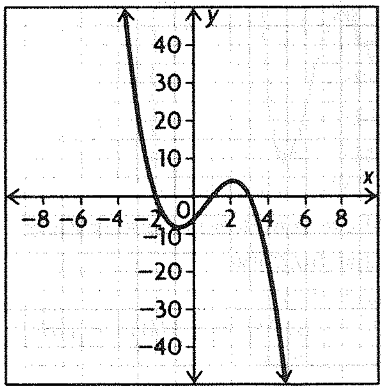
Family 4: D, J, L

These functions have double zeros at −3 and single zeros at −5.

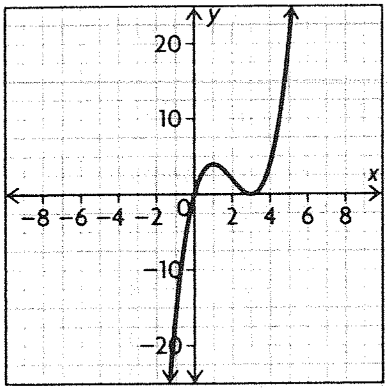
6. a)



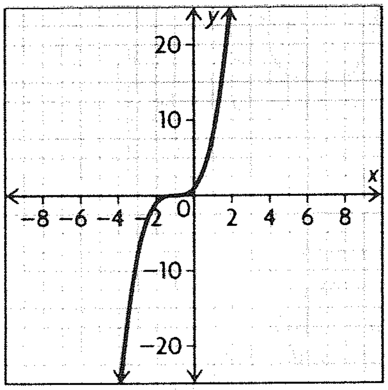
b)



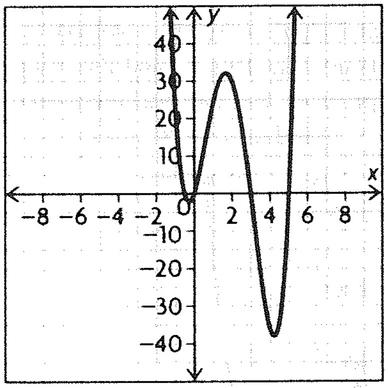
c)



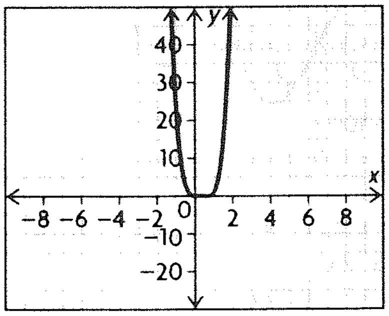
d)



e)

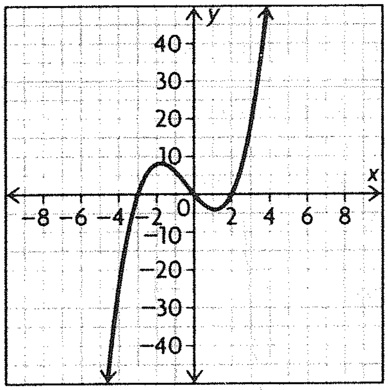


f)

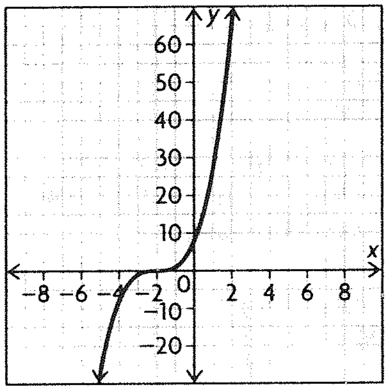


7. Answers may vary. For example:

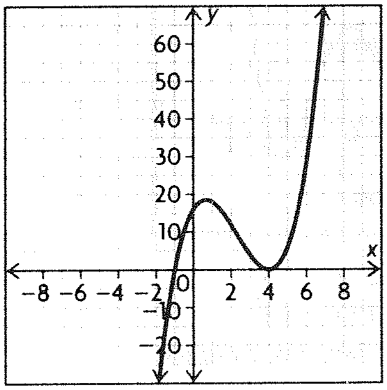
a) i) $y = x(x + 3)(x - 2)$



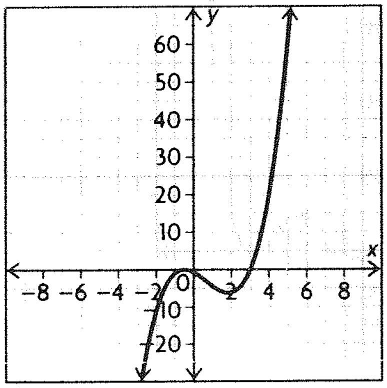
ii) $y = (x + 2)^3$



iii) $y = (x + 1)(x - 4)^2$



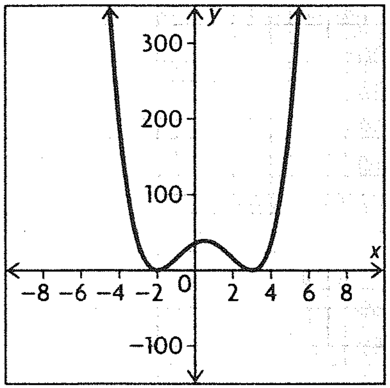
iv) $y = (x - 3)\left(x + \frac{1}{2}\right)^2$



b) No, all of the characteristics of the graphs are not unique because each equation belongs to a family of equations.

8. Answers may vary. For example:

a) $y = (x + 5)(x + 3)(x - 2)(x - 4)$

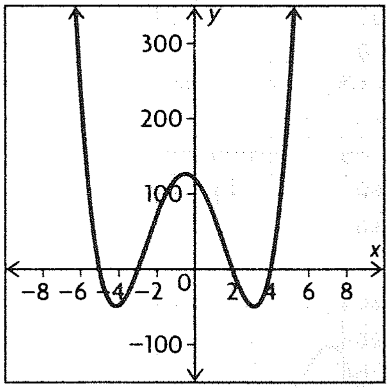


$y = 2(x + 5)(x + 3)(x - 2)(x - 4)$

$y = -5(x + 5)(x + 3)(x - 2)(x - 4)$

b) Answers may vary. For example:

$y = (x + 2)^2(x - 3)^2$

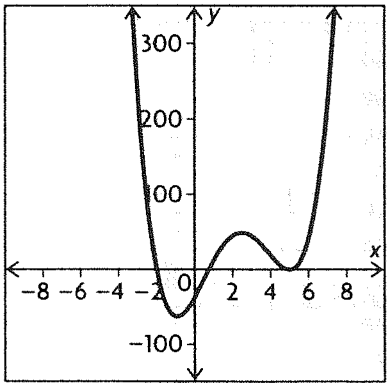


$y = 10(x + 2)^2(x - 3)^2$

$y = 7(x + 2)^2(x - 3)^2$

c) Answers may vary. For example:

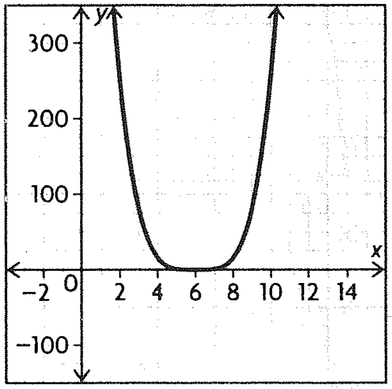
$$y = (x + 2)\left(x - \frac{3}{4}\right)(x - 5)^2$$



$$y = -(x + 2)\left(x - \frac{3}{4}\right)(x - 5)^2$$

$$y = \frac{2}{5}(x + 2)\left(x - \frac{3}{4}\right)(x - 5)^2$$

d) Answers may vary. For example: $y = (x - 6)^4$



$y = 15(x - 6)^4$

$y = -3(x - 6)^4$

9. a) $y = 3x^3 - 48x$

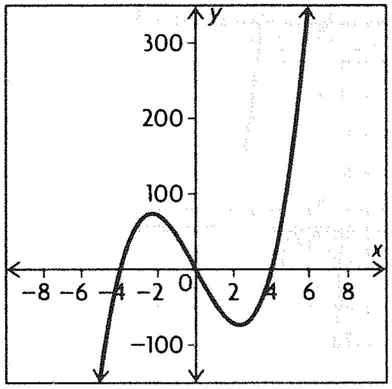
$y = 3x(x^2 - 16)$

$y = 3x(x + 4)(x - 4)$

The zeros are $x = 0, -4$, and 4.

The y-intercept is $f(0) = 3(0)^3 - 48(0) = 0$.

The function is an odd degree, and the leading coefficient is positive. So, the end behaviour is $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$.



b) $y = x^4 + 4x^3 + 4x^2$

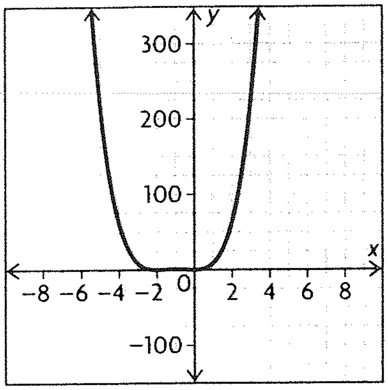
$y = x^2(x^2 + 4x + 4)$

$y = x^2(x + 2)^2$

The zeros are $x = 0$, and -2.

The y-intercept is $f(0) = 0^4 + 4(0)^3 + 4(0)^2 = 0$.

The function is an even degree, and the leading coefficient is positive. So, the end behaviour is $x \to +/-\infty$, $y \to \infty$.



c) $y = x^3 - 9x^2 + 27x - 27$

$y = (x - 3)(x^2 - 6x + 9)$

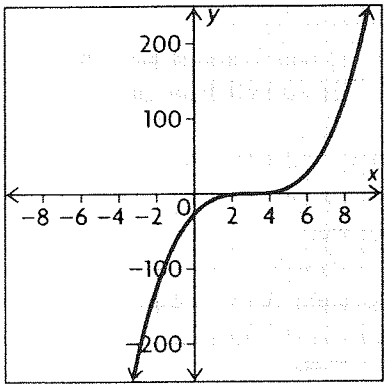
$y = (x - 3)(x - 3)(x - 3)$ or $(x - 3)^3$

$y = (0)^3 - 9(0)^2 + 27(0) - 27$

$y = -27$

The y-intercept is -27.

The function is an odd degree, and the leading coefficient is positive. So, the end behaviour is $x \to -\infty, y \to -\infty$ and as $x \to \infty, y \to \infty$.



d) $y = -x^4 - 15x^3 - 75x^2 - 125x$

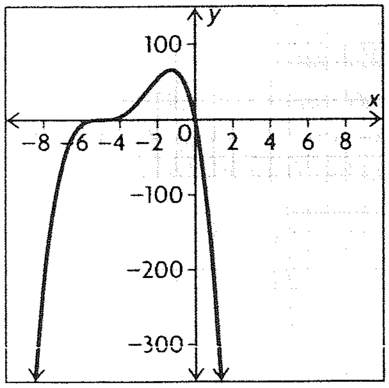
$y = -x(x^3 + 15x^2 + 75x + 125)$

$y = -x(x + 5)(x^2 + 10x + 25)$

$y = -x(x + 5)^3$

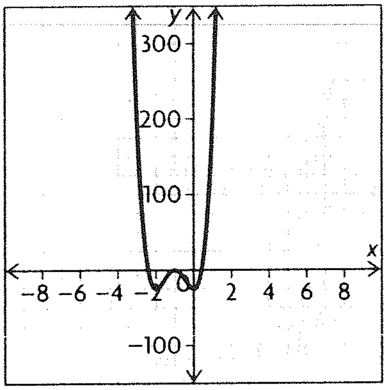
The y-intercept is 0.

The function is an even degree, and the leading coefficient is negative. So, the end behaviour is $x \to +/-\infty, y \to -\infty$.

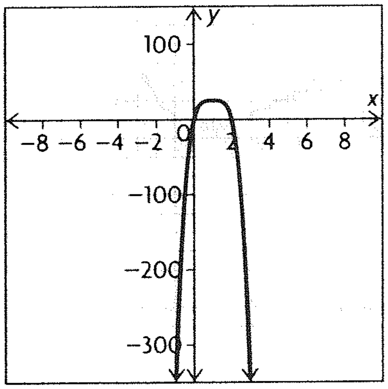


10. Answers may vary. For example:

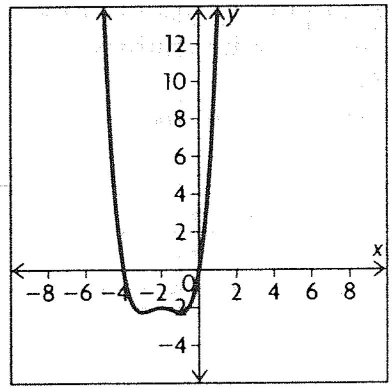
a) $y = 25x^4 + 100x^3 + 100x^2 - 25$



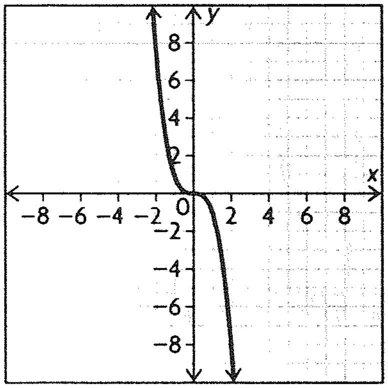
b) $y = -25x^4 - 100x^3 - 150x^2 + 100x$



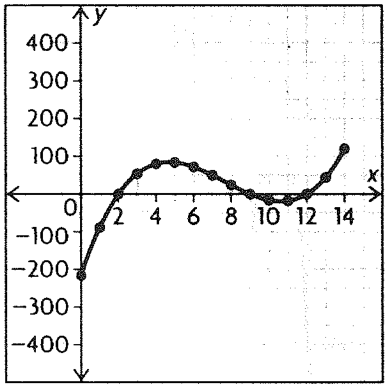
c) $y = \frac{1}{4}x^4 + 2x^3 + \frac{11}{2}x^2 + 6x$



d) $y = -x^3$



11. a)



b) The zeros are at $x = 2, 9, 12$.
$y = (x - 2)(x - 9)(x - 12)$

c) No, this is not likely to continue. The equation $y = (x - 2)(x - 9)(x - 12)$ only has 3 zeros, so after year 12, the company's profit will increase forever. The function's domain should be $\{x \in \mathbf{R} | 0 \le x \le 14\}$.

12. a) The zeros are at -2, -1, and 1.
$y = (x + 2)(x + 1)(x - 1)$
$y = x^3 + 2x^2 - x - 2$

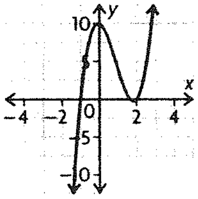
b) The zeros are at -4, -2, and 1.

$$y = a(x + 4)(x + 2)(x - 1)$$
$$-1.6 = a(-3 + 4)(-3 + 2)(-3 - 1)$$
$$-1.6 = a(1)(-1)(-4)$$
$$-1.6 = 4a$$
$$-0.4 = -\frac{2}{5} = a$$
$$y = -\frac{2}{5}(x - 1)(x + 2)(x + 4)$$

13. a)
$$f(x) = a(x + 3)(x + 5)$$
$$f(7) = a(7 + 3)(7 + 5)$$
$$-720 = 120a$$
$$-6 = a$$
$$f(x) = -6(x + 3)(x + 5)$$

b)
$$f(x) = a(x + 2)(x - 3)(x - 4)$$
$$f(5) = a(5 + 2)(5 - 3)(5 - 4)$$
$$28 = 14a$$
$$2 = a$$
$$f(x) = 2(x + 2)(x - 3)(x - 4)$$

14. $f(x) = kx^3 - 8x^2 - x + 3k + 1$
$f(2) = 0$, so
$$8k - 32 - 2 + 3k + 1 = 0$$
$$11k = 33$$
$$k = 3$$



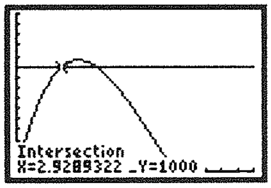
The zeros are $\frac{5}{3}$, -1, and 2.
$f(x) = (3x - 5)(x + 1)(x - 2)$

15. a) It has zeros at 2 and 4, and it has turning points at 2, 3, and 4. It extends from quadrant II to quadrant I.

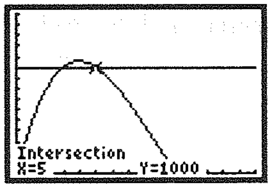
b) It has zeros at -4 and 3, and it has turning points at $-\frac{5}{3}$ and 3. It extends from quadrant III to quadrant I.

16. a)
$$V(2) = 2(30 - 4)(20 - 4)$$
$$= 2(26)(16)$$
$$= 832 \text{ cm}^3$$

b) $1000 = x(30 - 2x)(20 - 2x)$
Graph the left and right sides as separate functions on a graphing calculator and solve.



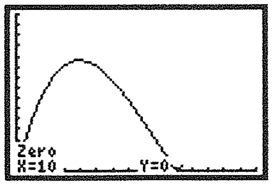
Intersection
X=2.9289322 Y=1000



Intersection
X=5 Y=1000

The volume is 1000 when $x = 2.93$ or $x = 5$. So the dimensions are 2.93 cm by 24.14 cm by 14.14 cm or 5 cm by 20 cm by 10 cm.

c)



Zero
X=10 Y=0

$V(x) > 0$ for $0 < x < 10$. The values of x are the side lengths of squares that can be cut from the sheet of cardboard to produce a box with positive volume. Since the sheet of cardboard is 30 cm by 20 cm, the side lengths of a square cut from each corner have to be less than 10 cm, or an entire edge would be cut away, leaving nothing to fold up.
d) The square that is cut from each corner must be larger than 0 cm by 0 cm but smaller than 10 cm by 10 cm.

3.4 Transformations of Cubic and Quartic Functions, pp. 155–158

1. a) B: $y = x^3$ has been vertically stretched by a factor of 2, horizontally translated 3 units to the right, and vertically translated 1 unit up.
b) C: $y = x^3$ has been reflected in the x-axis, vertically compressed by a factor of $\frac{1}{3}$, horizontally translated 1 unit to the left, and vertically translated 1 unit down.
c) A: $y = x^4$ has been vertically compressed by a factor of 0.2, horizontally translated 4 units to the right, and vertically translated 3 units down.
d) D: $y = x^4$ has been reflected in the x-axis, vertically stretched by a factor of 1.5, horizontally translated 3 units to the left, and vertically translated 4 units up.
2. a) $y = x^4$, vertical stretch by a factor of $\frac{5}{4}$ and vertical translation of 3 units up
b) $y = x$, vertical stretch by a factor of 3 and vertical translation of 4 units down
c) $y = x^3$, horizontal compression by a factor of $\frac{1}{3}$, horizontal translation of $\frac{4}{3}$ units to the left, and vertical translation of 7 units down
d) $y = x^4$, reflection in the x-axis and horizontal translation of 8 units to the left
e) $y = x^2$, reflection in the x-axis, vertical stretch by a factor of 4.8, and horizontal translation 3 units right
f) $y = x^3$, vertical stretch by a factor of 2, horizontal stretch by a factor of 5, horizontal translation of 7 units to the left, and vertical translation of 4 units down
3. a) $y = x^3$ has been translated 3 units to the left and 4 units down.
$y = (x + 3)^3 - 4$
b) $y = x^4$ has been reflected in the x-axis, vertically stretched by a factor of 2, horizontally translated 4 units to the left, and vertically translated 5 units up.
$y = -2(x + 4)^4 + 5$
c) $y = x^4$ has been vertically compressed by a factor of $\frac{1}{4}$, horizontally translated 1 unit to the right, and vertically translated 2 units down.
$y = \frac{1}{4}(x - 1)^4 - 2$
d) $y = x^3$ has been reflected in the x-axis, vertically stretched by a factor of 2, horizontally translated 3 units to the right, and vertically translated 4 units down.
$y = -2(x - 3)^3 - 4$
4. a) $y = x^3$ has been vertically stretched by a factor of 12, horizontally translated 9 units to the right, and vertically translated 7 units down.
b) $y = x^3$ has been horizontally stretched by a factor of $\frac{8}{7}$, horizontally translated 1 unit to the left, and vertically translated 3 units up.
c) $y = x^3$ has been vertically stretched by a factor of 2, reflected in the x-axis, horizontally translated 6 units to the right, and vertically translated 8 units down.
d) $y = x^3$ has been horizontally translated 9 units to the left.
e) $y = x^3$ has been reflected in the x-axis, vertically stretched by a factor of 2, reflected in the y-axis, horizontally compressed by a factor of $\frac{1}{3}$, horizontally translated 4 units to the right, and vertically translated 5 units down.
f) $y = x^3$ has been horizontally stretched by a factor of $\frac{4}{3}$ and horizontally translated 10 units to the right.
5. a) Since the vertex is at $(0, -1)$, $k = -11$, $h = 0$.
$y = a(x - 0)^2 - 11$
$y = ax^2 - 11$
Substitute $(1, -3)$ to determine a.
$-3 = a(1)^2 - 11$
$8 = a$
$y = 8x^2 - 11$
$y = x^2$ was vertically stretched by a factor of 8 and vertically translated 11 units down.
b) Since the vertex is at $(0, 1.25)$, $k = 1.25$, $h = 0$. Since the parabola opens downward, a is negative.
$y = -a(x - 0)^2 + 1.25$
$y = -ax^2 + 1.25$
Substitute $(5, -5)$ to determine a.
$-5 = a(5)^2 + 1.25$
$-6.25 = 25a$
$-0.25 = a$
$$y = -\frac{1}{4}x^2 + 1.25$$

$y = x^2$ was reflected in the x-axis, vertically compressed by a factor of $\frac{1}{4}$, and vertically translated 1.25 units up.

6. a) $y = \frac{1}{2}(5(x + 6)^3)$

$$(x, y) \rightarrow \left(\frac{x}{5} + (-6), \frac{1}{2}y + 0\right)$$

$$(-1, -1) \rightarrow \left(\frac{-1}{5} + (-6), \frac{1}{2}(-1) + 0\right)$$

$$(-1, -1) \rightarrow \left(-6\frac{1}{5}, -\frac{1}{2}\right)$$

$$(0, 0) \rightarrow \left(\frac{0}{5} + (-6), \frac{1}{2}(0) + 0\right)$$

$$(0, 0) \rightarrow (-6, 0)$$

$$(2, 8) \rightarrow \left(\frac{2}{5} + (-6), \frac{1}{2}(8) + 0\right)$$

$$(2, 8) \rightarrow \left(-5\frac{3}{5}, 4\right)$$

b) $y = \left(-\frac{1}{2}x\right)^3 + 3$

$$(x, y) \rightarrow (-2x + 0, y + 3)$$

$$(-1, -1) \rightarrow (-2(-1), -1 + 3)$$

$$(-1, -1) \rightarrow (2, 2)$$

$$(0, 0) \rightarrow (-2(0), 0 + 3)$$

$$(0, 0) \rightarrow (0, 3)$$

$$(2, 8) \rightarrow (-2(2), 8 + 3)$$

$$(2, 8) \rightarrow (-4, 11)$$

c) $y = -3(x - 4)^3 - \frac{1}{2}$

$$(x, y) \rightarrow \left(x + 4, -3y - \frac{1}{2}\right)$$

$$(-1, -1) \rightarrow \left(-1 + 4, -3(-1) - \frac{1}{2}\right)$$

$$(-1, -1) \rightarrow \left(3, 2\frac{1}{2}\right)$$

$$(0, 0) \rightarrow \left(0 + 4, -3(0) - \frac{1}{2}\right)$$

$$(0, 0) \rightarrow \left(4, -\frac{1}{2}\right)$$

$$(2, 8) \rightarrow \left(2 + 4, -3(8) - \frac{1}{2}\right)$$

$$(2, 8) \rightarrow \left(6, -24\frac{1}{2}\right)$$

d) $y = \frac{1}{10}\left(\frac{1}{7}x\right)^3 - 2$

$$(x, y) \rightarrow \left(7x + 0, \frac{1}{10}y - 2\right)$$

$$(-1, -1) \rightarrow \left(7(-1), \frac{1}{10}(-1) - 2\right)$$

$$(-1, -1) \rightarrow \left(-7, -2\frac{1}{10}\right)$$

$$(0, 0) \rightarrow \left(7(0), \frac{1}{10}(0) - 2\right)$$

$$(0, 0) \rightarrow (0, -2)$$

$$(2, 8) \rightarrow \left(7(2), \frac{1}{10}(8) - 2\right)$$

$$(2, 8) \rightarrow \left(14, -1\frac{1}{5}\right)$$

e) $y = -(-x)^3 + \frac{9}{10}$

$$(x, y) \rightarrow \left(-x + 0, -y + \frac{9}{10}\right)$$

$$(-1, -1) \rightarrow \left(-(-1), -(-1) + \frac{9}{10}\right)$$

$$(-1, -1) \rightarrow \left(1, 1\frac{9}{10}\right)$$

$$(0, 0) \rightarrow \left(-(0), -(0) + \frac{9}{10}\right)$$

$$(0, 0) \rightarrow \left(0, \frac{9}{10}\right)$$

$$(2, 8) \rightarrow \left(-(2), -(8) + \frac{9}{10}\right)$$

$$(2, 8) \rightarrow \left(-2, -7\frac{1}{10}\right)$$

f) $y = \left(\frac{1}{7}(x + 4)^3\right) - 2$

$$(x, y) \rightarrow (7x - 4, y - 2)$$

$$(-1, -1) \rightarrow (7(-1) - 4, -1 - 2)$$

$$(-1, -1) \rightarrow (-11, -8)$$

$$(0, 0) \rightarrow (7(0) - 4, 0 - 2)$$

$$(0, 0) \rightarrow (-4, -7)$$

$$(2, 8) \rightarrow (7(2) - 4, 8 - 2)$$

$$(2, 8) \rightarrow (10, 1)$$

7. Since the vertex is at (1, 3), $k = 3$, $h = 1$. Since the parabola opens downward, a is negative.

$$y = -a(x - 1)^4 + 3$$

$$y = -a(x - 1)^4 + 3$$

Substitute $(5, -61)$ to determine a.

$$-61 = a(5 - 1)^4 + 3$$
$$-61 = 256a + 3$$
$$-64 = 256a$$
$$-\frac{1}{4} = a$$
$$y = -\frac{1}{4}(x - 1)^4 + 3$$

8. $\left(11, -\frac{23}{3}\right) = \left(11 - 13, \frac{3}{2}\left(-\frac{23}{3} + 13\right)\right)$

$= (-2, 8)$

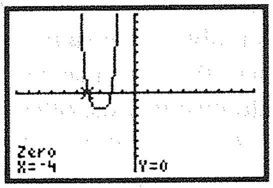
$(13, -13) = (13 - 13,) = (0, 0)$

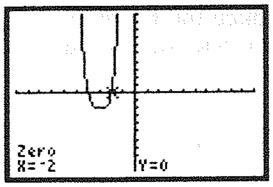
$\left(15, -\frac{55}{3}\right) = \left(15 - 3, \frac{3}{2}\left(-\frac{55}{3} + 13\right)\right)$

$= (2, -8)$

9. The x-intercepts are the zeros of the function:

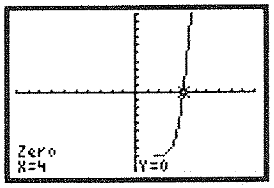
a)





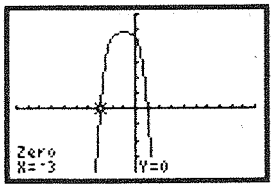
The x-intercepts are at -2 and -4.

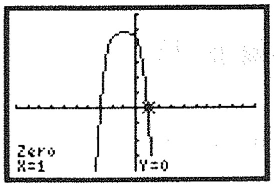
b)



The x-intercept is at 4.

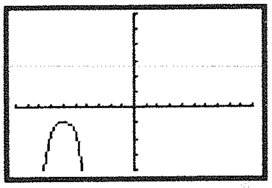
c)





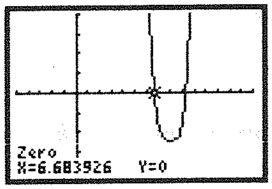
The x-intercepts are at -3 and 1.

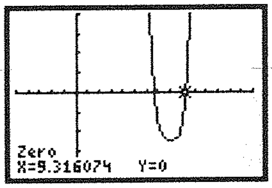
d)



There are no x-intercepts.

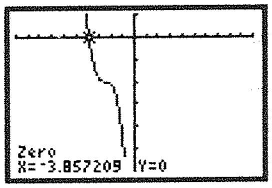
e)





The x-intercepts are at 6.68 and 9.32.

f)



The x-intercept is at -3.86.

10. a) It will have one zero. If the function is set equal to 0, there is only one solution for x.

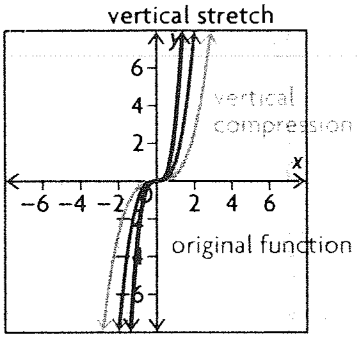
b) It will have no zeros. If the function is set equal to 0, there are no solutions for x because there is no real number that can be raised to the fourth power to produce a negative number.

c) The function will have one zero if n is odd, and it will have no zeros if n is even. This is because an odd root of a negative number is one negative number, and a negative number does not have any even roots.

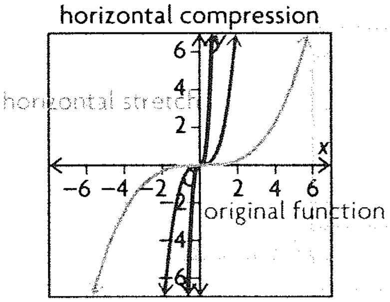
11. a) The reflection of the function $y = x^n$ in the x-axis will be the same as its reflection in the y-axis for odd values of n.

b) The reflections will be different for even values of n. The reflection in the x-axis will be $y = -x^n$, and the reflection in the y-axis will be $y = (-x)^n$. For odd values of n, $-x^n$ equals $(-x)^n$. For even values of n, $-x^n$ does not equal $(-x)^n$.

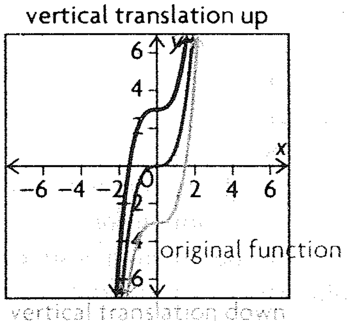
12. a) Vertical stretch and compression: $y = ax^3$



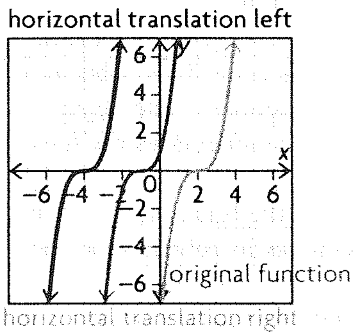
Horizontal stretch and compression: $y = (kx)^3$



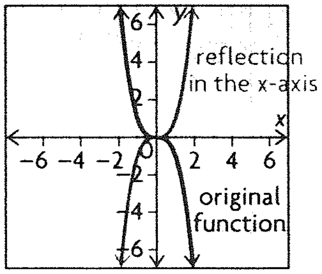
Vertical translation up or down: $y = x^3 + c$



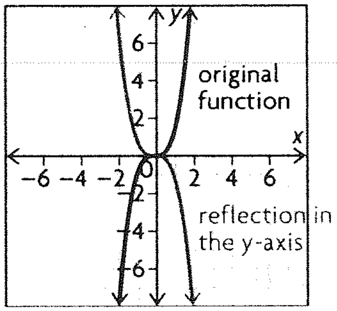
Horizontal translation left or right: $y = (x - d)^3$



Reflection in the x-axis: $y = -x^3$



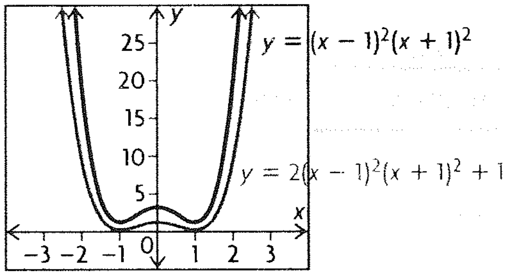
Reflection in the y-axis: $y = (-x)^3$



b) When using a table of values to sketch the graph of a function, you may not select a large enough range of values for the domain to produce an accurate representation of the function.

13. Yes, you can. The zeros of the first function have the same spacing between them as the zeros of the second function. Also, the ratio of the distances of the two curves above or below the x-axis at similar distances between the zeros is always the same. Therefore, the two curves have the same general shape, and one can be transformed into the other.

14. The graph of $y = (x - 1)^2(x + 1)^2$ must be vertically stretched by a factor of 2 and vertically translated 1 unit up to produce the graph of $y = 2(x - 1)^2(x + 1)^2 + 1$. The two graphs are shown below.



From the two graphs, it is apparent that the function $y = 2(x - 1)^2(x + 1)^2 + 1$ does not intersect the x-axis and, therefore, does not have any roots.

15. $f(x) = -4(4(x + 3))^2 - 5$

$$= \left(-\frac{5}{4}\right)(-4)\left(\left(\frac{1}{2}\right)(2)(x + 3)\right)^2 - 5 + 6$$

$$f(x) = 5(2(x + 3))^2 + 1$$

Mid-Chapter Review, p. 161

1. a) Yes.

b) No; it contains a rational exponent.

c) Yes.

d) No; it is a rational function.

2. a) Answers may vary. For example, $f(x) = x^3 + 2x^2 - 8x + 1$.

b) Answers may vary. For example, $f(x) = 5x^4 - x^2 - 7$.

c) Answers may vary. For example, $f(x) = 7x^6 + 3$.

d) Answers may vary. For example, $f(x) = -2x^5 - 4x^4 + 3x^3 - 2x^2 + 9$.

3. a) Since it is an odd degree and has a negative leading coefficient, the end behaviour is: as $x \rightarrow -\infty, y \rightarrow \infty$ and as $x \rightarrow \infty, y \rightarrow -\infty$.

b) Since it is an even degree and has a positive leading coefficient, the end behaviour is: as $x \rightarrow \pm\infty, y \rightarrow \infty$.

c) Since it is an odd degree and has a positive leading coefficient, the end behaviour is: as $x \rightarrow -\infty, y \rightarrow -\infty$ and as $x \rightarrow \infty, y \rightarrow \infty$.

d) Since it is an even degree and has a negative leading coefficient, the end behaviour is: as $x \rightarrow \pm\infty, y \rightarrow -\infty$.

4. a) $3 - 1 = 2 \rightarrow$ even

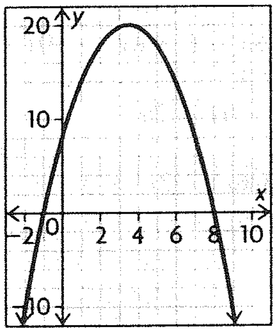
b) $6 - 1 = 5 \rightarrow$ odd

c) $4 - 1 = 3 \rightarrow$ odd

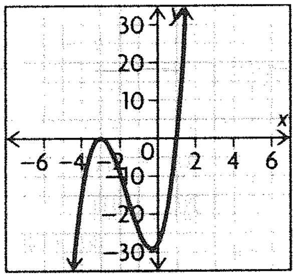
d) $5 - 1 = 4 \rightarrow$ even

5. Answers may vary. For example:

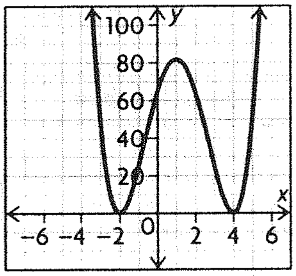
a)



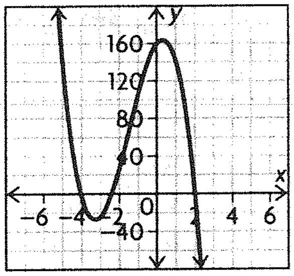
b)



c)



d)



6. The end behaviours cannot be determined if k is unknown because we must know the sign of the leading coefficient to determine the end behaviour.

7.

$$\begin{aligned} y &= a(x - 2)(x + 3)^2(x - 5) \\ 5000 &= a(7 - 2)(7 + 3)^2(7 - 5) \\ 5000 &= a(5)(10)^2(2) \\ 5000 &= 1000a \\ 5 &= a \\ y &= 5(x - 2)(x + 3)^2(x - 5) \end{aligned}$$

8. a) reflection in the x-axis, vertical stretch by a factor of 25, horizontal compression by a factor of $\frac{1}{3}$, horizontal translation 4 units to the left, vertical translation 60 units down

b) vertical stretch by a factor of 8, horizontal stretch by a factor of $\frac{4}{3}$, vertical translation 43 units up

c) reflection in the y-axis, horizontal compression by a factor of $\frac{1}{13}$, horizontal translation 2 units to the right, vertical translation 13 units up

d) vertical compression by a factor of $\frac{8}{11}$, reflection in the y-axis, vertical translation 1 unit down

9. $y = x^3$ was reflected in the x-axis, vertically stretched by a factor of 5, horizontally translated 4 units to the left, and vertically translated 2 units down.

3.5 Dividing Polynomials, pp. 168–170

1. a) i)

$$\begin{array}{r r}
 & x^3 - 14x^2 - 24x - 38 \\
 & x - 2\overline{)x^4 - 16x^3 + 4x^2 + 10x - 11} \\
x^3(x - 2) \rightarrow & \underline{x^4 + 2x^3} \quad \downarrow \qquad \downarrow \qquad \downarrow \\
 & -14x^3 + 4x^2 \qquad\qquad\qquad \\
-14x^2(x - 2) \rightarrow & \underline{-14x^3 + 28x^2} \qquad\qquad\qquad \\
 & -24x^2 + 10x \qquad\quad \\
-24x(x - 2) \rightarrow & \underline{-24x^2 + 48x} \qquad\quad \\
 & -38x - 11 \\
-38(x - 2) \rightarrow & \underline{-38x + 76} \\
 & -87
\end{array}$$

$x^3 - 14x^2 - 24x - 38$ remainder -87

ii)

$$\begin{array}{rr}
 & x^3 - 20x^2 - 84x - 326 \\
x + 4 & \overline{)x^4 - 16x^3 + 4x^2 + 10x - 11} \\
x^3(x + 4) \rightarrow & \underline{-x^4 + 4x^3} \quad \downarrow \quad \downarrow \quad \downarrow \\
 & -20x^3 + 4x^2 \qquad\qquad\quad \\
-20x^2(x + 4) \rightarrow & \underline{-20x^3 + 80x^2} \qquad\qquad\quad \\
 & 84x^2 + 10x \qquad\quad \\
84x(x + 4) \rightarrow & \underline{84x^2 + 336x} \qquad\quad \\
 & -326x - 11 \\
-326(x + 4) \rightarrow & \underline{-326x - 1304} \\
 & 1293
\end{array}$$

$x^3 - 20x^2 + 84x - 326$ remainder 1293

iii)

$$\begin{array}{rr}
 & x^3 - 15x^2 - 11x - 1 \\
x - 1 & \overline{)x^4 - 16x^3 + 4x^2 + 10x - 11} \\
x^3(x - 1) \rightarrow & \underline{x^4 - 1x^3} \quad \downarrow \quad \downarrow \quad \downarrow \\
 & -15x^3 + 4x^2 \qquad\qquad\quad \\
-15x^2(x - 1) \rightarrow & \underline{-15x^3 + 15x^2} \qquad\qquad\quad \\
 & -11x^2 + 10x \qquad\quad \\
-11x(x - 1) \rightarrow & \underline{-11x^2 + 118x} \qquad\quad \\
 & -1x - 11 \\
-1(x - 1) \rightarrow & \underline{-1x + 1} \\
 & -12
\end{array}$$

$x^3 - 15x^2 - 11x - 1$ remainder -12

b) No, they are not, because for each division problem there is a remainder.

a) degree of quotient: 2; degree of remainder: 1;
b) degree of quotient: 2; degree of remainder: 0;
c) degree of quotient: 1; degree of remainder: 3;
d) not possible

2. a) The degree of the quotient is 2 because $x^4 \div x^2 = x^2$.

b) The degree of the quotient is 2 because $x^3 \div x = x^2$.

c) The degree of the quotient is 1 because $x^4 \div x^3 = x$.

d) This is not possible because the degree of the divisor is greater than the degree of the dividend.

3. a)

$$\begin{array}{rr}
 & x^2 - 15x + 6 \\
x^2 - 4 & \overline{)x^4 - 15x^3 + 2x^2 + 12x - 10} \\
x^2(x^2 - 4) \rightarrow & \underline{x^4 \qquad - 4x^2} \quad \downarrow \quad \downarrow \\
 & -15x^3 + 6x^2 + 12x \qquad \\
-15x(x^2 - 4) \rightarrow & \underline{-15x^3 \qquad + 60x} \qquad \\
 & 6x^2 - 48x - 10 \\
6(x^2 - 4) \rightarrow & \underline{6x^2 \qquad - 24} \\
 & -48x + 14
\end{array}$$

$x^2 - 15x + 6$ remainder $-48x + 14$

b)

$$\begin{array}{r|rrrr}
-3 & 5 & -4 & 3 & -4 \\
 & \downarrow & -15 & 57 & -180 \\
\hline
 & 5 & -19 & 60 & -184
\end{array}$$

$5x^2 - 19x + 60$ remainder -184

c)

$$\begin{array}{rr}
 & x - 6 \\
x^3 - x^2 + 2x + 1 & \overline{)x^4 - 7x^3 + 2x^2 + 9x + 0} \\
x(x^3 - x^2 + 2x + 1) \rightarrow & \underline{x^4 - x^3 + 2x^2 - 1x \quad \downarrow} \\
 & -6x^3 \qquad + 10x + 0 \\
-6(x^3 - x^2 + 2x + 1) \rightarrow & \underline{-6x^3 + 6x^2 - 12x - 6} \\
 & -6x^2 + 22x + 6
\end{array}$$

$x - 6$ remainder $-6x^2 + 22x + 6$

d) Not possible

4.

$$\begin{array}{rr}
 & 2x^2 - 11x + 41 \\
x + 3 & \overline{)2x^3 - 5x^2 + 8x + 4} \\
2x^2(x + 3) & \underline{2x^3 + 6x^2} \quad \downarrow \quad \downarrow \\
 & -11x^2 + 8x \qquad \\
-11x(x + 3) & \underline{-11x^2 - 33x} \qquad \\
 & 41x + \quad 4 \\
41(x + 3) & \underline{41x + 123} \\
 & -119
\end{array}$$

$(2x + 4)(3x^3 - 5x + 8) - 3 = 6x^4 - 10x^2 + 16x + 12x^3 - 20x + 32 - 3$
$= 6x^4 + 12x^3 - 10x^2 - 4x + 29$

$$\begin{array}{rr}
 & 3x + 1 \\
2x^3 + 0x^2 + x - 4 & \overline{)6x^4 + 2x^3 + 3x^2 - 11x - 9} \\
3x(2x^3 + x - 4) \rightarrow & \underline{6x^4 + 0x^3 + 3x^2 - 12x} \quad \downarrow \\
 & 2x^3 + 0x^2 + \quad x - 9 \\
1(2x^3 + x - 4) \rightarrow & \underline{2x^3 + 0x^2 + \quad x - 4} \\
 & -5
\end{array}$$

$$\begin{array}{rr}
 & 3x^2 - 5x + 4 \\
x + 2 & \overline{)3x^3 + x^2 - 6x + 16} \\
3x^2(x + 2) \rightarrow & \underline{3x^3 + 6x^2} \quad \downarrow \quad \downarrow \\
 & -5x^2 + 6x \qquad \\
-5x(x + 2) \rightarrow & \underline{-5x^2 - 10x} \qquad \\
 & 4x + 16 \\
4(x + 2) & \underline{4x + 8} \\
 & 8
\end{array}$$

Dividend	Divisor	Quotient	Remainder
$2x^3 - 5x^2 + 8x + 4$	$x + 3$	$2x^2 - 11x - 41$	-119
$6x^4 + 12x^3 - 10x^2 - 4x + 29$	$2x + 4$	$3x^3 - 5x + 8$	-3
$6x^4 + 2x^3 + 3x^2 - 11x - 9$	$3x + 1$	$2x^3 + x - 4$	-5
$3x^3 + x^2 - 6x + 16$	$x + 2$	$3x^2 - 5x + 4$	8

5. a)

$$\begin{array}{rl} & \qquad\quad x^2+4x+14 \\ x-4 & \overline{)x^3+0x^2-2x+1} \\ x^2(x-4)\rightarrow & \underline{x^3-4x^2}\quad\downarrow\quad\downarrow \\ & \qquad 4x^2-2x \\ 4x(x-4)\rightarrow & \qquad \underline{4x^2-16x} \\ & \qquad\qquad 14x+1 \\ 14(x-4)\rightarrow & \qquad\qquad \underline{14x-56} \\ & \qquad\qquad\quad +57 \end{array}$$

$x^2+4x+14$ remainder 57

b)

$$\begin{array}{rl} & \qquad\qquad x^2-6 \\ x+2 & \overline{)x^3+2x^2-6x+1} \\ x^2(x+2)\rightarrow & \underline{x^3+2x^2}\quad\downarrow\quad\downarrow \\ & \qquad\quad -6x+1 \\ -6(x+2)\rightarrow & \qquad\quad \underline{-6x-12} \\ & \qquad\qquad +13 \end{array}$$

x^2-6 remainder 13

c)

$$\begin{array}{rl} & \qquad\quad x^2+2x-3 \\ 2x+1 & \overline{)2x^3+5x^2-4x-5} \\ x^2(2x+1)\rightarrow & \underline{2x^3+1x^2}\quad\downarrow\quad\downarrow \\ & \qquad 4x^2-4x \\ 2x(2x+1)\rightarrow & \qquad \underline{4x^2+2x} \\ & \qquad\qquad -6x-5 \\ -3(2x+1)\rightarrow & \qquad\qquad \underline{-6x-3} \\ & \qquad\qquad\quad -2 \end{array}$$

x^2+2x-3 remainder -2

d)

$$\begin{array}{rl} & \qquad\qquad x^2+3x-9 \\ x^2+7 & \overline{)x^4+3x^3-2x^2+5x-1} \\ x^2(x^2+7)\rightarrow & \underline{x^4\qquad +7x^2}\quad\downarrow\quad\downarrow \\ & \quad 3x^3-9x^2+5x \\ 3x(x^2+7)\rightarrow & \quad \underline{3x^3\qquad +21x} \\ & \qquad -9x^2-16x-1 \\ -9(x^2+7)\rightarrow & \qquad \underline{-9x^2\qquad -63} \\ & \qquad\qquad -16x+62 \end{array}$$

x^2+3x-9 remainder $-16x+62$

e)

$$\begin{array}{rl} & \qquad\qquad\qquad\qquad x+1 \\ x^3-x^2-x+1 & \overline{)x^4+0x^3+6x^2-8x+12} \\ x(x^3-x^2-x+1)\rightarrow & \underline{x^4-x^3-x^2+x}\quad\downarrow \\ & \quad 1x^3+7x^2-9x+12 \\ 1(x^3-x^2-x+1)\rightarrow & \quad \underline{x^3-x^2-x+1} \\ & \qquad 8x^2-8x+11 \end{array}$$

$x+1$ remainder $8x^2-8x+11$

f)

$$\begin{array}{rl} & \qquad\qquad\qquad\qquad\qquad x+3 \\ x^4+x^3+x^2+x-2 & \overline{)x^5+4x^4+0x^3+0x^2+9x+8} \\ x(x^4+x^3+x^2+x-2)\rightarrow & \underline{x^5+x^4+x^3+x^2-2x}\quad\downarrow \\ & \quad 3x^4-x^3-x^2+11x+8 \\ 3(x^4+x^3+x^2+x-2)\rightarrow & \quad \underline{3x^4+3x^3+3x^2+3x-6} \\ & \qquad -4x^3-4x^2+8x+14 \end{array}$$

$x+3$ remainder $-4x^3-4x^2+8x+14$

6. a)

$$\begin{array}{r|rrrr} 3 & 1 & 0 & -7 & -6 \\ & \downarrow & 3 & 9 & 6 \\ \hline & 1 & 3 & 2 & 0 \end{array}$$

x^2+3x+2 no remainder

b)

$$\begin{array}{r|rrrr} 1 & 2 & -7 & -7 & 19 \\ & \downarrow & 2 & -5 & -12 \\ \hline & 2 & -5 & -12 & 7 \end{array}$$

$2x^2-5x-12$ remainder 7

c)

$$\begin{array}{r|rrrrr} -3 & 6 & 13 & -34 & -47 & 28 \\ & \downarrow & -18 & 15 & 57 & -30 \\ \hline & 6 & -5 & -19 & 10 & -2 \end{array}$$

$6x^3-5x^2-19x+10$ remainder -2

d)

$$\begin{array}{r|rrrr} \frac{3}{2} & 2 & 1 & -22 & 20 \\ & \downarrow & 3 & 6 & -24 \\ \hline & 2 & 4 & -16 & -4 \\ & \div 2 & \div 2 & \div 2 & \div 2 \\ \hline & 1 & 2 & -8 & -2 \end{array}$$

x^2+2x-8 remainder -2

e)

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 12 & -56 & 59 & 9 & -18 \\ & \downarrow & -6 & 31 & -45 & 18 \\ \hline & 12 & -62 & 90 & -36 & 0 \\ & \div 2 & \div 2 & \div 2 & \div 2 & \\ \hline & 6 & -31 & 45 & -18 & \end{array}$$

$6x^3-31x^2+45x-18$ no remainder

f)

$$\begin{array}{r|rrrr} \frac{5}{2} & 6 & -15 & -2 & 5 \\ & \downarrow & 15 & 0 & -5 \\ \hline & 6 & 0 & -2 & 0 \\ & \div 2 & & \div 2 & \\ \hline & 3 & & -1 & \end{array}$$

$3x^2-1$ no remainder

7. a) Divisor: $x+10$; Quotient: x^2-6x+9; Remainder: -1

$= (x+10)(x^2-6x+9)-1$

$= x^3-6x^2+9x+10x^2-60x+90-1$

$= x^3+4x^2-51x+89$

b) Divisor: $3x - 2$; Quotient: $x^3 + x - 12$;
Remainder: 15

$= (3x - 2)(x^3 + x - 12) + 15$
$= 3x^4 + 3x^2 - 36x - 2x^3 - 2x + 24 + 15$
$= 3x^4 - 2x^3 + 3x^2 - 38x + 39$

c) Divisor: $5x + 2$; Quotient: $x^3 + 4x^2 - 5x + 6$;
Remainder: $x - 2$

$= (5x + 2)(x^3 + 4x^2 - 5x + 6) + (x - 2)$
$= 5x^4 + 20x^3 - 25x^2 + 30x + 2x^3 + 8x^2$
$\quad - 10x + 12 + x - 2$
$= 5x^4 + 22x^3 - 17x^2 + 21x + 10$

---

d) Divisor: $x^2 + 7x - 2$;
Quotient: $x^4 + x^3 - 11x + 4$;
Remainder: $x^2 - x + 5$

$= (x^2 + 7x - 2)(x^4 + x^3 - 11x + 4)$
$\quad + (x^2 - x + 5)$
$= x^6 + x^5 - 11x^3 + 4x^2 + 7x^5 + 7x^4 - 77x^2$
$\quad + 28x - 2x^4 - 2x^3 + 22x - 8 + x^2 - x + 5$
$= x^6 + 8x^5 + 5x^4 - 13x^3 - 72x^2 + 49x - 3$

8. a) $(2x - 3)(3x + 5) + r = 6x^2 + x + 5$
$6x^2 + 10x - 9x - 15 + r = 6x^2 + x + 5$
$6x^2 + x - 15 + r = 6x^2 + x + 5$
$r = 6x^2 + x + 5 - (6x^2 + x - 15)$
$r = 20$

b) $(x + 3)(x + 5) + r = x^2 + 9x - 7$
$x^2 + 5x + 3x + 15 + r = x^2 + 9x - 7$
$x^2 + 8x + 15 + r = x^2 + 9x - 7$
$r = x^2 + 9x - 7 - (x^2 + 8x + 15)$
$r = x - 22$

c) $(x + 3)(x^2 - 1) + r = x^3 + 3x^2 - x - 3$
$x^3 - x + 3x^2 - 3 + r = x^3 + 3x^2 - x - 3$
$r = x^3 + 3x^2 - x - 3 - (x^3 + 3x^2 - x - 3)$
$r = 0$

d) $(x^2 + 1)(2x^3 - 1) + r = 2x^5 + 2x^3 + x^2 + 1$
$2x^5 - x^2 + 2x^3 - 1 + r = 2x^5 + 2x^3 + x^2 + 1$
$r = 2x^5 + 2x^3 + x^2 + 1 - (2x^5 + 2x^3 - x^2 - 1)$
$r = 2x^2 + 2$

9. a) Dividend: $5x^3 + x^2 + 3$;
Quotient: $5x^2 - 14x + 42$; Remainder: -123
In order to determine the divisor, we need to determine:

$$\frac{5x^3 + x^2 + 3}{5x^2 - 14x + 42}.$$

$x + 3$ is the divisor.

b) Dividend: $10x^4 - x^2 + 20x - 2$;
Quotient: $10x^3 - 100x^2 + 999x - 9970$;
Remainder: 99 698
In order to determine the divisor, we need to determine:

$$\frac{10x^4 - x^2 + 20x - 2}{10x^3 - 100x^2 + 999x - 9970}.$$

$$
\begin{array}{rl}
 & \phantom{10x^4 + 0x^3 - x^2 + 20x - {}} x + 10 \\
10x^3 - 100x^2 + 999x - 9970 & \overline{)\,10x^4 + 0x^3 - x^2 + 20x - 2} \\
x(10x^3 - 100x^2 + 999x - 9970) \rightarrow & \underline{10x^4 - 100x^3 + 999x^2 - 9970x} \quad \downarrow \\
 & \phantom{10x^4 {}} 100x^3 - 1000x^2 + 9990x - 2 \\
10(10x^3 - 100x^2 + 999x - 9970) \rightarrow & \phantom{10x^4 {}} \underline{100x^3 - 1000x^2 + 9990x - 99\,700} \\
 & \phantom{10x^4 + 0x^3 - x^2 + 20x - {}} 99\,698
\end{array}
$$

$x + 10$ is the divisor.

c) Dividend: $x^4 + x^3 - 10x^2 - 1$;
Quotient: $x^3 - 3x^2 + 2x - 8$
In order to determine the divisor, we need to determine:

$$\frac{x^4 + x^3 - 10x^2 - 1}{x^3 - 3x^2 + 2x - 8}.$$

$$
\begin{array}{rl}
 & \phantom{x^4 + x^3 - 10x^2 + {}} x + 4 \\
x^3 - 3x^2 + 2x - 8 & \overline{)\,x^4 + x^3 - 10x^2 + 0x - 1} \\
x(x^3 - 3x^2 + 2x - 8) \rightarrow & \underline{x^4 - 3x^3 + 2x^2 - 8x} \quad \downarrow \\
 & \phantom{x^4 {}} 4x^3 - 12x^2 + 8x - 1 \\
4(x^3 - 3x^2 + 2x - 8) \rightarrow & \phantom{x^4 {}} \underline{4x^3 - 12x^2 + 8x - 32} \\
 & \phantom{x^4 + x^3 - 10x^2 + 0x - {}} 31
\end{array}
$$

$x + 4$ is the divisor.

d) Dividend: $x^3 + x^2 + 7x - 7$;
Quotient: $x^2 + 3x + 13$; Remainder: 19
In order to determine the divisor, we need to determine:

$$\frac{x^3 + x^2 + 7x - 7}{x^2 + 3x + 13}.$$

$$
\begin{array}{rl}
 & \phantom{x^3 + x^2 + {}} x - 2 \\
x^2 + 3x + 13 & \overline{)\,x^3 + x^2 + 7x - 7} \\
(x^2 + 3x + 13) \rightarrow & \underline{x^3 + 3x^2 + 13x} \ \downarrow \\
 & \phantom{x^3 {}} -2x^2 - 6x - 7 \\
-2(x^2 + 3x + 13) \rightarrow & \phantom{x^3 {}} \underline{-2x^2 - 6x - 26} \\
 & \phantom{x^3 + x^2 + 7x - {}} 19
\end{array}
$$

$x - 2$ is the divisor.

10. a)
$$\begin{array}{r|rrrr} -5 & 1 & 6 & -1 & -30 \\ & \downarrow & -5 & -5 & 30 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$
$x + 5$ is a factor since there is no remainder.

b)
$$\begin{array}{r|rrrrr} -2 & 1 & 0 & -5 & 0 & 4 \\ & \downarrow & -2 & 4 & 2 & -4 \\ \hline & 1 & -2 & -1 & 2 & 0 \end{array}$$
$x + 2$ is a factor since there is no remainder.

c)
$$\begin{array}{r|rrrrr} 2 & 1 & 0 & -5 & 0 & 6 \\ & \downarrow & 2 & 4 & -2 & -4 \\ \hline & 1 & 2 & -1 & -2 & 2 \end{array}$$
$x - 2$ is not a factor since there is a remainder of 2.

d)
$$\begin{array}{r|rrrrr} \frac{1}{2} & 2 & -1 & -4 & 2 & 1 \\ & \downarrow & 1 & 0 & -2 & 0 \\ \hline & 2 & 0 & -4 & 0 & 1 \end{array}$$
$2x - 1$ is not a factor since there is a remainder of 1.

e)
$$\begin{array}{r|rrrrrrr} -\frac{5}{3} & 3 & 5 & 0 & 0 & 9 & 17 & -1 \\ & \downarrow & -5 & 0 & 0 & 0 & -15 & -\frac{10}{3} \\ \hline & 3 & 0 & 0 & 0 & 9 & 2 & -\frac{13}{3} \end{array}$$
$3x + 5$ is not a factor since there is a remainder of $-\frac{13}{3}$.

f)
$$\begin{array}{r|rrrrr} \frac{1}{5} & 5 & -1 & 0 & 10 & -10 \\ & \downarrow & 1 & 0 & 0 & 2 \\ \hline & 5 & 0 & 0 & 10 & -8 \end{array}$$
$5x - 1$ is not a factor since there is a remainder of -8.

11. $V = \ell wh$

$x^3 + 6x^2 + 11x + 6 = (x + 2)(x + 3)h$

$$\frac{x^3 + 6x^2 + 11x + 6}{(x + 2)(x + 3)} = h$$

$$\frac{x^3 + 6x^2 + 11x + 6}{x^2 + 5x + 6} = h$$

$$\begin{array}{rl} & \phantom{x^3 + 6x^2 +{}} x + 1 \\ x^2 + 5x + 6 & \overline{)x^3 + 6x^2 + 11x + 6} \\ x(x^2 + 5x + 6) \rightarrow & \underline{x^3 + 5x^2 + 6x} \quad \downarrow \\ & \phantom{x^3 +{}} 1x^2 + 5x + 6 \\ 1(x^2 + 5x + 6) \rightarrow & \phantom{x^3 +{}} \underline{x^2 + 5x + 6} \\ & \phantom{x^3 + 6x^2 + 11x +{}} 0 \end{array}$$

The height of the box is $(x + 1)$ cm.

12. a) Using synthetic division, the problem looks like:

$$\begin{array}{r|rrrr} -\frac{1}{2} & 8 & 10 & -p & -5 \\ & \downarrow & -4 & -3 & 5 \\ \hline & 8 & 6 & x & 0 \end{array}$$

In order for there to be no remainder, x must be -10 because $-\frac{1}{2}(-10) = 5$.

So, $-p - 3 = -10$.

$-p = -7$

$p = 7$

To check this:

$$\begin{array}{r|rrrr} -\frac{1}{2} & 8 & 10 & -7 & -5 \\ & \downarrow & -4 & -3 & 5 \\ \hline & 8 & 6 & -10 & 0 \end{array}$$

b) Using long division, the problem looks like:

$$\begin{array}{rl} & \phantom{x^6 - 0x^5 + x^4 + 0x^3 - 2x^2 +{}} x^4 - 2 \\ x^2 + 1 & \overline{)x^6 - 0x^5 + x^4 + 0x^3 - 2x^2 + 0x + k} \\ x^4(x^2 + 1) \rightarrow & \underline{x^6 + 0x^5 + 1x^4} \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ & \phantom{x^6 - 0x^5 +{}} 0x^4 + 0x^3 - 2x^2 + 0x + k \\ -2(x^2 + 1) \rightarrow & \phantom{x^6 - 0x^5 + x^4 + 0x^3 {}} \underline{-2x^2 + 0x - 2} \\ & \phantom{x^6 - 0x^5 + x^4 + 0x^3 - 2x^2 + 0x + {}} 5 \end{array}$$

So, $k + 2 = 5$

$k = 3$

13. Using synthetic division and filling in what we know or can simply figure out, the problem looks like:

$$\begin{array}{r|rrrr} 1 & 1 & p & -1 & -2 \\ & \downarrow & 1 & 3 & 2 \\ \hline & 1 & 3 & 2 & 0 \end{array}$$

So, $p + 1 = 3$.

$p = 2$

To check this:

$$\begin{array}{r|rrrr} 1 & 1 & 2 & -1 & -2 \\ & \downarrow & 1 & 3 & 2 \\ \hline & 1 & 3 & 2 & 0 \end{array}$$

14. Yes, $f(x)$ is always divisible by $x - 1$. Regardless of the value of n, $f(x) = x^n - 1$ can always be written as

$f(x) = x^n + 0x^{n-1} + 0x^{n-2} + \ldots 0x - 1.$

Therefore, the same pattern continues when dividing $x^n - 1$ by $x - 1$, regardless of how large n is, and there is never a remainder.

15. a) $(x - 4)(x^2 + x - 6) + 7$

$x^3 + x^2 - 6x - 4x^2 - 4x + 24 + 7$

$x^3 - 3x^2 - 10x + 31$

So, $f(x) = (x^3 - 3x^2 - 10x + 31) = (x - 4)$ $(x^2 + x - 6)$ remainder 7

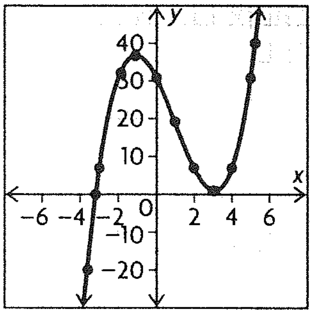
b) $x^2 + x - 6$

$(x + 3)(x - 2)$

So, $f(x) = (x^3 - 3x^2 - 10x + 31) = (x - 4)$

$(x + 3)(x - 2)$ remainder 7

c)



16.

$$\begin{array}{r|rrrrr} 3 & 2 & 3 & -25 & -7 & -14 \\ & \downarrow & 6 & 27 & 6 & -3 \\ \hline & 2 & 9 & 2 & -1 & \boxed{-17} \end{array}$$

$$\underbrace{2 \quad 9 \quad 2 \quad -1}_{\text{Coefficients of quotient}} \qquad \text{Remainder}$$

$$\overbrace{2x^3 + 9x^2 + 2x - 1}$$

$$\begin{array}{rl} & \quad 2x^3 + 9x^2 + 2x - 1 \\ x - 3 \,\big) & \overline{2x^4 + 3x^3 - 25x^2 - 7x - 14} \\ 2x^3(x - 3) \rightarrow & \underline{2x^4 - 6x^3} \\ & \qquad 9x^3 - 25x^2 \\ 9x^2(x - 3) \rightarrow & \qquad \underline{9x^3 - 27x^2} \\ & \qquad\qquad 2x^2 - 7x \\ 2x(x - 3) \rightarrow & \qquad\qquad \underline{2x^2 - 6x} \\ & \qquad\qquad\qquad -1x - 14 \\ -1(x - 3) \rightarrow & \qquad\qquad\qquad \underline{-1x + \ \ 3} \\ & \qquad\qquad\qquad\qquad -17 \end{array}$$

17. $V = \pi r^2 h$

$4\pi x^3 + 28\pi x^2 + 65\pi x + 50\pi = \pi r^2(x + 2)$

$\cancel{\pi}(4x^3 + 28x^2 + 65x + 50) = \cancel{\pi} r^2(x + 2)$

$$\frac{4x^3 + 28x^2 + 65x + 50}{x + 2} = r^2$$

$$\begin{array}{r|rrrr} -2 & 4 & 28 & 65 & 50 \\ & \downarrow & -8 & -40 & -50 \\ \hline & 4 & 20 & 25 & 0 \end{array}$$

$4x^2 + 20x + 25 = r^2$

$\sqrt{4x^2 + 20x + 25} = r$

$r = 2x + 5$ cm

18. a)

$$\begin{array}{rl} & \qquad\qquad\qquad x^2 + xy + y^2 \\ x^2 - y^2 \,\big) & \overline{x^4 + x^3y + 0x^2y^2 - xy^3 - y^4} \\ x^2(x^2 - y^2) \rightarrow & \underline{x^4 \qquad\quad - x^2y^2} \quad \downarrow \quad \downarrow \\ & \qquad x^3y + x^2y^2 - xy^3 \\ xy(x^2 - y^2) \rightarrow & \qquad \underline{x^3y \qquad\quad - xy^3} \\ & \qquad\qquad\quad x^2y^2 \qquad - y^4 \\ y^2(x^2 - y^2) \rightarrow & \qquad\qquad\quad \underline{x^2y^2 \qquad - y^4} \\ & \qquad\qquad\qquad\qquad\quad 0 \end{array}$$

b)

$$\begin{array}{rl} & \qquad\qquad\qquad x^2 - 2xy + y^2 \\ x^2 + y^2 \,\big) & \overline{x^4 - 2x^3y + 2x^2y^2 - 2xy^3 + y^4} \\ x^2(x^2 + y^2) \rightarrow & \underline{x^4 \qquad\quad + x^2y^2} \quad \downarrow \quad \downarrow \\ & \qquad -2x^3y + x^2y^2 - 2xy^3 \\ -2xy(x^2 + y^2) \rightarrow & \qquad \underline{-2x^3y \qquad\quad - 2xy^3} \\ & \qquad\qquad\quad x^2y^2 \qquad + y^4 \\ y^2(x^2 + y^2) \rightarrow & \qquad\qquad\quad \underline{x^2y^2 \qquad + y^4} \\ & \qquad\qquad\qquad\qquad\quad 0 \end{array}$$

19.

$$\begin{array}{rl} & \qquad\qquad x^2 + xy + y^2 \\ x - y \,\big) & \overline{x^3 - 0x^2y + 0xy^2 + y^3} \\ x^2(x - y) \rightarrow & \underline{x^3 - x^2y} \quad \downarrow \quad \downarrow \\ & \qquad x^2y + 0x^2y^2 \\ xy(x - y) \rightarrow & \qquad \underline{x^2y - xy^2} \\ & \qquad\qquad xy^2 - y^3 \\ y^2(x - y) \rightarrow & \qquad\qquad \underline{x^2y^2 - y^3} \\ & \qquad\qquad\qquad 0 \end{array}$$

$x - y$ is a factor because there is no remainder.

20. $q(x)(x + 5) < f(x)$

$[q(x) + 1](x + 5) = q(x)(x + 5) + (x + 5)$, which is greater than

$f(x) = q(x)(x + 5) + (x + 3)$.

The first multiple of $(x + 5)$ that is greater than $f(x)$ is $[q(x) + 1](x + 5)$.

3.6 Factoring Polynomials, pp. 176–177

1. a) i) $f(2) = 2^4 + 5(2)^3 + 3(2)^2 - 7(2) + 10$

$= 16 + 40 + 12 - 14 + 10$

$= 64$

ii) $f(-4) = (-4)^4 + 5(-4)^3 + 3(-4)^2$

$- 7(-4) + 10$

$= 256 - 320 + 48 + 28 + 10$

$= 22$

iii) $f(1) = 1^4 + 5(1)^3 + 3(1)^2 - 7(1) + 10$

$= 1 + 5 + 3 - 7 + 10$

$= 12$

b) No, according to the factor theorem, $x - a$ is a factor of $f(x)$ if and only if $f(a) = 0$.

2. a) $f(1) = 1^4 - 15(1)^3 + 2(1)^2 + 12(1) - 10$
$= 1 - 15 + 2 + 12 - 10$
$= -10$

$x - 1$ is not a factor of $f(x)$ because there is a remainder.

b) $g(1) = 5(1)^3 - 4(1)^2 + 3(1) - 4$
$= 5 - 4 + 3 - 4$
$= 0$

$x - 1$ is a factor of $g(x)$ because there is not a remainder.

c) $h(1) = 1^4 - 7(1)^3 + 2(1)^3 + 9(1)$
$= 1 - 7 + 2 + 9$
$= 5$

$x - 1$ is not a factor of $h(x)$ because there is a remainder.

d) $j(1) = 1^3 - 1$
$= 1 - 1$
$= 0$

$x - 1$ is a factor of $j(x)$ because there is not a remainder.

3. $f(x) = x^3 + 2x^2 - 5x - 6$
$f(-1) = (-1)^3 + 2(-1)^2 - 5(-1) - 6$
$= -1 + 2 + 5 - 6$
$= 0$

$x + 1$ is a factor.

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & \downarrow & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

$= (x + 1)(x^2 + x - 6)$
$= (x + 1)(x + 3)(x - 2)$

4. a) $f(-2) = (-2)^2 + 7(-2) + 9$
$= 4 - 14 + 9$
$= -1$

b) $f(-2) = 6(-2)^3 + 19(-2)^2 + 11(-2) - 11$
$= -48 + 76 - 22 - 11$
$= -5$

c) $f(-2) = (-2)^4 - 5(-2)^2 + 4$
$= 16 - 20 + 4$
$= 0$

d) $f(-2) = (-2)^4 - 2(-2)^3 - 11(-2)^2 + 10(-2) - 2$
$= 16 + 16 - 44 - 20 - 2$
$= -34$

e) $f(-2) = (-2)^3 + 3(-2)^2 - 10(-2) + 6$
$= -8 + 12 + 20 + 6$
$= 30$

f) $f(-2) = 4(-2)^4 + 12(-2)^3 - 13(-2)^2 - 33(-2) + 18$
$= 64 - 96 - 52 + 66 + 18$
$= 0$

5. a)
$$\begin{array}{r|rrrr} \frac{5}{2} & 2 & -5 & -2 & 5 \\ & \downarrow & 5 & 0 & -5 \\ \hline & 2 & 0 & -2 & 0 \end{array}$$

$2x - 5$ is a factor because there is no remainder.

b)
$$\begin{array}{r|rrrr} \frac{5}{2} & 3 & 2 & -3 & -2 \\ & \downarrow & \frac{15}{2} & \frac{95}{4} & \frac{415}{8} \\ \hline & 3 & \frac{19}{2} & \frac{83}{4} & \frac{399}{8} \end{array}$$

$2x - 5$ is not a factor because there is a remainder.

c)
$$\begin{array}{r|rrrrr} \frac{5}{2} & 2 & -7 & -13 & 63 & -45 \\ & \downarrow & 5 & -5 & -45 & 45 \\ \hline & 2 & -2 & -18 & 18 & 0 \end{array}$$

$2x - 5$ is a factor because there is no remainder.

d)
$$\begin{array}{r|rrrrr} \frac{5}{2} & 6 & 1 & -7 & -1 & 1 \\ & \downarrow & 15 & 40 & \frac{165}{2} & \frac{815}{4} \\ \hline & 6 & 16 & 33 & \frac{163}{2} & \frac{819}{4} \end{array}$$

$2x - 5$ is not a factor because there is a remainder.

6. a) $x^3 - 3x^2 - 10x + 24$
$f(2) = 2^3 - 3(2)^2 - 10(2) + 24$
$= 8 - 12 - 20 + 24$
$= 0$

$x - 2$ is a factor.

$$\begin{array}{r|rrrr} 2 & 1 & -3 & -10 & 24 \\ & \downarrow & 2 & -2 & -24 \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

$= (x - 2)(x^2 - x - 12)$
$= (x - 2)(x - 4)(x + 3)$

b) $4x^3 + 12x^2 - x - 15$
$f(1) = 4(1)^3 + 12(1)^2 - 1 - 15$
$= 4 + 12 - 1 - 15$
$= 0$

$x - 1$ is a factor.

$$\begin{array}{r|rrrr} 1 & 4 & 12 & -1 & -15 \\ & \downarrow & 4 & 16 & 15 \\ \hline & 4 & 16 & 15 & 0 \end{array}$$

$= (x - 1)(4x^2 + 16x + 15)$
$= (x - 1)(4x^2 + 10x + 6x + 15)$

$= (x - 1)(2x(2x + 5) + 3(2x + 5))$

$= (x - 1)(2x + 3)(2x + 5)$

c) $x^4 + 8x^3 + 4x^2 - 48x$

$x(x^3 + 8x^2 + 4x - 48)$

For $(x^3 + 8x^2 + 4x - 48)$

$f(2) = 2^3 + 8(2)^2 + 4(2) - 48$

$= 8 + 32 + 8 - 48$

$= 0$

$x - 2$ is a factor.

$$\begin{array}{r|rrrr} 2 & 1 & 8 & 4 & -48 \\ & \downarrow & 2 & 20 & 48 \\ \hline & 1 & 10 & 24 & 0 \end{array}$$

$= x(x - 2)(x^2 + 10x + 24)$

$= x(x - 2)(x + 4)(x + 6)$

d) $4x^4 + 7x^3 - 80x^2 - 21x + 270$

$f(-2) = 4(-2)^4 + 7(-2)^3 - 80(-2)^2 - 21(-2) + 270$

$= 64 - 56 - 320 + 42 + 270$

$= 0$

$x + 2$ is a factor.

$$\begin{array}{r|rrrrr} -2 & 4 & 7 & -80 & -21 & 270 \\ & \downarrow & -8 & 2 & 156 & -270 \\ \hline & 4 & -1 & -78 & 135 & 0 \end{array}$$

$= (x + 2)(4x^3 - x^2 - 78x + 135)$

For $(4x^3 - x^2 - 78x + 135)$

$f(3) = 4(3)^3 - (3)^2 - 78(3) + 135$

$= 108 - 9 - 234 + 135$

$= 0$

$x - 3$ is a factor.

$$\begin{array}{r|rrrr} 3 & 4 & -1 & -78 & 135 \\ & \downarrow & 12 & 33 & -135 \\ \hline & 4 & 11 & -45 & 0 \end{array}$$

$= (x + 2)(x - 3)(4x^2 + 11x - 45)$

$= (x + 2)(x - 3)(4x^2 + 20x - 9x - 45)$

$= (x + 2)(x - 3)(4x(x + 5) - 9(x + 5))$

$= (x + 2)(x + 5)((4x - 9)(x - 3))$

e) $x^5 - 5x^4 - 7x^3 + 29x^2 + 30x$

$x(x^4 - 5x^3 - 7x^2 + 29x + 30)$

For $(x^4 - 5x^3 - 7x^2 + 29x + 30)$

$f(-1) = (-1)^4 - 5(-1)^3 - 7(-1)^2 + 29(-1) + 30$

$= 1 + 5 - 7 - 29 + 30$

$= 0$

$x + 1$ is a factor.

$$\begin{array}{r|rrrrr} -1 & 1 & -5 & -7 & 29 & 30 \\ & \downarrow & -1 & 6 & 1 & -30 \\ \hline & 1 & -6 & -1 & 30 & 0 \end{array}$$

$= x(x + 1)(x^3 - 6x^2 - x + 30)$

For $(x^3 - 6x^2 - x + 30)$

$f(3) = 3^3 - 6(3)^2 - 3 + 30$

$= 27 - 54 - 3 + 30$

$= 0$

$x - 3$ is a factor.

$$\begin{array}{r|rrrr} 3 & 1 & -6 & -1 & 30 \\ & \downarrow & 3 & -9 & -30 \\ \hline & 1 & -3 & -10 & 0 \end{array}$$

$= x(x + 1)(x - 3)(x^2 - 3x - 10)$

$= x(x + 2)(x + 1)(x - 3)(x - 5)$

f) $x^4 + 2x^3 - 23x^2 - 24x + 144$

$f(3) = 3^4 + 2(3)^3 - 23(3)^2 - 24(3) + 144$

$= 81 + 54 - 207 - 72 + 144$

$= 0$

$x - 3$ is a factor.

$$\begin{array}{r|rrrrr} 3 & 1 & 2 & -23 & -24 & 144 \\ & \downarrow & 3 & 15 & -24 & -144 \\ \hline & 1 & 5 & -8 & -48 & 0 \end{array}$$

$= (x - 3)(x^3 + 5x^2 - 8x - 48)$

For $x^3 + 5x^2 - 8x - 48$

$f(-4) = (-4)^3 + 5(-4)^2 - 8(-4) - 48$

$= -64 + 80 + 32 - 48$

$= 0$

$x + 4$ is a factor.

$$\begin{array}{r|rrrr} -4 & 1 & 5 & -8 & -48 \\ & \downarrow & -4 & -4 & 48 \\ \hline & 1 & 1 & -12 & 0 \end{array}$$

$(x - 3)(x + 4)(x^2 + x - 12)$

$= (x - 3)(x - 3)(x + 4)(x + 4)$

7. a) $x^3 + 9x^2 + 8x - 60$

$f(2) = 2^3 + 9(2)^2 + 8(2) - 60$

$= 8 + 36 + 16 - 60$

$= 0$

$$\begin{array}{r|rrrr} 2 & 1 & 9 & 8 & -60 \\ & \downarrow & 2 & 22 & 60 \\ \hline & 1 & 11 & 30 & 0 \end{array}$$

$= (x - 2)(x^2 + 11x + 30)$

$= (x - 2)(x + 5)(x + 6)$

b) $x^3 - 7x - 6$

$f(-1) = (-1)^3 - 7(-1) - 6$

$= -1 + 7 - 6$

$= 0$

$x + 1$ is a factor.

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & \downarrow & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$= (x + 1)(x^2 - x - 6)$

$= (x + 1)(x - 3)(x + 2)$

c) $x^4 - 5x^2 + 4$
$= (x^2 - 1)(x^2 - 4)$
$= (x + 1)(x - 1)(x - 2)(x + 2)$

d) $x^4 + 3x^3 - 38x^2 + 24x + 64$
$f(2) = 2^4 + 3(2)^3 - 38(2)^2 + 24(2) + 64$
$= 16 + 24 - 152 + 48 + 64$
$= 0$
$x - 2$ is a factor.

$$\begin{array}{r|rrrrr} 2 & 1 & 3 & -38 & 24 & 64 \\ & \downarrow & 2 & 10 & -56 & -64 \\ \hline & 1 & 5 & -28 & -32 & 0 \end{array}$$

$= (x - 2)(x^3 + 5x^2 - 28x - 32)$
For $x^3 + 5x^2 - 28x - 32$
$f(-1) = (-1)^3 + 5(-1)^2 - 28(-1) - 32$
$= -1 + 5 + 28 - 32$
$= 0$
$x + 1$ is a factor.

$$\begin{array}{r|rrrr} -1 & 1 & 5 & -28 & -32 \\ & \downarrow & -1 & -4 & 32 \\ \hline & 1 & 4 & -32 & 0 \end{array}$$

$= (x - 2)(x + 1)(x^2 + 4x - 32)$
$= (x - 2)(x + 1)(x + 8)(x - 4)$

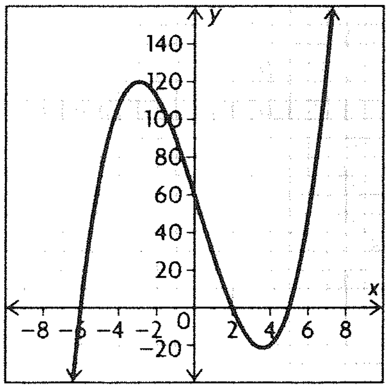
e) $x^3 - x^2 + x - 1$
$x^2(x - 1) + 1(x - 1)$
$(x - 1)(x^2 + 1)$

f) $x^5 - x^4 + 2x^3 - 2x^2 + x - 1$
$f(1) = 1^5 - 1^4 + 2(1)^3 - 2(1)^2 + 1 - 1$
$= 1 - 1 + 2 - 2 + 1 - 1$
$= 0$
$x - 1$ is a factor.

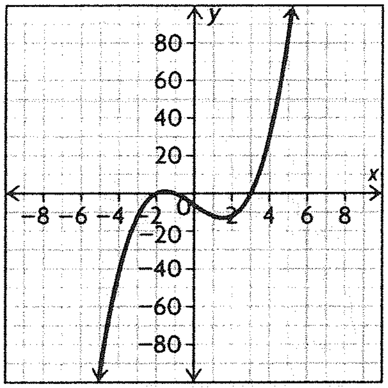
$$\begin{array}{r|rrrrrr} 1 & 1 & -1 & 2 & -2 & 1 & -1 \\ & \downarrow & 1 & 0 & 2 & 0 & 1 \\ \hline & 1 & 0 & 2 & 0 & 1 & 0 \end{array}$$

$= (x - 1)(x^4 + 2x^2 + 1)$
$= (x - 1)(x^2 + 1)(x^2 + 1)$

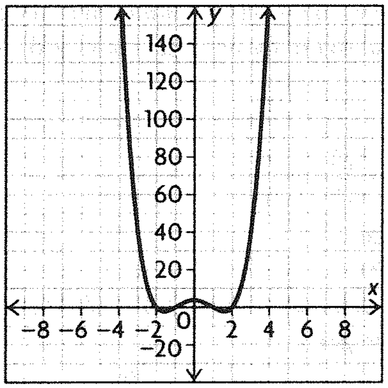
8. a)



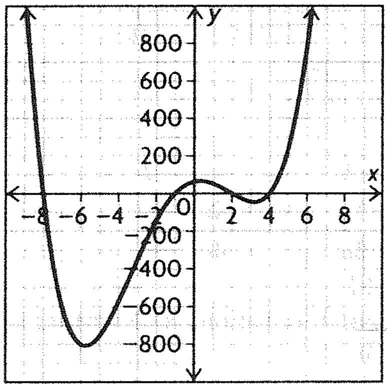
b)



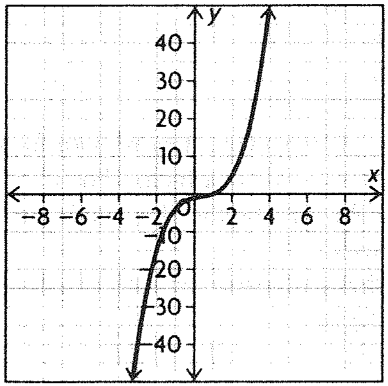
c)



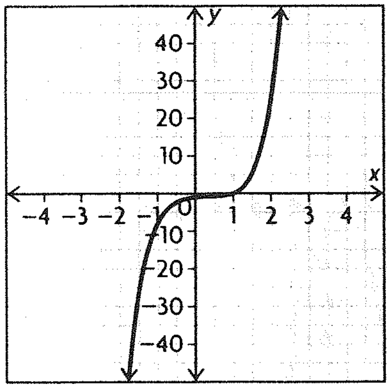
d)



e)



f)



9. Using synthetic division and filling in everything we know or can figure by working backwards, we get:

$$\begin{array}{r|rrrr} \frac{1}{2} & 12 & k & -1 & -6 \\ & \downarrow & 6 & 13 & 6 \\ \hline & 12 & 26 & 12 & 0 \end{array}$$

So, $k + 6 = 26$

$k = 20$

10. Using synthetic division for both divisors and remainders, we get:

$$\begin{array}{r|rrrr} 1 & a & -1 & 2 & b \\ & \downarrow & a & -1 + a & 1 + a \\ \hline & a & -1 + a & 1 + a & 10 \end{array}$$

So, $b + 1 + a = 10$

$a + b = 9$

and

$$\begin{array}{r|rrrr} 2 & a & -1 & 2 & b \\ & \downarrow & 2a & -2 + 4a & 8a \\ \hline & a & -1 + 2a & 4a & 51 \end{array}$$

So, $8a + b = 51$

Solve the system of equations to determine a and b.

$a + b = 9$

$\underline{-(8a + b = 51)}$

$-7a = -42$

$a = 6$

$6 + b = 9$

$b = 3$

11. For $x^n - a^n$, if n is even, they're both factors. If n is odd, only $(x - a)$ is a factor. For $x^n + a^n$, if n is even, neither is a factor. If n is odd, only $(x + a)$ is a factor.

12. Using synthetic division for both divisors and remainders, we get:

$$\begin{array}{r|rrrr} 2 & a & -1 & b & -24 \\ & \downarrow & 2a & -2 + 4a & 24 \\ \hline & a & -1 + 2a & 12 & 0 \end{array}$$

So, $b + (-2 + 4a) = 12$

$4a + b = 14$

and

$$\begin{array}{r|rrrr} -4 & a & -1 & b & -24 \\ & \downarrow & -4a & 4 + 16a & 24 \\ \hline & a & -1 - 4a & -6 & 0 \end{array}$$

So, $4 + 16a + b = -6$

$16a + b = -10$

Solve the system of equations to determine a and b.

$4a + b = 14$

$\underline{-(16a + b = -10)}$

$-12a = 24$

$a = -2$

$4(-2) + b = 14$

$b = 22$

$f(x) = -2x^3 - x^2 + 22x - 24$

Two factors are $x - 2$ and $x + 4$, so $f(x)$ is divisible by $(x - 2)(x + 4) = x^2 + 2x - 8$.

$$\begin{array}{r} -2x + 3 \\ x^2 + 2x - 8\overline{)-2x^3 - x^2 + 22x - 24} \\ \underline{-2x^3 - 4x^2 + 16x} \\ 3x^2 + 6x - 24 \\ \underline{3x^2 + 6x - 24} \end{array}$$

The other factor is $-2x + 3$.

13. Using synthetic division for both divisors and remainders, we get:

$$\begin{array}{r|rrrr} -2 & 1 & 4 & k & -4 \\ & \downarrow & -2 & -4 & -2k + 8 \\ \hline & 1 & 2 & k - 4 & 2r \end{array}$$

So, $-2k + 8 - 4 = 2r$

$-2k - 2r = -4$

and

$$\begin{array}{r|rrrr} 2 & 1 & 4 & k & -4 \\ & \downarrow & 2 & 12 & 2k + 24 \\ \hline & 1 & 6 & k + 12 & r \end{array}$$

So, $2k + 24 - 4 = r$

$2k - r = -20$

Solve the system of equations to determine a and b.

$-2k - 2r = -4$

$\underline{2k - r = -20}$

$-3r = -24$

$r = 8$

$2(k) - 8 = -20$

$k = -6$

14. $x^4 - a^4$

$= (x^2)^2 - (a^2)^2$

$= (x^2 + a^2)(x^2 - a^2)$

$= (x^2 + a^2)(x + a)(x - a)$

$x - a$ is a factor of $x^4 - a^4$.

15. Answers may vary. For example:
If $f(x) = k(x - a)$, then
$f(a) = k(a - a) = k(0) = 0$.

16. $x^2 - x - 2 = (x + 1)(x - 2)$
Check that both of these factors work for
$x^3 - 6x^2 + 3x + 10$ using the factor theorem.
$f(-1) = (-1)^3 - 6(-1)^2 + 3(-1) + 10$
$= -1 - 6 - 3 + 10$
$= 0$
$f(2) = (2)^3 - 6(2)^2 + 3(2) + 10$
$= 8 - 24 + 6 + 10$
$= 0$

17. Let $f(x) = (x + a)^5 + (x + c)^5 + (a - c)^5$.
$x + a$ is a factor of $f(x)$ if and only if $f(-a) = 0$.
$f(-a) = (-a + a)^5 + (-a + c)^5 + (a - c)^5$
$= 0^5 + [-1(a - c)]^5 + (a - c)^5$
$= -(a - c)^5 + (a - c)^5$
$= 0$
So $x + a$ is a factor of
$(x + a)^5 + (x + c)^5 + (a - c)^5$.

3.7 Factoring a Sum or Difference of Cubes, p. 182

1. Let $f(x) = x^3 + b^3$
$f(-b) = 0$, so $(x + b)$ is a factor.
$(x^3 + b^3) \div (x + b) = (x^2 - bx + b^2)$
$x^3 + b^3 = (x + b)(x^2 - bx + b^2)$

2. a) $x^3 - 64$
$= (x)^3 - (4)^3$
$= (x - 4)(x^2 + 4x + 16)$

b) $x^3 - 125$
$= (x)^3 - (5)^3$
$= (x - 5)(x^2 + 5x + 25)$

c) $x^3 + 8$
$= (x)^3 + (2)^3$
$= (x + 2)(x^2 - 2x + 4)$

d) $8x^3 - 27$
$= (2x)^3 - (3)^3$
$= (2x - 3)(4x^2 + 6x + 9)$

e) $64x^3 - 125$
$= (4x)^3 - (5)^3$
$= (4x - 5)(16x^2 + 20x + 25)$

f) $x^3 + 1$
$= (x)^3 + (1)^3$
$= (x + 1)(x^2 - x + 1)$

g) $27x^3 + 8$
$= (3x)^3 + (2)^3$
$= (3x + 2)(9x^2 - 6x + 4)$

h) $1000x^3 + 729$
$= (10x)^3 + (9)^3$
$= (10x + 9)(100x^2 - 90x + 81)$

i) $216x^3 - 8$
$= (6x)^3 - (2)^3$
$= (6x - 2)(36x^2 + 12x + 4)$
$= 8(3x - 1)(9x^2 + 3x + 1)$

3. a) $64x^3 + 27y^3$
$= (4x)^3 + (3y)^3$
$= (4x + 3y)(16x^2 - 12xy + 9y^2)$

b) $-3x^4 + 24x$
$= -3x(x^3 - 8)$
$= -3x((x)^3 - (2)^3)$
$= (-3x)(x - 2)(x^2 + 2x + 4)$

c) $(x + 5)^3 - (2x + 1)^3$
$= ((x + 5) - (2x + 1))$
$\times ((x + 5)^2 + (x + 5)(2x + 1) + (2x + 1)^2)$
$= (4 - x)(x^2 + 10x + 25 + 2x^2 + 11x$
$+ 5 + 4x^2 + 4x + 1)$
$= (4 - x)(7x^2 + 25x + 31)$

d) $x^6 + 64$
$= (x^2)^3 + (4)^3$
$= (x^2 + 4)(x^4 - 4x^2 + 16)$

4. a) $x^3 - 343$
$= (x)^3 - (7)^3$
$= (x - 7)(x^2 + 7x + 49)$

b) $216x^3 - 1$
$= (6x)^3 - (1)^3$
$= (6x - 1)(36x^2 + 6x + 1)$

c) $x^3 + 1000$
$= (x)^3 + (10)^3$
$= (x + 10)(x^2 - 10x + 100)$

d) $125x^3 - 512$
$= (5x)^3 - (8)^3$
$= (5x - 8)(25x^2 + 40x + 64)$

e) $64x^3 - 1331$
$= (4x)^3 - (11)^3$
$= (4x - 11)(16x^2 + 44x + 121)$

f) $343x^3 + 27$
$= (7x)^3 + (3)^3$
$= (7x + 3)(49x^2 - 21x + 9)$

g) $512x^3 + 1$
$= (8x)^3 + (8)^3$
$= (8x + 1)(64x^2 - 8x + 1)$

h) $1331x^3 + 1728$
$= (11x)^3 + (12)^3$
$= (11x + 12)(121x^2 - 132x + 144)$

i) $512 - 1331x^3$
$= (8)^3 - (11x)^3$
$= (8 - 11x)(64 + 88x + 121x^2)$
$= 8(3x - 1)(9x^2 + 3x + 1)$

5. a) $\frac{1}{27}x^3 - \frac{8}{125}$
$= \left(\frac{1}{3}x\right)^3 - \left(\frac{2}{5}\right)^3$
$= \left(\frac{1}{3}x - \frac{2}{5}\right)\left(\frac{1}{9}x^2 + \frac{2}{15}x + \frac{4}{25}\right)$

b) $-432x^5 - 128x^2$
$= (-16x^2)(27x^3 + 8)$
$= (-8x^2)((3x)^3 + (2)^3)$
$= -16x^2(3x + 2)(9x^2 - 6x + 4)$

c) $(x - 3)^3 + (3x - 2)^3$
$= (x - 3 + 3x - 2)$
$\times ((x - 3)^2 - (x - 3)(3x - 2) + (3x - 2)^2)$
$= (4x - 5)((x^2 - 6x + 9) - (3x^2 - 11x + 6)$
$+ (9x^2 - 12x + 4))$
$= (4x - 5)(7x^2 - 7x + 7)$
$= 7(4x - 5)(x^2 - x + 1)$

d) $\frac{1}{512}x^9 - 512$
$= \left(\frac{1}{8}x^3\right)^3 - (8)^3$
$= \left(\frac{1}{8}x^3 - 8\right)\left(\frac{1}{64}x^6 + x^3 + 64\right)$
$= \left(\left(\frac{1}{2}x\right)^3 - (2)^3\right)\left(\frac{1}{64}x^6 + x^3 + 64\right)$
$= \left(\frac{1}{2}x - 2\right)\left(\frac{1}{4}x^2 + x + 4\right)\left(\frac{1}{64}x^6 + x^3 + 64\right)$

6. Agree; by the formulas for factoring the sum and difference of cubes, the numerator of the fraction is equivalent to $(a^3 + b^3) + (a^3 - b^3)$. Since $(a^3 + b^3) + (a^3 - b^3) = 2a^3$, the entire fraction is equal to 1.

7. $1^3 + 12^3$
$= (1 + 12)(1^2 - (1)(12) + 12^2)$
$= (1 + 12)(1 - 12 + 144)$
$= (13)(133)$
$= 1729$
$9^3 + 10^3$
$= (9 + 10)(9^2 - (9)(10) + 10^2)$
$= (9 + 10)(81 - 90 + 100)$
$= (19)(91)$
$= 1729$

8. $x^9 + y^9$
$= x^{18} + 2x^9y^9 + y^{18}$
$= (x^{18} + y^{18}) + 2x^9y^9$
$= (x^6 + y^6)(x^{12} - x^6y^6 + y^{12}) + 2x^9y^9$
$= (x^2 + y^2)(x^4 - x^2y^2 + y^4)(x^{12} - x^6y^6 + y^{12})$
$+ 2x^9y^9$

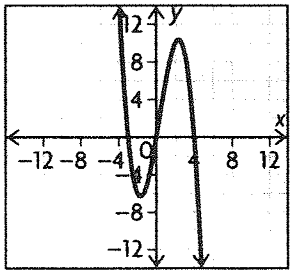
9. Answers may vary. For example, this statement is true because $a^3 - b^3$ is the same as $a^3 + (-b)^3$.

10. a) A taxicab number (TN) is the smallest number that can be expressed as a sum of two positive cubes in *n* distinct ways.

b) Yes;
TN(1) $= 2 = 1^3 + 1^3$
TN(2) $= 1729 = 1^3 + 12^3$
$= 9^3 + 10^3$
TN(3) $= 87\ 539\ 319$
$= 167^3 + 436^3$
$= 228^3 + 423^3$
$= 255^3 + 414^3$
TN(4) $= 6\ 963\ 472\ 309\ 248$
$= 2421^3 + 19\ 083^3$
$= 5436^3 + 18\ 948^3$
$= 10\ 200^3 + 18\ 072^3$
$= 13\ 322^3 + 16\ 630^3$
TN(5) $= 48\ 988\ 659\ 276\ 962\ 496$
$= 38\ 787^3 + 365\ 757^3$
$= 107\ 839^3 + 362\ 753^3$
$= 205\ 292^3 + 342\ 952^3$
$= 221\ 424^3 + 336\ 588^3$
$= 231\ 518^3 + 331\ 954^3$
TN(6) $= 24\ 153\ 319\ 581\ 254\ 312\ 065\ 344$
$= 582\ 162^3 + 28\ 906\ 206^3$
$= 3\ 064\ 173^3 + 28\ 894\ 806^3$
$= 8\ 519\ 281^3 + 28\ 657\ 487^3$
$= 16\ 218\ 068^3 + 27\ 093\ 208^3$
$= 17\ 492\ 496^3 + 26\ 590\ 452^3$
$= 18\ 289\ 922^3 + 26\ 224\ 366^3$

Chapter Review, pp. 184–185

1.



2. Since it is of odd degree and the leading coefficient is positive, the end behaviour is: as $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$.

3. a) There are 2 turning points which means that the degree is 2 + 1 or 3. Based on the end behaviour of the function, which is as $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$, the leading coefficient is positive.

b) There are 3 turning points which means that the degree is 3 + 1 or 4. Based on the end behaviour of the function which is $x \to \pm\infty$, $y \to \infty$ the leading coefficient is positive.

4. a) Answers may vary. For example,
$f(x) = (x + 3)(x - 6)(x - 4)$,
$f(x) = 10(x + 3)(x - 6)(x - 4)$, and
$f(x) = -4(x + 3)(x - 6)(x - 4)$.

b) Answers may vary. For example,
$f(x) = (x - 5)(x + 1)(x + 2)$,
$f(x) = -6(x - 5)(x + 1)(x + 2)$, and
$f(x) = 9(x - 5)(x + 1)(x + 2)$.

c) Answers may vary. For example,
$f(x) = (x + 7)(x - 2)(x - 3)$,
$f(x) = \frac{1}{4}(x + 7)(x - 2)(x - 3)$, and
$f(x) = 3(x + 7)(x - 2)(x - 3)$.

d) Answers may vary. For example,
$f(x) = (x - 9)(x + 5)(x + 4)$,
$f(x) = 7(x - 9)(x + 5)(x + 4)$, and
$f(x) = -\frac{1}{3}(x - 9)(x + 5)(x + 4)$.

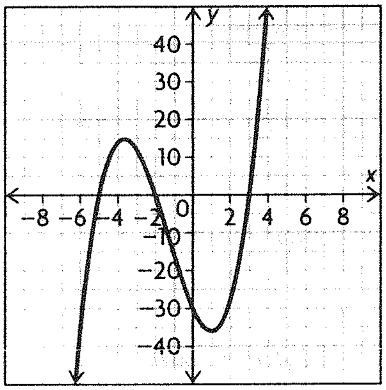
5. a) Answers may vary. For example,
$f(x) = (x + 6)(x - 2)(x - 5)(x - 8)$,
$f(x) = 2(x + 6)(x - 2)(x - 5)(x - 8)$, and
$f(x) = -8(x + 6)(x - 2)(x - 5)(x - 8)$.

b) Answers may vary. For example,
$f(x) = (x - 4)(x + 8)(x - 1)(x - 2)$,
$f(x) = \frac{3}{4}(x - 4)(x + 8)(x - 1)(x - 2)$, and
$f(x) = -12(x - 4)(x + 8)(x - 1)(x - 2)$.

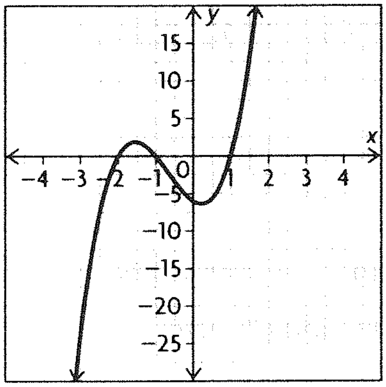
c) Answers may vary. For example,
$f(x) = x(x + 1)(x - 9)(x - 10)$,
$f(x) = 5x(x + 1)(x - 9)(x - 10)$, and
$f(x) = -3x(x + 1)(x - 9)(x - 10)$.

d) Answers may vary. For example,
$f(x) = (x + 3)(x - 3)(x + 6)(x - 6)$,
$f(x) = \frac{2}{5}(x + 3)(x - 3)(x + 6)(x - 6)$, and
$f(x) = -10(x + 3)(x - 3)(x + 6)(x - 6)$.

6. The zeros are 3, -2, and -5. The function is a cubic with a positive leading coefficient (1).



7. $y = a(x - 1)(x + 1)(x + 2)$
$-6 = a(0 - 1)(0 + 1)(0 + 2)$
$-6 = -2a$
$3 = a$
$y = 3(x - 1)(x + 1)(x + 2)$



8. a) $y = x^2$ has been reflected in the x-axis, vertically stretched by a factor of 2, horizontally translated 1 unit to the right, and vertically translated 23 units up.

b) $y = x^2$ has been horizontally stretched by a factor of $\frac{13}{12}$, horizontally translated 9 units to the left, and vertically translated 14 units down.

c) $y = x^2$ has been horizontally translated 4 units to the right.

d) $y = x^2$ has been horizontally translated $\frac{3}{7}$ units to the left.

e) $y = x^2$ has been vertically stretched by a factor of 40, reflected in the y-axis, horizontally compressed by a factor of $\frac{1}{7}$, horizontally translated 10 units to the right, and vertically translated 9 units up.

9. a) $y = 25\left(\frac{6}{5}(x - 3)\right)^3$

Answers may vary. For example:

$x = -2$

$y = 25\left(\frac{6}{5}((-2) - 3)\right)^3 = 25(-6)^3 = -5400$

$x = 3$

$y = 25\left(\frac{6}{5}(3 - 3)\right)^3 = 25(0)^3 = 0$

$x = 8$

$y = 25\left(\frac{6}{5}(8 - 3)\right)^3 = 25(6)^3 = 5400$

$(-2, -5400)$, $(3, 0)$, and $(8, 5400)$

b) $y = -\left(\frac{1}{7}x\right)^3 - 19$

Answers may vary. For example:

$x = -7$

$y = -\left(\frac{1}{7}(-7)\right)^3 - 19 = 1 - 19 = -18$

$x = 0$

$y = -\left(\frac{1}{7}(0)\right)^3 - 19 = 0 - 19 = -19$

$x = 7$

$y = -\left(\frac{1}{7}(7)\right)^3 - 19 = -1 - 19 = -20$

$(-7, -18)$, $(0, -19)$, and $(7, -20)$

c) $y = \frac{6}{11}(-(x + 5))^3 + 16$

Answers may vary. For example:

$x = -6$

$y = \frac{6}{11}(-(-6 + 5))^3 + 16 = \frac{6}{11} + 16 = \frac{182}{11}$

$x = -5$

$y = \frac{6}{11}(-(-5 + 5))^3 + 16 = 0 + 16 = 16$

$x = -4$

$y = \frac{6}{11}(-(-4 + 5))^3 + 16 = -\frac{6}{11} + 16 = \frac{170}{11}$

$\left(-6, \frac{182}{11}\right)$, $(-5, 16)$, and $\left(-4, \frac{170}{11}\right)$

d) $y = 100\left(\frac{1}{2}x\right)^3 + 14$

Answers may vary. For example:

$x = -2$

$y = 100\left(\frac{1}{2}(-2)\right)^3 + 14 = -100 + 14 = -86$

$x = 0$

$y = 100\left(\frac{1}{2}(0)\right)^3 + 14 = 0 + 14 = 14$

$x = 2$

$y = 100\left(\frac{1}{2}(2)\right)^3 + 14 = 100 + 14 = 114$

$(-2, -86)$, $(0, 14)$, and $(2, 114)$

e) $y = -(x)^3 - 45$

Answers may vary. For example:

$x = -1$

$y = -(-1)^3 - 45 = 1 - 45 = -44$

$x = 0$

$y = -(0)^3 - 45 = 0 - 45 = -45$

$x = 1$

$y = -(1)^3 - 45 = -1 - 45 = -46$

$(-1, -44)$, $(0, -45)$, and $(1, -46)$

f) $y = \left(-\frac{10}{7}(x - 12)\right)^3 + 6$

Answers may vary. For example:

$x = 5$

$y = \left(-\frac{10}{7}(5 - 12)\right)^3 + 6 = 1000 + 6$

$x = 12$

$y = \left(-\frac{10}{7}(12 - 12)\right)^3 + 6 = 0 + 6 = 6$

$x = 19$

$y = \left(-\frac{10}{7}(19 - 12)\right)^3 + 6 = -1000 + 6 = -994$

$(5, 1006)$, $(12, 6)$, and $(19, -994)$

10. a)

$$\begin{array}{rl}
 & 2x^2 - 5x + 28 \\
x + 5 & \overline{)2x^3 + 5x^2 + 3x - 4} \\
2x^2(x + 5) \rightarrow & \underline{2x^3 + 10x^2} \quad \downarrow \quad \downarrow \\
 & -5x^2 + 3x \\
-5x(x + 5) \rightarrow & \underline{-5x^2 - 25x} \\
 & 28x - 4 \\
28(x + 5) \rightarrow & \underline{28x + 140} \\
 & -144
\end{array}$$

$2x^2 - 5x + 28$ remainder -144

b)

$$\begin{array}{rl}
 & x^2+4x+5 \\
 & x^2-8\overline{)x^4+4x^3-3x^2-6x-7} \\
x^2(x^2-8)\rightarrow & \underline{x^4 -8x^2} \quad \downarrow \quad \downarrow \\
 & 4x^3+5x^2-6x \\
4x(x^2-8)\rightarrow & \underline{4x^3 -32x} \\
 & 5x^2+26x-7 \\
5(x^2-8)\rightarrow & \underline{5x^2 -40} \\
 & 26x+33
\end{array}$$

$x^2 + 4x + 5$ remainder $26x + 33$

c)

$$\begin{array}{rl}
 & 2x-6 \\
 & x^3+3x^2+3x-3\overline{)2x^4+0x^3-2x^2+3x-16} \\
2x(x^3+3x^2+3x-3)\rightarrow & \underline{2x^4+6x^3+6x^2-6x} \quad \downarrow \\
 & -6x^3-8x^2+9x-16 \\
-6(x^3+3x^2+3x-3)\rightarrow & \underline{-6x^3-18x^2-18x+18} \\
 & 10x^2+27x-34
\end{array}$$

$2x - 6$ remainder $10x^2 + 27x - 34$

d)

$$\begin{array}{rl}
 & x-4 \\
 & x^4+4x^3+4x^2-x-3\overline{)x^5+0x^4-8x^3+0x^2-7x-6} \\
x(x^4+4x^3+4x^2-x-3)\rightarrow & \underline{x^5+4x^4+4x^3-x^2-3x} \quad \downarrow \\
 & -4x^4-12x^3+x^2-4x-6 \\
-4(x^4+4x^3+4x^2-x-3)\rightarrow & \underline{-4x^4-16x^3-16x^2+4x+12} \\
 & 4x^3+17x^2-8x-18
\end{array}$$

$x - 4$ remainder $4x^3 + 17x^2 - 8x - 18$

11. a)

$$\begin{array}{r|rrrr}
-2 & 2 & 5 & -1 & -5 \\
 & \downarrow & -4 & -2 & 6 \\
\hline
 & 2 & 1 & -3 & 1
\end{array}$$

$(x + 2)(2x^2 + x - 3)$ remainder 1

b)

$$\begin{array}{r|rrrr}
-2 & 3 & 13 & 17 & 3 \\
 & \downarrow & -6 & -14 & -6 \\
\hline
 & 3 & 7 & 3 & -3
\end{array}$$

$(x + 2)(3x^2 + 7x + 3)$ remainder -3

c)

$$\begin{array}{r|rrrrr}
-2 & 2 & 5 & -16 & -45 & -18 \\
 & \downarrow & -4 & -2 & 36 & 18 \\
\hline
 & 2 & 1 & -18 & -9 & 0
\end{array}$$

$(x + 2)(2x^3 + x^2 - 18x - 9)$ remainder 0

d)

$$\begin{array}{r|rrrr}
-2 & 2 & 4 & -5 & -4 \\
 & \downarrow & -4 & 0 & 10 \\
\hline
 & 2 & 0 & -5 & 6
\end{array}$$

$(x + 2)(2x^2 - 5)$ remainder 6

12. a) Divisor: $x - 9$; Quotient: $2x^2 + 11x - 8$;
Remainder: 3
$= (x - 9)(2x^2 + 11x - 8) + 3$
$= 2x^3 + 11x^2 - 8x - 18x^2 - 99x + 72 + 3$
$= 2x^3 - 7x^2 - 107x + 75$

b) Divisor: $4x + 3$; Quotient: $x^3 - 2x + 7$;
Remainder: -4
$= (4x + 3)(x^3 - 2x + 7) - 4$
$= 4x^4 - 8x^2 + 28x + 3x^3 - 6x + 21 - 4$
$= 4x^4 + 3x^3 - 8x^2 + 22x + 17$

c) Divisor: $3x - 4$; Quotient: $x^3 + 6x^2 - 6x - 7$;
Remainder: 5
$= (3x - 4)(x^3 + 6x^2 - 6x - 7) + 5$
$= 3x^4 + 18x^3 - 18x^2 - 21x - 4x^3 - 24x^2 + 24x + 28 + 5$
$= 3x^4 + 14x^3 - 42x^2 + 3x + 33$

d) Divisor: $3x^2 + x - 5$;
Quotient: $x^4 - 4x^3 + 9x - 3$; Remainder: $2x - 1$
$= (3x^2 + x - 5)(x^4 - 4x^3 + 9x - 3) + (2x - 1)$
$= (3x^6 - 12x^5 + 27x^3 - 9x^2 + x^5 - 4x^4 + 9x^2 - 3x - 5x^4 + 20x^3 - 45x + 15) + (2x - 1)$
$= 3x^6 - 11x^5 - 9x^4 + 47x^3 - 46x + 14$

13. $f(-2) = (-2)^3 + 2(-2)^2 - 6(-2) + 1$
$= -8 + 8 + 12 + 1$
$= 13$

14. a) $x^3 - 5x^2 - 22x - 16$
$f(-1) = (-1)^3 - 5(-1)^2 - 22(-1) - 16$
$= -1 - 5 + 22 - 16$
$= 0$
$x + 1$ is a factor.

$$\begin{array}{r|rrrr} -1 & 1 & -5 & -22 & -16 \\ & \downarrow & -1 & 6 & 16 \\ \hline & 1 & -6 & -16 & 0 \end{array}$$

$= (x + 1)(x^2 - 6x - 16)$
$= (x + 1)(x - 8)(x + 2)$

b) $2x^3 + x^2 - 27x - 36$
$f(4) = 2(4)^3 + (4)^2 - 27(4) - 36$
$= 128 + 16 - 108 - 36$
$= 0$
$x - 4$ is a factor.

$$\begin{array}{r|rrrr} 4 & 2 & 1 & -27 & -36 \\ & \downarrow & 8 & 36 & 36 \\ \hline & 2 & 9 & 9 & 0 \end{array}$$

$= (x - 4)(2x^2 + 9x + 9)$
$= (x - 4)(2x^2 + 6x + 3x + 9)$
$= (x - 4)(2x(x + 3) + 3(x + 3))$
$= (x - 4)(2x + 3)(x + 3)$

c) $3x^4 - 19x^3 + 38x^2 - 24x$
$x(3x^3 - 19x^2 + 38x - 24)$
For $3x^3 - 19x^2 + 38x - 24$:
$f(2) = 3(2)^3 - 19(2)^2 + 38(2) - 24$
$= 24 + 76 + 76 - 24$
$= 0$
$x - 2$ is a factor.

$$\begin{array}{r|rrrr} 2 & 3 & -19 & 38 & -24 \\ & \downarrow & 6 & -26 & 24 \\ \hline & 3 & -13 & 12 & 0 \end{array}$$

$= x(x - 2)(3x^2 - 13x + 12)$
$= x(x - 2)(3x^2 - 9x - 4x + 12)$
$= x(x - 2)(3x(x - 3) - 4(x - 3))$
$= x(x - 2)(x - 3)(3x - 4)$

d) $x^4 + 11x^3 + 36x^2 + 16x - 64$
$f(1) = 1^4 + 11(1)^3 + 36(1)^2 + 16(1) - 64$
$= 1 + 11 + 36 + 16 - 64$
$= 0$
$x - 1$ is a factor.

$$\begin{array}{r|rrrrr} 1 & 1 & 11 & 36 & 16 & -64 \\ & \downarrow & 1 & 12 & 48 & 64 \\ \hline & 1 & 12 & 48 & 64 & 0 \end{array}$$

$= (x - 1)(x^3 + 12x^2 + 48x + 64)$
For $x^3 + 12x^2 + 48x + 64$:
$f(-4) = (-4)^3 + 12(-4)^2 + 48(-4) + 64$
$= -64 + 192 - 192 + 64$
$= 0$

$$\begin{array}{r|rrrr} -4 & 1 & 12 & 48 & 64 \\ & \downarrow & -4 & -32 & -64 \\ \hline & 1 & 8 & 16 & 0 \end{array}$$

$= (x - 1)(x + 4)(x^2 + 8x + 16)$
$= (x - 1)(x + 4)(x + 4)(x + 4)$

15. a) $8x^3 - 10x^2 - 17x + 10$
$f(2) = 8(2)^3 - 10(2)^2 - 17(2) + 10$
$= 64 - 40 - 34 + 10$
$= 0$
$x - 2$ is a factor.

$$\begin{array}{r|rrrr} 2 & 8 & -10 & -17 & 10 \\ & \downarrow & 16 & 12 & -10 \\ \hline & 8 & 6 & -5 & 0 \end{array}$$

$= (x - 2)(8x^2 + 6x - 5)$
$= (x - 2)(8x^2 + 10x - 4x - 5)$
$= (x - 2)(2x(4x + 5) - 1(4x + 5))$
$= (x - 2)(4x + 5)(2x - 1)$

b) $2x^3 + 7x^2 - 7x - 30$
$f(2) = 2(2)^3 + 7(2)^2 - 7(2) - 30$
$= 16 + 28 - 14 - 30$
$= 0$
$x - 2$ is a factor.

$$\begin{array}{r|rrrr} 2 & 2 & 7 & -7 & -30 \\ & \downarrow & 4 & 22 & 30 \\ \hline & 2 & 11 & 15 & 0 \end{array}$$

$= (x - 2)(2x^2 + 11x + 15)$
$= (x - 2)(2x^2 + 6x + 5x + 15)$
$= (x - 2)(2x(x + 3) + 5(x + 3))$
$= (2x + 5)(x - 2)(x + 3)$

c) $x^4 - 7x^3 + 9x^2 + 27x - 54$
$f(3) = 3^4 - 7(3)^3 + 9(3)^2 + 27(3) - 54$
$= 81 - 189 + 81 + 81 - 54$
$= 0$
$x - 3$ is a factor.

$$\begin{array}{r|rrrrr} 3 & 1 & -7 & 9 & 27 & -54 \\ & \downarrow & 3 & -12 & -9 & 54 \\ \hline & 1 & -4 & -3 & 18 & 0 \end{array}$$

$= (x - 3)(x^3 - 4x^2 - 3x + 18)$
For $x^3 - 4x^2 - 3x + 18$:
$f(-2) = (-2)^3 - 4(-2)^2 - 3(-2) + 18$
$= -8 - 16 + 6 + 18$
$= 0$
$x + 2$ is a factor.

$$\begin{array}{r|rrrr} -2 & 1 & -4 & -3 & 18 \\ & \downarrow & -2 & 12 & -18 \\ \hline & 1 & -6 & 9 & 0 \end{array}$$

$= (x - 3)(x + 2)(x^2 - 6x + 9)$
$= (x - 3)(x - 3)(x - 3)(x + 2)$

d) $4x^4 + 4x^3 - 35x^2 - 36x - 9$
$f(3) = 4(3)^4 + 4(3)^3 - 35(3)^2 - 36(3) - 9$
$= 324 + 108 - 315 - 108 - 9$
$= 0$
$x - 3$ is a factor.

$$\begin{array}{r|rrrrr} 3 & 4 & 4 & -35 & -36 & -9 \\ & \downarrow & 12 & 48 & 39 & 9 \\ \hline & 4 & 16 & 13 & 3 & 0 \end{array}$$

$= (x - 3)(4x^3 + 16x^2 + 13x + 3)$
For $4x^3 + 16x^2 + 13x + 3$:
$f(-3) = 4(-3)^3 + 16(-3)^2 + 13(-3) + 3$
$= -108 + 144 - 39 + 3$
$= 0$
$x + 3$ is a factor.

$$\begin{array}{r|rrrr} -3 & 4 & 16 & 13 & 3 \\ & \downarrow & -12 & -12 & -3 \\ \hline & 4 & 4 & 1 & 0 \end{array}$$

$= (x - 3)(x + 3)(4x^2 + 4x + 1)$
$= (x - 3)(x + 3)(4x^2 + 2x + 2x + 1)$
$= (x - 3)(x + 3)(2x(2x + 1) + 1(2x + 1))$
$= (2x + 1)(2x + 1)(x - 3)(x + 3)$

16. a) $64x^3 - 27$
$= (4x)^3 - (3)^3$
$= (4x - 3)(16x^2 + 12x + 9)$

b) $512x^3 - 125$
$= (8x)^3 - (5)^3$
$= (8x - 5)(64x^2 + 40x + 25)$

c) $343x^3 - 1728$
$= (7x)^3 - (12)^3$
$= (7x - 12)(49x^2 + 84x + 144)$

d) $1331x^3 - 1$
$= (11x)^3 - (1)^3$
$= (11x - 1)(121x^2 + 11x + 1)$

17. a) $1000x^3 + 343$
$= (10x)^3 + (7)^3$
$= (10x + 7)(100x^2 - 70x + 49)$

b) $1728x^3 + 125$
$= (12x)^3 + (5)^3$
$= (12x + 5)(144x^2 - 60x + 25)$

c) $27x^3 + 1331$
$= (3x)^3 + (11)^3$
$= (3x + 11)(9x^2 - 33x + 121)$

d) $216x^3 + 2197$
$= (6x)^3 + (13)^3$
$= (6x + 13)(36x^2 - 78x + 169)$

18. a) $(x^6 - y^6) = (x^3 - y^3)(x^3 + y^3)$
$= (x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2)$

b) $(x^6 - y^6) = (x^2 - y^2)(x^4 + x^2y^2 + y^4)$
$= (x - y)(x + y)(x^4 + x^2y^2 + y^4)$

c) Both methods produce factors of $(x - y)$ and $(x + y)$; however, the other factors are different. Since the two factorizations must be equal to each other, this means that $(x^4 + x^2y^2 + y^4)$ must be equal to $(x^2 + xy + y^2)(x^2 - xy + y^2)$.

Chapter Self-Test, p. 186

1. a) $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers and n is a whole number. The degree of the function is n; the leading coefficient is a_n.
$y = x^4 - 2x^3 + x^2 - 2x + 8.$
The degree of this function is 4, and the leading coefficient is 1.

b) The maximum number of turning points is $n - 1$.

c) The function may have at most n zeros (the same as the degree).

d) If the least number of zeros is one, it is an odd degree function.

e) The function is an even degree function with a negative leading coefficient.

2. $y = a(x + 4)(x + 2)(x - 2)$
$-16 = a(0 + 4)(0 + 2)(0 - 2)$
$-16 = a(4)(2)(-2)$
$-16 = -16a$
$1 = a$
$y = (x + 4)(x + 2)(x - 2)$

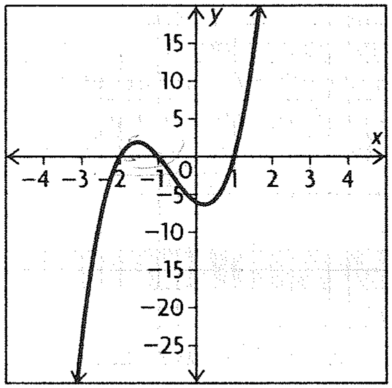
3. a) $2x^3 - x^2 - 145x - 72$
$f(9) = 2(9)^3 - (9)^2 - 145(9) - 72$
$= 1458 - 81 - 1305 - 72$
$= 0$
$x - 9$ is a factor.

$$\begin{array}{r|rrrr} 9 & 2 & -1 & -145 & -72 \\ & \downarrow & 18 & 153 & 72 \\ \hline & 2 & 17 & 8 & 0 \end{array}$$

$= (x - 9)(2x^2 + 17x + 8)$
$= (x - 9)(2x^2 + 16x + 1x + 8)$
$= (x - 9)(2x(x + 8) + 1(x + 8))$
$= (x - 9)(x + 8)(2x - 1)$

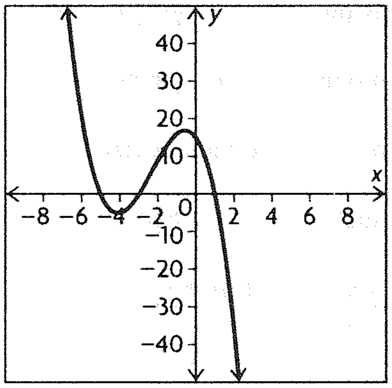
b) $(x-7)^3+(2x+3)^3$
$= (x-7+2x+3)((x-7)^2-(x-7)$
$(2x+3)+(2x+3)^2)$
$= (3x-4)((x^2-14x+49)-$
$(2x^2-11x-21)+(4x^2+12x+9))$
$= (3x-4)(3x^2+9x+79)$

4.



By shifting the original function up, the new function has more zeros.

5.



x is below the x-axis in the intervals of $-5 < x < -3; x > 1$.

6.
$$\begin{array}{c|cccc} \frac{1}{2} & 6 & 1 & -12 & 5 \\ & \downarrow & 3 & 2 & -5 \\ \hline & 6 & 4 & -10 & 0 \end{array}$$

$2x-1$ is a factor because there is no remainder.

7. a) $y = 5(2(x-2))^3+4$

b) $\left(\frac{x}{k}+d, ay+c\right)$

$= \left(\frac{1}{2}+2, (5)(1)+4\right) = (2.5, 9)$

8.
$$\begin{array}{rr} & x+5 \\ & x^3-2x^2+x-5\overline{)x^4+3x^3-9x^2+0x+6} \\ x(x^3-2x^2+x-5)\rightarrow & \underline{x^4-2x^3-x^2-5x\quad\downarrow} \\ & 5x^3-10x^2+5x+6 \\ 5(x^3-2x^2+x-5)\rightarrow & \underline{5x^3-10x^2+5x-25} \\ & 31 \end{array}$$

Julie divided by $x+5$.

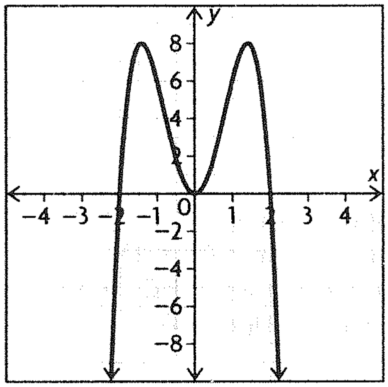
9. To determine a, use synthetic division and work backwards.

$$\begin{array}{c|ccccc} 2 & a & 0 & 8 & 0 & 0 \\ & \downarrow & -4 & -8 & 0 & 0 \\ \hline & -2 & -4 & 0 & 0 & 0 \end{array}$$

$a = -2$

$f(x) = -2x^4+8x^2$
$= -2x^2(x^2-4)$
$= -2x^2(x+2)(x-2)$

The zeros are located at 0, −2, and 2.



Chapters 1–3 Cumulative Review, pp. 188–191

1. The domain is all real numbers except 5, since 5 would make the denominator 0.
So, the domain is $\{x \in \mathbf{R} | x \neq 5\}$.
The correct answer is **b)**.

2. $x = -y^2$ is not a function of x because the points $(-1, 1)$ and $(-1, -1)$ are both in the relation.
The correct answer is **a)**.

3. The parent graph is $f(x) = |x|$. This function is horizontally compressed by a factor of $\frac{1}{2}$ and then translated 2 units to the right.
The new function after the transformations is $f(x) = |2(x-2)|$ or $|2x-4|$.
The correct answer is **c)**.

4. $f(-x) = (-x - 2)(-x + 2)$
$= -1(x + 2)(-1)(x - 2)$
$= (x + 2)(x - 2)$

$f(-x) = f(x)$, so the function is even.
The correct answer is **b**).

5. The function was horizontally stretched by a factor of 3 and then translated 2 units to the right.
The correct answer is **b**).

6. The parent function is $g(x) = \frac{1}{x}$
So, the correct answer is **d**).

7. The horizontal stretch transforms the function to $2^{\frac{1}{5}x}$, and the translation 3 units down transforms the function to $2^{\frac{1}{5}x} - 3$.
The correct answer is **d**).

8. $y = 2x^2 + 5$
Determine the inverse.

$$x = 2y^2 + 5$$
$$x - 5 = 2y^2$$
$$\frac{x - 5}{2} = y^2$$
$$\pm\sqrt{\frac{x - 5}{2}} = y$$

So, the domain is $\{x \in \mathbf{R} | x \geq 5\}$
The correct answer is **a**).

9. $f(x) = 2x^2 - 4$

$$y = 2x^2 - 4$$
$$x = 2y^2 - 4$$
$$x + 4 = 2y^2$$
$$\frac{x + 4}{2} = y^2$$
$$\pm\sqrt{\frac{x + 4}{2}} = y$$

The correct answer is **c**).

10. Look at the endpoints to determine which function is continuous.
For the first function, there is an open circle at (1, 2) and a closed circle at (1, 2).
For the second function, there is an closed circle at (2, 0) and a open circle at (2, 0).
For the third function, there is an open circle at (−1, −1) and a closed circle at (−1, −1).
For the fourth function, there is an open circle at (2, 5) and a closed circle at (2, 4).
So, the fourth function is not continuous.
The correct answer is **d**).

11. $f(-1) = (-1)^3 - 2(-1)^2 + 7$
$= -1 - 2 + 7$
$= 4$

So, the ordered pair is (−1, 4).

$f(3) = (3)^3 - 2(3)^2 + 7$
$= 27 - 18 + 7$
$= 16$

The ordered pair is (3, 16).
So, the rate of change is
$(16 - 4) \div (3 - (-1)) = 12 \div 4$ or 3.
The correct answer is **a**).

12. Kristin's rate of change: $(5 - 0.1) \div 3 \doteq 1.63$
Husain's rate of change: $(15 - 0.1) \div 10 = 1.49$
Kristin's grew faster.
The correct answer is **a**).

13. rate of change:
$(27.015\,002 - 27) \div 0.001 = 15.002$
This is the average rate of change over 0.001 s, so the best estimate for the instantaneous rate of change is 15 m/s.
The correct answer is **c**).

14. $f(-1) = 2^{-1} - 2(-1) + 1$
$= 0.5 + 2 + 1$
$= 3.5$

$f(-1.0001) = 2^{-1.0001} - 2(-1.0001) + 1$
$\doteq 3.500\,165$

The rate of change is
$(3.500\,165 - 3.5) \div -0.0001 = -1.65$.
The correct answer is **d**).

15. Draw the tangent line and find the slope of the line. The slope appears to be about −2.
The correct answer is **c**).

16. The slope of the first part should be greater than the slope of the second part. The slope of the last part should be the greatest. None of the slopes should be 0.
The correct answer is **c**).

17. $\frac{-b}{2a} = \frac{-13}{2(-1.3)}$
$= \frac{-13}{-2.6}$
$= 5$

The coefficient of x^2 is negative, so the graph opens downward. The point is a maximum.
The correct answer is **a**).

18. $\frac{f(3 + h) - f(3)}{h}$

$= \frac{2(3 + h)^2 - 3(3 + h) + 9 - (2(3)^2 - 3(3) + 9)}{h}$

$= \frac{2((3)^2 + 2(3)h + h^2) - 3(3) - 3h + 9 - 2(3)^2 + 3(3) - 9}{h}$

$= \frac{2(9 + 6h + h^2) - 9 - 3h + 9 - 18 + 9 - 9}{h}$

$= \frac{12h + 2h^2 - 3h}{h}$

$= 9 + 2h$

The correct answer is **d)**.

19. The maximum will be at $y = 35(1.7)^8$ or 2 441.5.
The correct answer is **b)**.

20. The function $f(x) = 2^x - 3$ is not a polynomial function.
The correct answer is **c)**.

21. A cubic function cannot be represented by the graph because the graph does not show the same end behaviour as a cubic, for which the end behaviours are opposite.
The correct answer is **b)**.

22. In a cubic function, as $x \rightarrow \pm\infty$, the signs of y are opposite.
The correct answer is **b)**.

23. The function shows a cubic function that has been translated 3 units to the right, so the best match is choice **b)**.

24. $f(x) = a(x + 2)x(x - 1)(x - 3)$

$16 = a(2 + 2)(2)(2 - 1)(2 - 3)$

$16 = a(4)(2)(1)(-1)$

$16 = -8a$

$-2 = a$

So, the equation is $-2(x + 2)x(x - 1)(x - 3)$.

$-2(x + 2)x(x - 1)(x - 3)$

$= -2(x^2 + 2x)(x^2 - 4x + 3)$

$= -2(x^4 - 4x^3 + 3x^2 + 2x^3 - 8x^2 + 6x)$

$= -2x^4 + 4x^3 + 10x^2 - 12x$

The correct answer is **a)**.

25. Since the function is stretched horizontally by a factor of 2, $y = \left(\frac{1}{2}x\right)^3$. Since the function is translated 3 units to the right, the new function is $y = \left(\frac{1}{2}(x - 3)\right)^3$. The correct answer is **c)**.

26.

$$
\begin{array}{rr}
 & x + 1 \\
x^2 - 3x + 4 & \overline{)x^3 - 2x^2 + 7x + 12} \\
x(x^2 - 3x + 4) \rightarrow & \underline{x^3 - 3x^2 + 4x \quad \downarrow} \\
 & x^2 + 3x + 12 \\
1(x^2 - 3x + 4) \rightarrow & \underline{x^2 - 3x + 4} \\
 & 6x + 8
\end{array}
$$

The remainder is $6x + 8$.
The correct answer is **c)**.

27. By the remainder theorem, when $x^4 - 5x^2 + 12x + 16$ is divided by $x + 3$ or $x - (-3)$, the remainder is the value of the polynomial evaluated at -3.

$(-3)^4 - 5(-3)^2 + 12(-3) + 16$

$= 81 - 5(9) - 36 + 16$

$= 81 - 45 - 36 + 16$

$= 16$

The remainder is 16.
The correct answer is **d)**.

28. Because $x - 3$ is a factor of the polynomial, there is no remainder when the polynomial is divided by $x - 3$.
Show the synthetic division and fill in what is known. Work backward from the 0 remainder.

$$
\begin{array}{c|cccc}
3 & 2 & k & -3 & 18 \\
 & \downarrow & 6 & -3 & -18 \\
\hline
 & 2 & -1 & -6 & 0
\end{array}
$$

$k + 6 = -1$

$k = -7$

The correct answer is **b)**.

29. $27x^3 - 216 = 27(x^3 - 8)$

$= 27(x - 2)(x^2 + 2x + 4)$

The correct answer is **c)**.

30. $(x + 3)^3 + 8 = (x + 3)^3 + 2^3$

$= (x + 3 + 2)((x + 3)^2 - 2(x + 3) + 4)$

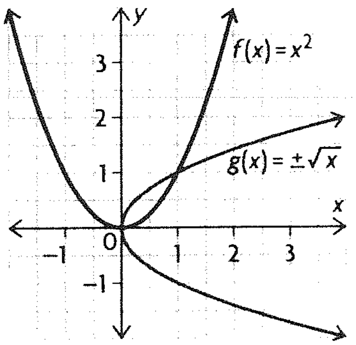
$= (x + 5)(x^2 + 6x + 9 - 2x - 6 + 4)$

$= (x + 5)(x^2 + 4x + 7)$

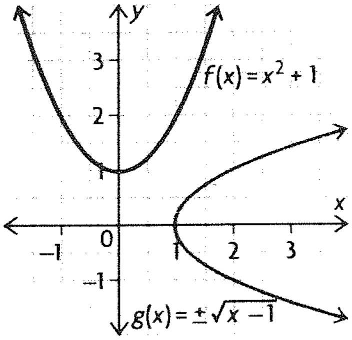
The correct answer is **c)**.

31. The graph pictured in choice c represents the height of the water. The height will change slowly at first, then speed up, then become constant.
The correct answer is **c)**.

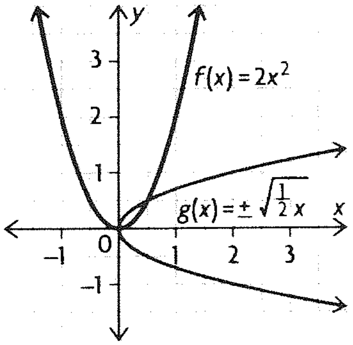
32. a)



b) Answers may vary. For example, vertical translation up produces horizontal translation of the inverse to the right.



Vertical stretch produces horizontal stretch of inverse.



c) Answers may vary. For example, if the vertex of the inverse is (a, b), restrict the value of y to either $y \geq b$ or $y \leq b$.

33. Answers may vary. For example, average rates of change vary between -2 and 4, depending on the interval; instantaneous rates of change are 9 at (0, 1), 0 at (1, 5), -3 at (2, 3), 0 at (3, 1), 9 at (4, 5); instantaneous rate of change is 0 at maximum (1, 5) and at minimum (3, 1).

34. a) $-24 = k(1 + 1)^2(1 - 2)(1 - 4)$

$$-24 = k(2)^2(-1)(-3)$$
$$-24 = 12k$$
$$-2 = k$$

So, the equation is

$f(x) = -2(x + 1)^2(x - 2)(x - 4)$.

b) $(3, p)$ is a point on the graph of $f(x)$

$f(x) = -2(x + 1)^2(x - 2)(x - 4)$

$$p = -2(4)^2(1)(-1)$$
$$p = 32$$

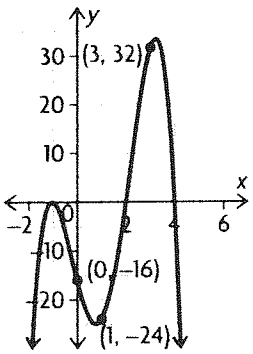
(3, 32) is a point on the graph of $f(x)$.

c) Quartic with the same end behaviours; $k < 0$, so as $x \to \pm\infty$, $f(x) \to -\infty$; from the given factors, the zeros are -1, 2, and 4

d) $f(0) = -2(0 + 1)^2(0 - 2)(0 - 4)$

$$= -2(1)^2(-2)(-4)$$
$$= -16$$

e) $f(x)=k(x+1)^2(x-2)(x-4)$



CHAPTER 4

Polynomial Equations and Inequalities

Getting Started, pp. 194–195

1. a) $5x - 7 = -3x + 17$

$5x + 3x = 17 + 7$

$8x = 24$

$x = 3$

b) $12x - 9 - 6x = 5 + 3x + 1$

$12x - 6x - 3x = 5 + 1 + 9$

$3x = 15$

$x = 5$

c) $2(3x - 5) = -4(3x - 2)$

$6x - 10 = -12x + 8$

$6x + 12x = 8 + 10$

$18x = 18$

$x = 1$

d) $\dfrac{2x + 5}{3} = 7 - \dfrac{x}{4}$

$12\left(\dfrac{2x + 5}{3} = 7 - \dfrac{x}{4}\right)$

$4(2x + 5) = 84 - 3x$

$8x + 20 = 84 - 3x$

$8x + 3x = 84 - 20$

$11x = 64$

$x = \dfrac{64}{11}$

2. a) $x^3 + x^2 - 30x = x(x^2 + x - 30)$

$= x(x + 6)(x - 5)$

b) $x^3 - 64 = (x - 4)(x^2 + 4x + 16)$

c) $24x^4 + 81x = 3x(8x^3 + 27)$

$= 3x(2x + 3)(4x^2 - 6x + 9)$

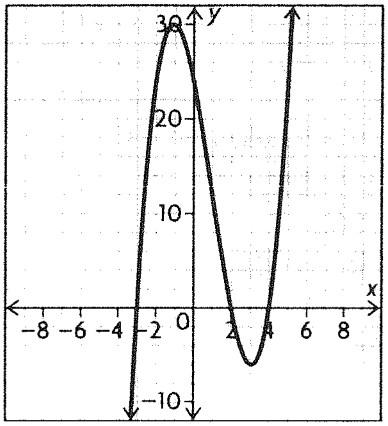
d) $2x^3 + 7x^2 - 18x - 63$

$= x^2(2x + 7) - 9(2x + 7)$

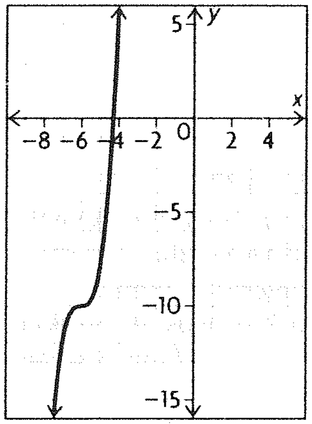
$= (x^2 - 9)(2x + 7)$

$= (x + 3)(x - 3)(2x + 7)$

3. a)



b)



4. The graph crosses the x-axis at $x = 2$ and at $x = 5$. The roots of the equation are 2 and 5.

5. a) $2x^2 = 18$

$x^2 = 9$

$x = \pm\sqrt{9}$

$x = \pm 3$

The roots are 3 and −3.

b) $(x^2 + 8x - 20) = 0$

$(x + 10)(x - 2) = 0$

$(x + 10) = 0$ and $(x - 2) = 0$

$x = -10, 2$

The roots are −10 and 2.

c) $6x^2 = 11x + 10$

$6x^2 - 11x - 10 = 0$

$6x^2 - 15x + 4x - 10 = 0$

$3x(2x - 5) + 2(2x - 5) = 0$

$(3x + 2)(2x - 5) = 0$

$(3x + 2) = 0$ and $(2x - 5) = 0$

$$x = -\frac{2}{3}, \frac{5}{2}$$

The roots are $-\frac{2}{3}$ and $\frac{5}{2}$.

d) $$x(x + 3) = 3 - 5x - x^2$$
$$x^2 + 3x = 3 - 5x - x^2$$
$$2x^2 + 8x - 3 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 2$, $b = 8$, and $c = -3$

$$x = \frac{-8 \pm \sqrt{8^2 - 4(2)(-3)}}{2(2)}$$
$$= \frac{-8}{4} \pm \frac{\sqrt{64 - (-24)}}{4}$$
$$= -2 \pm \frac{\sqrt{88}}{4}$$
$$= -2 \pm \frac{2\sqrt{22}}{4}$$
$$= -2 \pm \frac{\sqrt{22}}{2}$$
$$\doteq 0.3452, -4.345$$

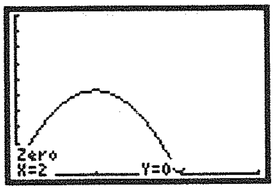
The roots are 0.3452, −4.345.

6. a) (3, 7); Answers may vary. For example, the change in distance over time from $t = 3$ to $t = 7$ is greater than at other intervals of time.

b) From 0 seconds to 3 seconds, she walks a total of 1 metre for a rate of $\frac{1}{3}$ m/s. From 3 seconds to 7 seconds, she walks a total of 3 metres for a rate of $\frac{3}{4}$ m/s.

c) Answers may vary. For example, away; Erika's displacement, or distance from the sensor, is increasing.

7. a) Determine the value of t for which $h(t) = 0$. Use a graphing calculator.



The ball is in the air for 2 s.

b) $$h(0) = 0.5$$
$$h(1) = -5 + 9.75 + 0.5$$
$$= 5.25$$
$$\frac{5.25 - 0.5}{1 - 0} = 4.75 \text{ m/s}$$

c) Estimate the instantaneous rate of change by calculating the average rate of change from $t = 1.999$ s to $t = 2$ s.

$$\frac{h(2) - h(1.999)}{2 - 1.999} = \frac{0 - 0.010\,245}{2 - 1.999}$$
$$= \frac{-0.010\,245}{0.001}$$
$$= -10.245 \text{ m/s}$$

8. Answers given for justfication may vary. Sample answers are provided.

	Statement	**Agree**	**Disagree**	**Justification**
a)	The quadratic formula can only be used when solving a quadratic equation.		x	You could use the quadratic formula to solve $y = x^3 + 4x^2 + 3x$ because it equals $x(x^2 + 4x + 3)$.
b)	Cubic equations always have three real roots.		x	$y = (x + 3)^2(x - 2)$ is a cubic equation that will have two roots.
c)	The graph of a cubic function always passes through all four quadrants.		x	The equation $y = x^3$ will only pass through two quadrants.
d)	The graphs of all polynomial functions must pass through at least two quadrants.	x		All polynomials are continuous and all polynomials have a y-intercept.
e)	The expression $x^2 > 4$ is only true if $x > 2$.		x	$f(-3) = 9$
f)	If you know the instantaneous rates of change for a function at $x = 2$ and $x = 3$, you can predict fairly well what the function looks like in between.	x		The instantaneous rates of change will tell you whether the graph is increasing, decreasing, or not changing at those points.

4.1 Solving Polynomial Equations, pp. 204–206

1. a) $y = 2x(x - 1)(x + 2)(x - 2)$

$2x = 0$ and $x - 1 = 0$ and

$x + 2 = 0$ and $x - 2 = 0$

$x = 0, 1, -2, 2$

b) $y = 5(2x + 3)(4x - 5)(x + 7)$

$2x + 3 = 0$ and $4x - 5 = 0$ and $x + 7 = 0$

$x = -\frac{3}{2}, \frac{5}{4}, -7$

c) $y = 2(x - 3)^2(x + 5)(x - 4)$

$x - 3 = 0$ and $x + 5 = 0$ and $x - 4 = 0$

$x = 3, -5, 4$

d) $y = (x + 6)^3(2x - 5)$

$x + 6 = 0$ and $2x - 5 = 0$

$x = -6, \frac{5}{2}$

e) $y = -5x(x^2 - 9)$

$-5x = 0$ and $x^2 - 9 = 0$

$x = 0$ and $x^2 = 9$

$x = 0$ and $x = \pm 3$

$x = 0, -3, 3$

f) $y = (x + 5)(x^2 - 4x - 12)$

$y = (x + 5)(x + 2)(x - 6)$

$x + 5 = 0$ and $x + 2 = 0$ and $x - 6 = 0$

$x = -5, -2, 6$

2. a) $3x^3 = 27x$

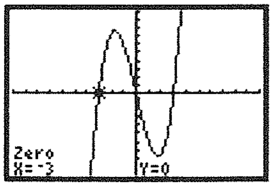
$3x^3 - 27x = 0$

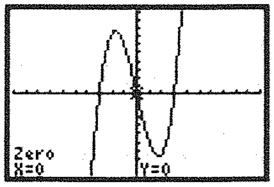
$3x(x^2 - 9) = 0$

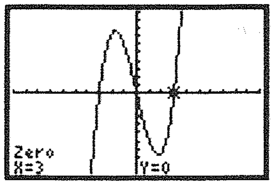
$3x(x + 3)(x - 3) = 0$

$3x = 0$ and $x + 3 = 0$ and $x - 3 = 0$

$x = 0, -3, 3$







b) $4x^4 = 24x^2 + 108$

$4x^4 - 24x^2 - 108 = 0$

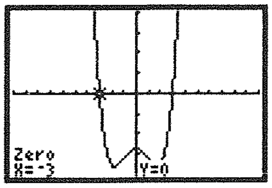
$x^4 - 6x^2 - 27 = 0$

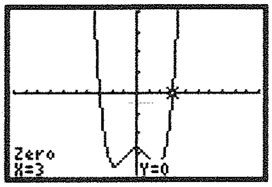
$(x^2 - 9)(x^2 + 3) = 0$

$x^2 - 9 = 0$ and $x^2 + 3 = 0$

$x^2 = 9$ and $x^2 = -3$

$x = \pm 3$ no real solutions





c) $3x^4 + 5x^3 - 12x^2 - 20x = 0$

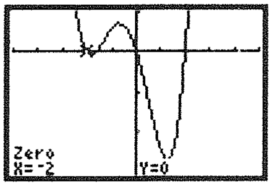
$x(3x^3 + 5x^2 - 12x - 20) = 0$

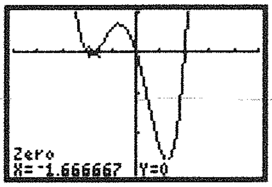
$x(x^2(3x + 5) - 4(3x + 5)) = 0$

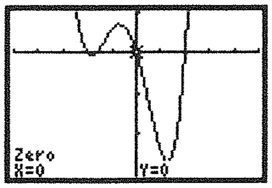
$x(x^2 - 4)(3x + 5) = 0$

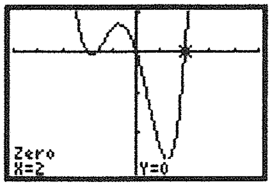
$x = 0$ and $x^2 - 4 = 0$ and $3x + 5 = 0$

$x = 0, 2, -2, -\frac{5}{3}$



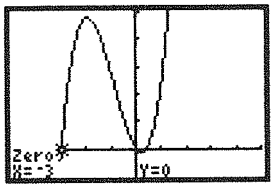


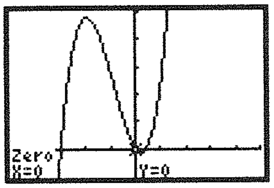


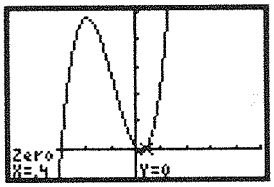


d)
$$\begin{aligned} 10x^3 + 26x^2 - 12x &= 0 \\ 2x(5x^2 + 13x - 6) &= 0 \\ 2x(5x^2 + 15x - 2x - 6) &= 0 \\ 2x(5x(x + 3) - 2(x + 3)) &= 0 \\ 2x(5x - 2)(x + 3) &= 0 \end{aligned}$$
$2x = 0$ and $5x - 2 = 0$ and $x + 3 = 0$

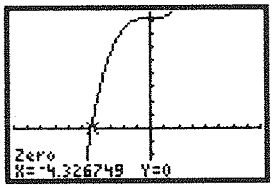
$x = 0, \frac{2}{5}, -3$







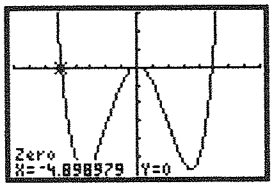
e) $2x^3 + 162 = 0$
$$\begin{aligned} 2x^3 &= -162 \\ x^3 &= -81 \\ \sqrt[3]{x^3} &= \sqrt[3]{-81} \\ x &= -3\sqrt[3]{3} \end{aligned}$$

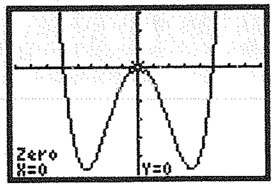


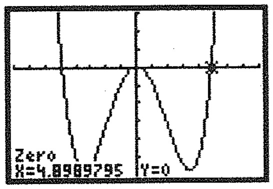
f)
$$\begin{aligned} 2x^4 &= 48x^2 \\ 2x^4 - 48x^2 &= 0 \\ 2x^2(x^2 - 24) &= 0 \end{aligned}$$
$2x^2 = 0$ and $x^2 - 24 = 0$

$x^2 = 0$ and $x^2 = 24$

$x = 0, \pm 2\sqrt{6}$







3. a) $y = 2x^3 - 17x^2 + 23x + 42$

$f(1) = 50$

$f(6) = 0$

$$\begin{array}{r|rrrr} 6 & 2 & -17 & 23 & 42 \\ & \downarrow & 12 & -30 & -42 \\ \hline & 2 & -5 & -7 & 0 \end{array}$$

$$\begin{aligned} (x - 6)(2x^2 - 5x - 7) &= 0 \\ (x - 6)(2x^2 - 7x + 2x - 7) &= 0 \\ (x - 6)(x(2x - 7) + 1(2x - 7)) &= 0 \\ (x - 6)(x + 1)(2x - 7) &= 0 \end{aligned}$$
$x - 6 = 0$ and $x + 1 = 0$ and $2x - 7 = 0$

$x = 6, -1, \frac{7}{2}$

b) From part a) $2x^3 - 17x^2 + 23x + 42 = 0$ or $(x - 6)(x + 1)(2x - 7) = 0$. The roots of either of these equations are the zeros of the function in part a).

4. Algebraically: Collect all terms on one side of the equation and solve by factoring.

$x^3 + 12x^2 + 21x - 4 = x^4 - 2x^3 - 13x^2 - 4$

$0 = x^4 - 3x^3 - 25x^2 - 21x$

$x(x^3 - 3x^2 - 25x - 21) = 0$

$x = 0$

$x^3 - 3x^2 - 25x - 21 = 0$

$f(-1) = 0$

$$\begin{array}{r|rrrr} -1 & 1 & -3 & -25 & -21 \\ & \downarrow & -1 & 4 & 21 \\ \hline & 1 & -4 & -21 & 0 \end{array}$$

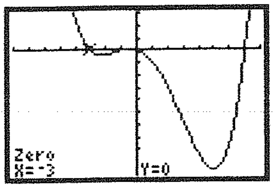
$(x + 1)(x^2 - 4x - 21) = 0$

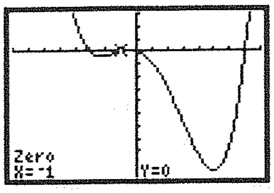
$(x + 1)(x + 3)(x - 7) = 0$

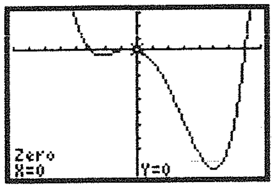
$x + 1 = 0$ and $x + 3 = 0$ and $x - 7 = 0$

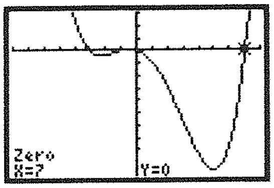
$x = -1, -3, 7, 0$

Graphically: Collect all terms on one side of the equation. Graph the resulting expression and use the calculator to determine the zeros.









$x = -3, -1, 0, 7$

5. $f(x) = 2x^4 - 11x^3 - 37x^2 + 156x$

$x(2x^3 - 11x^2 - 37x + 156) = 0$

$x = 0$

$2x^3 - 11x^2 - 37x + 156 = 0$

$f(1) = 110$

$f(2) = 54$

$f(3) = 0$

$$\begin{array}{r|rrrr} 3 & 2 & -11 & -37 & 156 \\ & \downarrow & 6 & -15 & -156 \\ \hline & 2 & -5 & -52 & 0 \end{array}$$

$(x - 3)(2x^2 - 5x - 52) = 0$

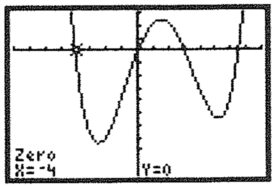
$(x - 3)(2x^2 - 13x + 8x - 52) = 0$

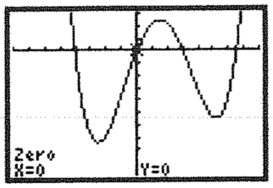
$(x - 3)(x(2x - 13) + 4(2x - 13)) = 0$

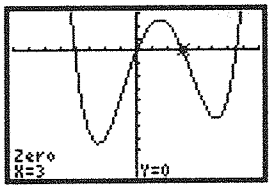
$(x - 3)(x + 4)(2x - 13) = 0$

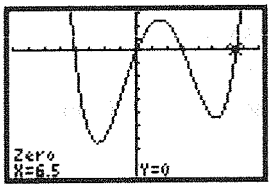
$x - 3 = 0$ and $x + 4 = 0$ and $2x - 13 = 0$

$x = 0, 3, -4, \frac{13}{2}$









6. a) $f(x) = x(x - 2)^2(x + 5)$

$x = 0$ and $(x - 2)^2 = 0$ and $x + 5 = 0$

$x = 0, 2, -5$

b) $f(x) = (x^3 + 1)(x - 17)$

$x^3 + 1 = 0$ and $x - 17 = 0$

$x = -1, 17$

c) $f(x) = (x^2 + 36)(8x - 16)$

$x^2 + 36 = 0$ and $8x - 16 = 0$

$x^2 = -36$ and $x = 2$

$x = 2$ (Since there is no number that, when squared, will equal -36.)

d) $f(x) = -3x^3(2x + 4)(x^2 - 25)$

$-3x^3 = 0$ and $2x + 4 = 0$ and $x^2 - 25 = 0$

$x = 0$ and $x = -2$ and $x = \pm 5$

$x = 0, -2, -5, 5$

e) $f(x) = (x^2 - x - 12)(3x)$

$x^2 - x - 12 = 0$ and $3x = 0$

$(x - 4)(x + 3) = 0$ and $x = 0$

$x - 4 = 0$ and $x + 3 = 0$

$x = 0, -3, 4$

f) $f(x) = (x + 1)(x^2 + 2x + 1)$

$x + 1 = 0$ and $x^2 + 2x + 1 = 0$

$x = -1$ and $(x + 1)(x + 1) = 0$

$x = -1$

7. a) $x^3 - 8x^2 - 3x + 90 = 0$

$f(-3) = 0$

$$\begin{array}{r|rrrr} -3 & 1 & -8 & -3 & 90 \\ & \downarrow & -3 & 33 & -90 \\ \hline & 1 & -11 & 30 & 0 \end{array}$$

$(x + 3)(x^2 - 11x + 30) = 0$

$(x + 3)(x - 6)(x - 5) = 0$

$x + 3 = 0$ and $x - 6 = 0$ and $x - 5 = 0$

$x = -3, 6, 5$

b) $x^4 + 9x^3 + 21x^2 - x - 30 = 0$

$f(1) = 0$

$$\begin{array}{r|rrrrr} 1 & 1 & 9 & 21 & -1 & -30 \\ & \downarrow & 1 & 10 & 31 & 30 \\ \hline & 1 & 10 & 31 & 30 & 0 \end{array}$$

$(x - 1)(x^3 + 10x^2 + 31x + 30) = 0$

For $x^3 + 10x^2 + 31x + 30 = 0$

$f(-2) = 0$

$$\begin{array}{r|rrrr} -2 & 1 & 10 & 31 & 30 \\ & \downarrow & -2 & -16 & -30 \\ \hline & 1 & 8 & 15 & 0 \end{array}$$

$(x - 1)(x + 2)(x^2 + 8x + 15) = 0$

$(x - 1)(x + 2)(x + 3)(x + 5) = 0$

$x - 1 = 0$ and $x + 2 = 0$ and $x + 3 = 0$ and $x + 5 = 0$

$x = 1, -2, -3, -5$

c) $2x^3 - 5x^2 - 4x + 3 = 0$

$f(-1) = 0$

$$\begin{array}{r|rrrr} -1 & 2 & -5 & -4 & 3 \\ & \downarrow & -2 & 7 & -3 \\ \hline & 2 & -7 & 3 & 0 \end{array}$$

$(x + 1)(2x^2 - 7x + 3) = 0$

$(x + 1)(2x^2 - 6x - 1x + 3) = 0$

$(x + 1)(2x(x - 3) - 1(x - 3)) = 0$

$(x + 1)(2x - 1)(x - 3) = 0$

$x + 1 = 0$ and $2x - 1 = 0$ and $x - 3 = 0$

$x = -1, \frac{1}{2}, 3$

d) $2x^3 + 3x^2 = 5x + 6$

$2x^3 + 3x^2 - 5x - 6 = 0$

$f(-1) = 0$

$$\begin{array}{r|rrrr} -1 & 2 & 3 & -5 & -6 \\ & \downarrow & -2 & -1 & 6 \\ \hline & 2 & 1 & -6 & 0 \end{array}$$

$(x + 1)(2x^2 + x - 6) = 0$

$(x + 1)(2x^2 + 4x - 3x - 6) = 0$

$(x + 1)(2x(x + 2) - 3(x + 2)) = 0$

$(x + 1)(2x - 3)(x + 2) = 0$

$x + 1 = 0$ and $2x - 3 = 0$ and $x + 2 = 0$

$x = -1, \frac{3}{2}, -2$

e) $4x^4 - 4x^3 - 51x^2 + 106x = 40$

$4x^4 - 4x^3 - 51x^2 + 106x - 40 = 0$

$f(2) = 0$

$$\begin{array}{r|rrrrr} 2 & 4 & -4 & -51 & 106 & -40 \\ & \downarrow & 8 & 8 & -86 & 40 \\ \hline & 4 & 4 & -43 & 20 & 0 \end{array}$$

$(x - 2)(4x^3 + 4x^2 - 43x + 20) = 0$

For $4x^3 + 4x^2 - 43x + 20 = 0$,

$f(-4) = 0$

$$\begin{array}{r|rrrr} -4 & 4 & 4 & -43 & 20 \\ & \downarrow & -16 & 48 & -20 \\ \hline & 4 & -12 & 5 & 0 \end{array}$$

$(x - 2)(x + 4)(4x^2 - 12x + 5) = 0$

$(x - 2)(x + 4)(4x^2 - 10x - 2x + 5) = 0$

$(x - 2)(x + 4)(2x(2x - 5) - 1(2x - 5)) = 0$

$(x - 2)(x + 4)(2x - 1)(2x - 5) = 0$

$x - 2 = 0$ and $x + 4 = 0$ and $2x - 1 = 0$ and $2x - 5 = 0$

$x = 2, -4, \frac{1}{2}, \frac{5}{2}$

f) $12x^3 - 44x^2 = -49x + 15$

$12x^3 - 44x^2 + 49x - 15 = 0$

$f\left(\frac{1}{2}\right) = 0$

$$\begin{array}{r|rrrr} \frac{1}{2} & 12 & -44 & 49 & -15 \\ & \downarrow & 6 & -19 & 15 \\ \hline & 12 & -38 & 30 & 0 \end{array}$$

$\left(x - \frac{1}{2}\right)(12x^2 - 38x + 30) = 0$

For $12x^2 - 38x + 30 = 0$,

$f\left(\frac{5}{3}\right) = 0$

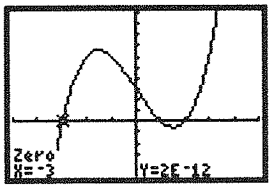
$$\begin{array}{r|rrr} \frac{5}{3} & 12 & -38 & 30 \\ & \downarrow & 20 & -30 \\ \hline & 12 & -18 & 0 \end{array}$$

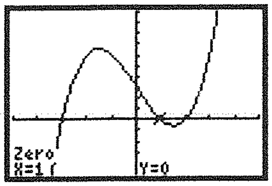
$\left(x - \frac{1}{2}\right)\left(x - \frac{5}{3}\right)(12x - 18) = 0$

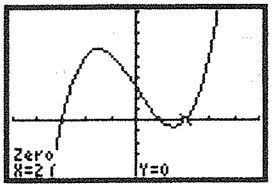
$x - \frac{1}{2} = 0$ and $x - \frac{5}{3} = 0$ and $12x - 18 = 0$

$x = \frac{1}{2}, \frac{5}{3}, \frac{3}{2}$

8. a) $x^3 - 7x + 6 = 0$

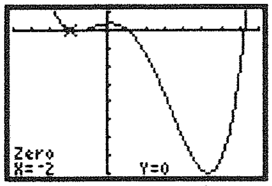


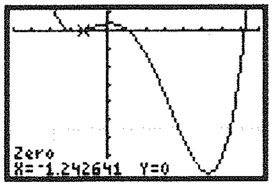


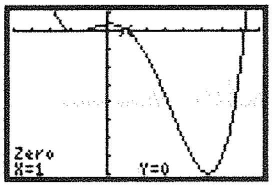


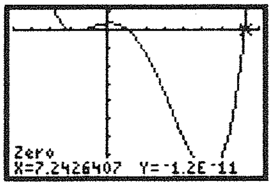
$x = -3, 1, 2$

b) $x^4 - 5x^3 - 17x^2 + 3x + 18 = 0$





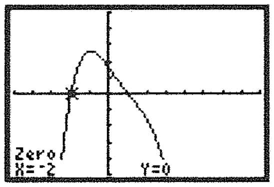


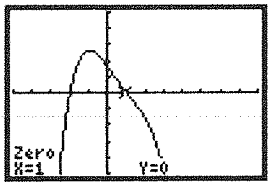


$x = -2, -1.24, 1, 7.24$

c) $3x^3 - 2x^2 + 16 = x^4 + 16x$

$-x^4 + 3x^3 - 2x^2 - 16x + 16 = 0$

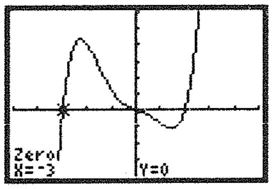


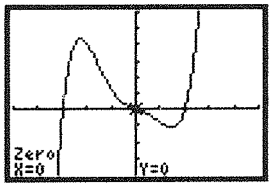


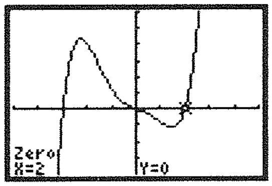
$x = -2, 1$

d) $x^5 + x^4 = 5x^3 - x^2 + 6x$

$x^5 + x^4 - 5x^3 + x^2 - 6x = 0$



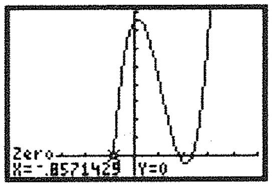


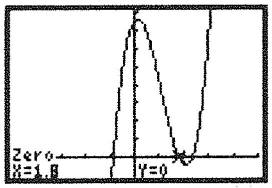


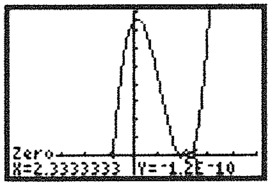
$x = -3, 0, 2$

e) $105x^3 = 344x^2 - 69x - 378$

$105x^3 - 344x^2 + 69x + 378 = 0$

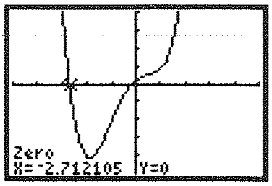


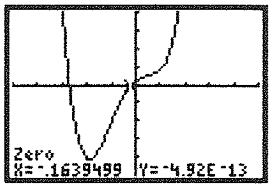




$x = -0.86, 1.8, 2.33$

f) $21x^3 - 58x^2 + 10 = -18x^4 - 51x$
$18x^4 + 21x^3 - 58x^2 + 51x + 10 = 0$





$x = -2.71, -0.16$

9. a) $x^3 - 6x^2 - x + 30 = 0$
$f(3) = 0$

3	1	−6	−1	−30
	↓	3	−9	30
	1	−3	−10	0

$(x - 3)(x^2 - 3x - 10) = 0$
$(x - 3)(x + 2)(x - 5) = 0$
$x - 3 = 0$ and $x + 2 = 0$ and $x - 5 = 0$
$x = 3, -2, 5$

b) $9x^4 - 42x^3 + 64x^2 - 32x = 0$
$x(9x^3 - 42x^2 + 64x - 32) = 0$
For $9x^3 - 42x^2 + 64x - 32 = 0$
$f(2) = 0$

2	9	−42	64	−32
	↓	18	−48	32
	9	−24	16	0

$x(x - 2)(9x^2 - 24x + 16) = 0$
$x(x - 2)(9x^2 - 12x - 12x + 16) = 0$
$x(x - 2)(3x(3x - 4) - 4(3x - 4)) = 0$
$x(x - 2)(3x - 4)(3x - 4) = 0$
$x = 0$ and $x - 2 = 0$ and $3x - 4 = 0$
$x = 0, 2, \frac{4}{3}$

c) $6x^4 - 13x^3 - 29x^2 + 52x = -20$
$6x^4 - 13x^3 - 29x^2 + 52x + 20 = 0$
$f(2) = 0$

2	6	−13	−29	52	20
	↓	12	−2	−62	−20
	6	−1	−31	−10	0

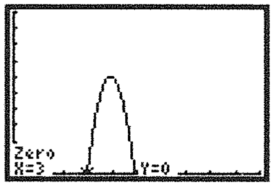
$(x - 2)(6x^3 - x^2 - 31x - 10) = 0$
For $6x^3 - x^2 - 31x - 10 = 0$
$f(-2) = 0$

−2	6	−1	−31	−10
	↓	−12	26	10
	6	−13	−5	0

$(x - 2)(x + 2)(6x^2 - 13x - 5) = 0$
$(x - 2)(x + 2)(6x^2 - 15x + 2x - 5) = 0$
$(x - 2)(x + 2)(3x(2x - 5) + 1(2x - 5)) = 0$
$(x - 2)(x + 2)(3x + 1)(2x - 5) = 0$
$x - 2 = 0$ and $x + 2 = 0$ and $3x + 1 = 0$ and $2x - 5 = 0$
$x = 2, -2, -\frac{1}{3}, \frac{5}{2}$

d) $x^4 - 6x^3 + 10x^2 - 2x = x^2 - 2x$
$x^4 - 6x^3 + 9x^2 = 0$
$x^2(x^2 - 6x + 9) = 0$
$x^2(x - 3)(x - 3) = 0$
$x^2 = 0$ and $x - 3 = 0$
$x = 0, 3$

10. $V = lwh$
$(30 - 2x)(20 - 2x)(x) = 1008$
$(4x^2 - 100x + 600)(x) = 1008$
$4x^3 - 100x^2 + 600x - 1008 = 0$
$x^3 - 25x^2 + 150x - 252 = 0$



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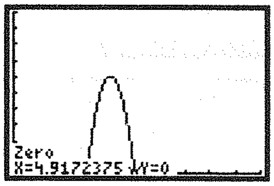
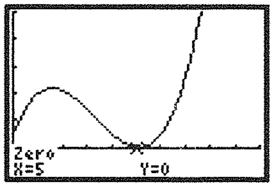


$x = 3, 4.92$
The dimensions of the cut square could be either 3 cm by 3 cm or 4.92 cm by 4.92 cm.

11. a) $x(x - 4)(x - 6) = 0$
$x = 0$ and $x - 4 = 0$ and $x - 6 = 0$
$x = 0, 4, 6$
Since there was not a game number 0, Maya's score was equal to zero after games 4 and 6.

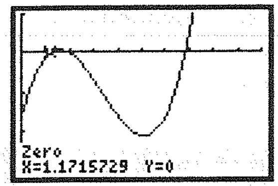
b) $x(x - 4)(x - 6) = -5$
$x(x^2 - 10x + 24) = -5$
$x^3 - 10x^2 + 24x + 5 = 0$



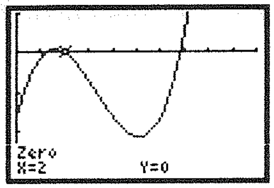
$x = 5$. Maya's score was −5 after game 5.

c) $x(x-4)(x-6) = 16$

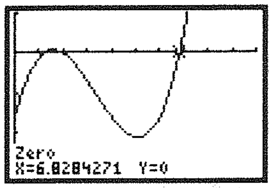
$x^3 - 10x^2 + 24x - 16 = 0$



Zero
X=1.1715729 Y=0



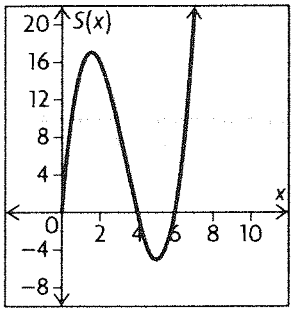
Zero
X=2 Y=0



Zero
X=6.8284271 Y=0

Since $x \in \mathbf{W}$, $x = 2$. Maya's score was 16 after game 2.

d)

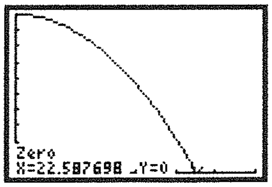


This is not a good model to represent Maya's score because the graph is shown for real numbers, but the number of games can only be a whole number.

12. $s(t) = -\frac{1}{2}(3.92)t^2 + 1000$

$= -1.96t^2 + 1000$

Use a graphing calculator to determine the value of t for which $s(t) = 0$.



Zero
X=22.587698 Y=0

It takes the object about 22.59 s to hit the surface.

13. a) $d(t) = -3t^3 + 3t^2 + 18t$

$= -3t(t^2 - t - 6)$

$= -3t(t + 2)(t - 3)$

b) The ship is in the harbour when the distance equals 0.

$-3t(t + 2)(t - 3) = 0$

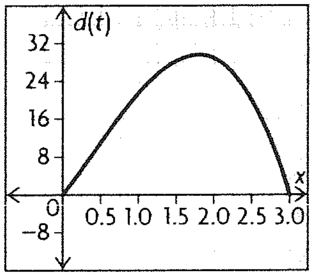
$-3t = 0$ and $t + 2 = 0$ and $t - 3 = 0$

$t = 0, -2, 3$

The ship returns to the harbour after 3 hours.

c) -2 is the other zero. It is not relevant because time cannot be negative.

d)



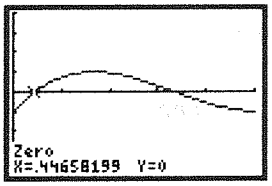
e) By examining the graph, it appears that the ship will begin its return trip to the harbour about 1.8 hours after departure.

14. a) $0 \leq t \leq 5$ because time cannot be negative and the model is only for a 5 s respiratory cycle.

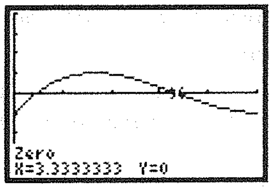
b) Answers may vary. For example, because the function involves decimals, graphing technology would be the better strategy for answering the question.

c) $0.027t^3 - 0.27t^2 + 0.675t = 0.25$

$0.027t^3 - 0.27t^2 + 0.675t - 0.25 = 0$



Zero
X=.44658199 Y=0



Zero
X=3.3333333 Y=0

At 0.45 s and 3.33 s, the person's lungs have a volume of air of 0.25 L.

15. All powers are even, which means every term is positive for all real numbers. Thus, the polynomial is always positive.

16. The zeros are 2, 3, and -5, so the polynomial function is of the form

$f(x) = a(x - 2)(x - 3)(x + 5)$. Since the graph of the function passes through (4, 36),

$36 = a(4 - 2)(4 - 3)(4 + 5)$

$36 = 18a$

$2 = a$

So $f(x) = 2(x - 2)(x - 3)(x + 5)$

$= 2(x^2 - 5x + 6)(x + 5)$

$= 2(x^3 - 19x + 30)$

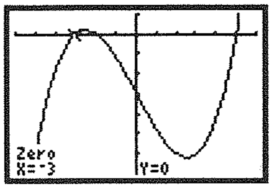
To determine the value of x for which $f(x) = 120$, set the polynomial equal to 120 and solve for x.

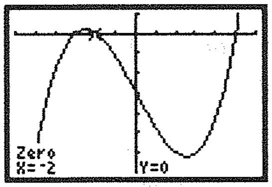
$2(x^3 - 19x + 30) = 120$

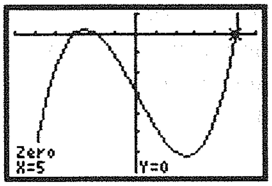
$x^3 - 19x + 30 = 60$

$x^3 - 19x - 30 = 0$

Graph $y = x^3 - 19x - 30$ on a graphing calculator and use the calculator to determine the zeros.







The fuction has a value of 120 at $x = -3$, $x = -2$, and $x = 5$.

17. Answers may vary. For example:

a) $x^3 + x^2 - x - 1 = 0$; $f(1) = 0$, so it is simple to solve using the factor theorem.

b) $x^2 - 2x = 0$; The common factor, x, can be factored out to solve the equation.

c) $x^3 - 2x^2 - 9x + 18$; An x can be factored out of the first two terms and a -2 out of the second two terms leaving you with the factors $(x - 2)(x^2 - 9)$.

d) $10x^2 - 7x + 1 = 0$; The roots are fractional which makes using the quadratic formula the most sensible approach.

e) $x^3 - 8 = 0$; This is the difference of two cubes.

f) $0.856x^3 - 2.74x^2 + 0.125x - 2.89 = 0$; The presence of decimals makes using graphing technology the most sensible strategy.

18. a) $0 = x^4 + 10$; x^4 is non-negative for all real x, so $x^4 + 10$ is always positive.

b) A degree 5 polynomial function $y = f(x)$ has opposite end behaviour, so somewhere in the middle it must cross the x-axis. This means its corresponding equation $0 = f(x)$ will have at least one real root.

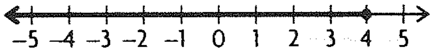
19. $y = x^5 + x + 1$; By the factor theorem, the only possible rational zeros are 1 and -1. Neither works. Because the degree is odd, the polynomial has opposite end behaviour, and hence must have at least one zero, which must be irrational.

4.2 Solving Linear Inequalities, pp. 213–215

1. a) $3x - 1 \le 11$

$3x \le 12$

$x \le 4$

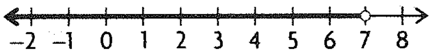


$\{x \in \mathbf{R} \mid x \le 4\}$

b) $-x + 5 > -2$

$-x > -7$

$x < 7$

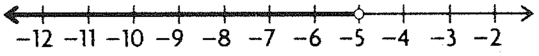


$\{x \in \mathbf{R} \mid x < 7\}$

c) $x - 2 > 3x + 8$

$-2x > 10$

$x < -5$



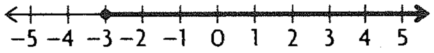
$\{x \in \mathbf{R} \mid x < -5\}$

d) $3(2x + 4) \ge 2x$

$6x + 12 \ge 2x$

$4x \ge -12$

$x \ge -3$



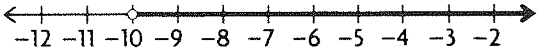
$\{x \in \mathbf{R} \mid x \ge -3\}$

e) $-2(1 - 2x) < 5x + 8$

$-2 + 4x < 5x + 8$

$-1x < 10$

$x > -10$



$\{x \in \mathbf{R} \mid x > -10\}$

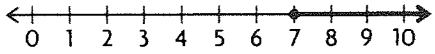
f) $\frac{6x + 8}{5} \le 2x - 4$

$6x + 8 \le 5(2x - 4)$

$6x + 8 \le 10x - 20$

$-4x \le -28$

$x \ge 7$



$\{x \in \mathbf{R} \mid x \ge 7\}$

2. a) $2x - 5 \leq 4x + 1$

$-2x \leq 6$

$x \geq -3$

$x \in [-3, \infty)$

b) $2(x + 3) < -(x - 4)$

$2x + 6 < -x + 4$

$3x < -2$

$x < -\frac{2}{3}$

$x \in \left(-\infty, -\frac{2}{3}\right)$

c) $\frac{2x + 3}{3} \leq x - 5$

$2x + 3 \leq 3(x - 5)$

$2x + 3 \leq 3x - 15$

$-x \leq -18$

$x \geq 18$

$x \in [18, \infty)$

d) $2x + 1 \leq 5x - 2$

$-3x \leq -3$

$x \geq 1$

$x \in [1, \infty)$

e) $-x + 1 > x + 1$

$-2x > 0$

$x < 0$

$x \in (-\infty, 0)$

f) $\frac{x + 4}{2} \geq \frac{x - 2}{4}$

$4(x + 4) \geq 2(x - 2)$

$4x + 16 \geq 2x - 4$

$2x \geq -20$

$x \geq -10$

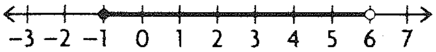
$x \in [-10, \infty)$

3. $3 \leq 2x + 5 < 17$

$3 - 5 \leq 2x + 5 - 5 < 17 - 5$

$-2 \leq 2x < 12$

$-1 \leq x < 6$



4. a) $x > -1$

$2 > -1$

Yes, $x = 2$ is contained in the solution set.

b) $5x - 4 > 3x + 2$

$5(2) - 4 > 3(2) + 2$

$10 - 4 > 6 + 2$

$6 > 8$

This is false. So, $x = 2$ is not contained in the solution set.

c) $4(3x - 5) \geq 6x$

$4(3(2) - 5) \geq 6(2)$

$4(6 - 5) \geq 12$

$4(1) \geq 12$

$4 \geq 12$

This is false. So, $x = 2$ is not contained in the solution set.

d) $5x + 3 \leq -3x + 1$

$5(2) + 3 \leq -3(2) + 1$

$10 + 3 \leq -6 + 1$

$13 \leq -5$

This is false. So, $x = 2$ is not contained in the solution set.

e) $x - 2 \leq 3x + 4 \leq x + 14$

$2 - 2 \leq 3(2) + 4 \leq 2 + 14$

$0 \leq 6 + 4 \leq 16$

$0 \leq 10 \leq 16$

Yes, $x = 2$ is contained in the solution set.

f) $33 < -10x + 3 < 54$

$33 < -10(2) + 3 < 54$

$33 < -20 + 3 < 54$

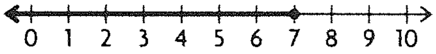
$33 < -17 < 54$

This is false. So, $x = 2$ is not contained in the solution set.

5. a) $2x - 1 \leq 13$

$2x \leq 14$

$x \leq 7$



Check $x = 6$ to verify.

$2(6) - 1 \leq 13$

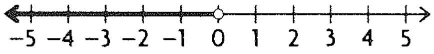
$12 - 1 \leq 13$

$11 \leq 13$

b) $-2x - 1 > -1$

$-2x > 0$

$x < 0$



Check $x = -1$ to verify.

$-2(-1) - 1 > -1$

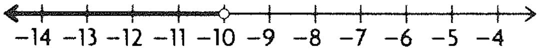
$2 - 1 > -1$

$1 > -1$

c) $2x - 8 > 4x + 12$

$-2x > 20$

$x < -10$

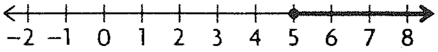


Check $x = -11$ to verify.

$2(-11) - 8 > 4(-11) + 12$

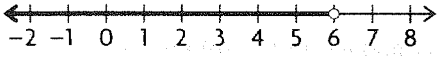
$-22 - 8 > -44 + 12$
$-30 > -32$

d) $5(x - 3) \geq 2x$
$5x - 15 \geq 2x$
$3x \geq 15$
$x \geq 5$



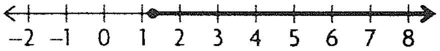
Check $x = 7$ to verify.
$5(7 - 3) \geq 2(7)$
$5(4) \geq 14$
$20 \geq 14$

e) $-4(5 - 3x) < 2(3x + 8)$
$-20 + 12x < 6x + 16$
$6x < 36$
$x < 6$



Check $x = 4$ to verify.
$-4(5 - 3(4)) < 2(3(4) + 8)$
$-4(5 - 12) < 2(12 + 8)$
$-20 + 48 < 24 + 16$
$28 < 40$

f) $\frac{x - 2}{3} \leq 2x - 3$
$x - 2 \leq 3(2x - 3)$
$x - 2 \leq 6x - 9$
$-5x \leq -7$
$x \geq \frac{7}{5}$



Check $x = 4$ to verify.
$\frac{4 - 2}{3} \leq 2(4) - 3$
$\frac{2}{3} \leq 8 - 3$
$\frac{2}{3} \leq 5$

6. a) $3x \leq 4x + 1$
$3(0) \leq 4(0) + 1$
$0 \leq 0 + 1$
$0 \leq 1$
Yes, 0 is contained in the solution set.

b) $-6x < x + 4 < 12$
$-6(0) < 0 + 4 < 12$
$0 < 4 < 12$
Yes, 0 is contained in the solution set.

c) $-x + 1 > x + 12$
$-0 + 1 > 0 + 12$
$1 > 12$
This is false. So, 0 is not contained in the solution set.

d) $3x \leq x + 1 \leq x - 1$
$3(0) \leq 0 + 1 \leq 0 - 1$
$0 \leq 1 \leq -1$
This is false. So, 0 is not contained in the solution set.

e) $x(2x - 1) \leq x + 7$
$0(2(0) - 1) \leq 0 + 7$
$0 \leq 7$
Yes, 0 is contained in the solution set.

f) $x + 6 < (x + 2)(5x + 3)$
$0 + 6 < (0 + 2)(5(0) + 3)$
$6 < (2)(3)$
$6 < 6$
This is false. So, 0 is not contained in the solution set.

7. a) $-5 < 2x + 7 < 11$
$-5 - 7 < 2x + 7 - 7 < 11 - 7$
$-12 < 2x < 4$
$-6 < x < 2$

b) $11 < 3x - 1 < 23$
$11 + 1 < 3x - 1 + 1 < 23 + 1$
$12 < 3x < 24$
$4 < x < 8$

c) $-1 \leq -x + 9 \leq 13$
$-1 - 9 \leq -x + 9 - 9 \leq 13 - 9$
$-10 \leq -x \leq 4$
$-4 \leq x \leq 10$

d) $0 \leq -2(x + 4) \leq 6$
$0 \leq -2x - 8 \leq 6$
$0 + 8 \leq -2x - 8 + 8 \leq 6 + 8$
$8 \leq -2x \leq 14$
$-7 \leq x \leq -4$

e) $59 < 7x + 10 < 73$
$59 - 10 < 7x + 10 - 10 < 73 - 10$
$49 < 7x < 63$
$7 < x < 9$

f) $18 \leq -12(x - 1) \leq 48$
$18 \leq -12x + 12 \leq 48$
$18 - 12 \leq -12x + 12 - 12 \leq 48 - 12$
$6 \leq -12x \leq 36$
$-3 \leq x \leq -\frac{1}{2}$

8. a) Answers may vary. For example:
$3x + 1 > 9 + x$
$2x > 8$
$x > 4$

f) $v(x) = 9$

Slope $= 0$

6. Instantaneous rate at $x = 3$

a) $f(x) = 3x + 1$

Slope $= \dfrac{f(a + h) - f(a)}{h}$, where $a = 3$; $h = 0.001$

$$= \frac{f(3.001) - f(3)}{0.001}$$

$$= \frac{(3(3.001) + 1) - (3(3) + 1)}{0.001}$$

$$= \frac{10.003 - 10}{0.001}$$

$$= 3$$

b) $t(x) = 3x^2 - 4x + 1$

$$= \frac{f(3.001) - f(3)}{0.001}$$

$$= \frac{(3(3.001)^2 - 4(3.001) + 1)}{0.001} - \frac{(3(3)^2 - 4(3) + 1)}{0.001}$$

$$= \frac{16.014\,003 - 16}{0.001}$$

$$\doteq 14$$

c) $g(x) = \dfrac{1}{x}$

$$= \frac{f(3.001) - f(3)}{0.001}$$

$$= \frac{\left(\frac{1}{3.001}\right) - \left(\frac{1}{3}\right)}{0.001}$$

$$\doteq -0.111$$

$$\doteq -\frac{1}{9}$$

d) $d(x) = -x^2 + 7$

$$= \frac{f(3.001) - f(3)}{0.001}$$

$$= \frac{(-(3.001)^2 + 7) - (-(3)^2 + 7)}{0.001}$$

$$= \frac{-2.006\,001 - (-2)}{0.001}$$

$$\doteq -6$$

e) $h(x) = 2^x$

$$= \frac{f(3.001) - f(3)}{0.001}$$

$$= \frac{(2^{3.001}) - (2^3)}{0.001}$$

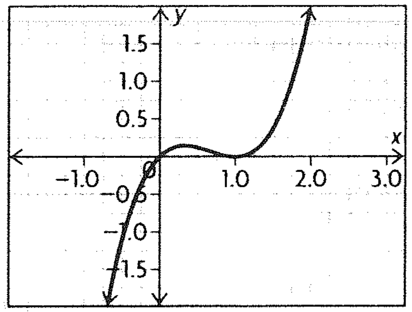
$$= \frac{8.005\,547\,1 - 8}{0.001}$$

$$\doteq 5.5$$

f) $v(x) = 9$

slope $= 0$

7.



Rate of change is positive on $(-\infty, \frac{1}{3})$ and $(1, \infty)$, negative on $(\frac{1}{3}, 1)$, and zero at $x = \frac{1}{3}$ and $x = 1$.

8. $s(t) = 320 - 5t^2, 0 \le t \le 8$

a) $3 \le t \le 8$

Average rate of change $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$$= \frac{f(8) - f(3)}{8 - 3}$$

$$= \frac{(320 - 5(8)^2) - (320 - 5(3)^2)}{5}$$

$$= \frac{0 - 275}{5}$$

$= -55$ m/s

b) $t = 2$

Slope $= \dfrac{f(a + h) - f(a)}{h}$, where $a = 2$; $h = 0.001$

$$= \frac{f(2.001) - f(2)}{0.001}$$

$$= \frac{(320 - 5(2.001)^2) - (320 - 5(2)^2)}{0.001}$$

$$= \frac{299.979\,995 - 300}{0.001}$$

$\doteq -20$ m/s

9. $f(x) = 3x^2 - 4x - 1$

a) $x = 1$

Slope $= \dfrac{f(a + h) - f(a)}{h}$, where $a = 1$; $h = 0.001$

$$= \frac{f(1.001) - f(1)}{0.001}$$

$$= \frac{(3(1.001)^2 - 4(1.001) - 1)}{0.001} - \frac{(3(1)^2 - 4(1) - 1)}{0.001}$$

$$= \frac{-1.997\,997 - (-2)}{0.001}$$

$$\doteq 2$$

b) $f(1) = 3(1)^2 - 4(1) - 1$
$= 3 - 4 - 1$
$= -2$

c) Slope = 2; point of tangency: $(1, -2)$

$y = 2x + b$
$-2 = 2(1) + b$
$-4 = b$
$y = 2x - 4$

10. $h(t) = -5t^2 + 50t$

a) $t = 4$

Slope $= \dfrac{f(a + h) - f(a)}{h}$, where $a = 4$; $h = 0.001$

$$= \frac{f(4.001) - f(4)}{0.001}$$

$$= \frac{(-5(4.001)^2 + 50(4.001))}{0.001} - \frac{(-5(4)^2 + 50(4))}{0.001}$$

$$= \frac{120.009\,995 - 120}{0.001}$$

$\doteq 10$ m/s.

b) $t = 10$

Slope $= \dfrac{f(a + h) - f(a)}{h}$, where $a = 10$; $h = 0.001$

$$= \frac{f(10.001) - f(10)}{0.001}$$

$$= \frac{(-5(10.001)^2 + 50(10.001))}{0.001} - \frac{(-5(10)^2 + 50(10))}{0.001}$$

$$= \frac{-0.050\,005 - 0}{0.001}$$

$\doteq -50$ m/s

c) interval from $t = 0$ to $t = 10$

Average rate of change $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$$= \frac{f(10) - f(0)}{10 - 0}$$

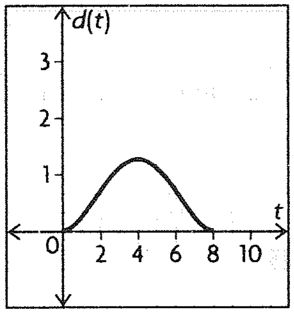
$$= \frac{(-5(10)^2 + 50(10)) - (-5(0)^2 + 50(0))}{10}$$

$$= \frac{0 - 0}{10}$$

$= 0$ m/s

11. $d(t) = \left(\dfrac{1}{200}\right)t^2(t - 8)^2$

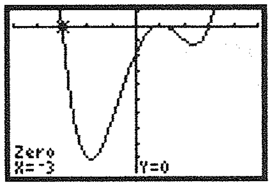
a)



The rate is positive for $t \in (0, 4)$, negative for $t \in (4, 8)$, and zero at $t = 0, 4,$ and 8

b) When the rate of change is zero, the boat stops. When the rate of change is negative, the boat is headed back to the dock.

12. $y = x^4 - 2x^3 - 8x^2 + 18x - 9$



at $x = -3$

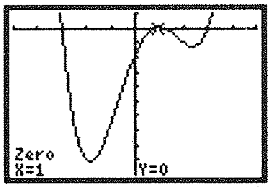
Slope $= \dfrac{f(a + h) - f(a)}{h}$, where $a = -3$;

$h = 0.001$

$$= \frac{f(-2.99) - f(-3)}{0.001}$$

$$= \frac{-0.095\,936\,014 - 0}{0.001}$$

$\doteq -96$

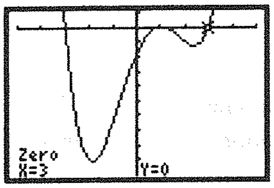


at $x = 1$

Slope $= \dfrac{f(a + h) - f(a)}{h}$, where $a = 1$; $h = 0.001$

$$= \frac{f(1.001) - f(1)}{0.001}$$

$$= \frac{-0.000\,007\,99 - 0}{0.001}$$
$$\doteq 0$$



at $x = 3$

$$\text{Slope} = \frac{f(a+h) - f(a)}{h}, \text{ where } a = 3;\ h = 0.001$$
$$= \frac{f(3.001) - f(3)}{0.001}$$
$$= \frac{0.024\,028\,01 - 0}{0.001}$$
$$\doteq 24$$

13. $f(x) = x^2 + 3x - 5$

a) Instantaneous rate of change at $x = 1$,

$$\text{Slope} = \frac{f(a+h) - f(a)}{h}, \text{ where } a = 1;\ h = 0.001$$
$$= \frac{f(1.001) - f(1)}{0.001}$$
$$= \frac{(1.001^2 + 3(1.001) - 5) - (1^2 + 3(1) - 5)}{0.001}$$
$$= \frac{-0.994999 - (-1)}{0.001}$$
$$\doteq 5$$

b) $\dfrac{f(x+h) - f(x)}{h}$

$$= \frac{((x+h)^2 + 3(x+h) - 5) - (x^2 + 3x - 5)}{h}$$
$$= \frac{(x^2 + 2xh + h^2 + 3x + 3h - 5)}{h}$$
$$- \frac{(x^2 + 3x - 5)}{h}$$
$$= \frac{\cancel{x^2} + 2xh + h^2 + \cancel{3x} + 3h \cancel{- 5 - x^2 - 3x + 5}}{h}$$
$$= \frac{2xh + 3h + h^2}{h}$$
$$= \frac{\cancel{h}(2x + 3 + h)}{\cancel{h}}$$
$$= 2x + 3 + h$$

c) It gets closer to $2x + 3$ as h becomes very close to 0.

d) The expression for the instantaneous rate of change is $2x + 3$. At $x = 1$: $2(1) + 3 = 5$

14. When the instantaneous rate of change is zero, the function potentially has a local maximum or a local minimum. If the rate is positive to the left and negative to the right, it has a local maximum. If the rate is negative to the left and positive to the right, it has a local minimum.

15. $f(x) = e^x$

a) Instantaneous rate of change at $x = 5$,

$$\text{Slope} = \frac{f(a+h) - f(a)}{h}, \text{ where } a = 5;\ h = 0.001$$
$$= \frac{f(5.001) - f(5)}{0.001}$$
$$= \frac{e^{5.001} - e^5}{0.001}$$
$$= \frac{148.561\,646\,5 - 148.413\,159\,1}{0.001}$$
$$\doteq 148.4$$
$$f(5) = e^5 \doteq 148.4$$

b) Answers may vary. For example,

i) Instantaneous rate of change at $x = 1$,

$$\text{Slope} = \frac{f(a+h) - f(a)}{h}, \text{ where } a = 1;\ h = 0.001$$
$$= \frac{f(1.001) - f(1)}{0.001}$$
$$= \frac{e^{1.001} - e^1}{0.001}$$
$$= \frac{2.721\,001\,47 - 2.718\,281\,828}{0.001}$$
$$\doteq 2.7$$
$$f(1) = e^1 \doteq 2.7$$

ii) Instantaneous rate of change at $x = 3$,

$$\text{Slope} = \frac{f(a+h) - f(a)}{h}, \text{ where } a = 3;\ h = 0.001$$
$$= \frac{f(3.001) - f(3)}{0.001}$$
$$= \frac{e^{3.001} - e^3}{0.001}$$
$$= \frac{20.105\,632\,51 - 20.085\,536\,92}{0.001}$$
$$\doteq 20.1$$
$$f(3) = e^3 \doteq 20.1$$

iii) Instantaneous rate of change at $x = 4$,

$$\text{Slope} = \frac{f(a+h) - f(a)}{h}, \text{ where } a = 4;\ h = 0.001$$
$$= \frac{f(4.001) - f(4)}{0.001}$$

$$= \frac{e^{4.001} - e^4}{0.001}$$

$$= \frac{54.652\,775\,49 - 54.598\,150\,03}{0.001}$$

$$\doteq 54.6$$

$f(4) = e^4 \doteq 54.6$

c) The instantaneous rate of change of e^x for any value of x is e^x.

16. $f(x) = x^3 - 4x$

a) Instantaneous rate of change at $x = 1$,

$$\text{Slope} = \frac{f(a + h) - f(a)}{h}, \text{ where } a = 1;\ h = 0.001$$

$$= \frac{f(1.001) - f(1)}{0.001}$$

$$= \frac{((1.001)^3 - 4(1.001)) - ((1)^3 - 4(1))}{0.001}$$

$$= \frac{-3.000\,996\,999 - (-3)}{0.001}$$

$$\doteq -1$$

b) $f(1) = 1^3 - 4(1) = -3$

Point of tangency: $(1, -3)$

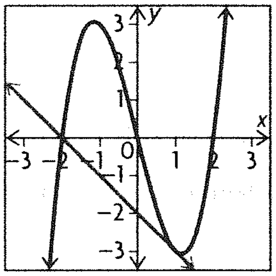
$y = -1x + b$

$-3 = -1(1) + b$

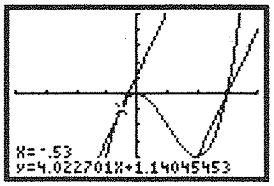
$-2 = b$

$y = -x - 2$

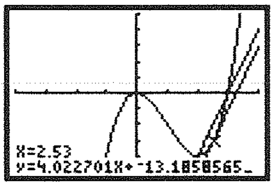
c) $(-2, 0)$



17. The slope of the secant line is $\frac{0 - (-4)}{3 - 2}$ or 4. Graph the curve and the secant using a graphing calculator. Estimate where the slope of the tangent to the curve is 4. Use the calculator to draw the tangents at various x-values until your estimate is accurate to two decimal places.



```
X=-.53
Y=4.022701X+1.14045453
```



```
X=2.53
Y=4.022701X+-13.1858565_
```

The slope of the tangent lines when $x = -0.53$ and $x = 2.53$ are the same as the slope of the given secant line.

Chapter Review, pp. 240–241

1. a) $x^4 - 16x^2 + 75 = 2x^2 - 6$

$x^4 - 18x^2 + 81 = 0$

$(x^2 - 9)(x^2 - 9) = 0$

$x^2 - 9 = 0$

$x^2 = 9$

$x^2 = 9$

$\sqrt{x^2} = \pm\sqrt{9}$

$x = \pm 3$

b) $2x^2 + 4x - 1 = x + 1$

$2x^2 + 3x - 2 = 0$

$2x^2 + 4x - 1x - 2 = 0$

$2x(x + 2) - 1(x + 2) = 0$

$(2x - 1)(x + 2) = 0$

$2x - 1 = 0$ and $x + 2 = 0$

$x = \frac{1}{2}, -2$

c) $4x^3 - x^2 - 2x + 2 = 3x^3 - 2(x^2 - 1)$

$4x^3 - x^2 - 2x + 2 = 3x^3 - 2x^2 + 2$

$x^3 + x^2 - 2x = 0$

$x(x^2 + x - 2) = 0$

$x(x + 2)(x - 1) = 0$

$x = 0$ and $x + 2 = 0$ and $x - 1 = 0$

$x = 0, -2, 1$

d) $-2x^2 + x - 6 = -x^3 + 2x - 8$

$x^3 - 2x^2 - x + 2 = 0$

$x^2(x - 2) - 1(x - 2) = 0$

$(x^2 - 1)(x - 2) = 0$

$x^2 - 1 = 0$ and $x - 2 = 0$

$x^2 = 1$ and $x = 2$

$x = \pm 1, 2$

2. $18x^4 - 53x^3 + 52x^2 - 14x - 8$

$= 3x^4 - x^3 + 2x - 8$

$15x^4 - 52x^3 + 52x^2 - 16x = 0$

$x(15x^3 - 52x^2 + 52x - 16) = 0$

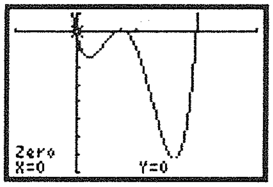
For $15x^3 - 52x^2 + 52x - 16 = 0, f(2) = 0.$

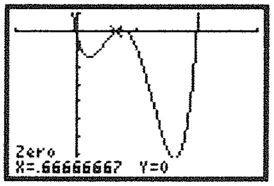
$$\begin{array}{r|rrrr} 2 & 15 & -52 & 52 & -16 \\ & \downarrow & 30 & -44 & 16 \\ \hline & 15 & -22 & 8 & 0 \end{array}$$

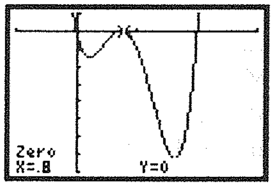
$$x(x-2)(15x^2 - 22x + 8) = 0$$
$$x(x-2)(15x^2 - 12x - 10x + 8) = 0$$
$$x(x-2)(3x(5x-4) - 2(5x-4)) = 0$$
$$x(x-2)(3x-2)(5x-4) = 0$$

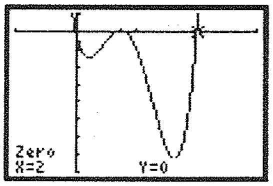
$x = 0$ and $x - 2 = 0$ and $3x - 2 = 0$ and $5x - 4 = 0$

$x = 0, 2, \frac{2}{3}, \frac{4}{5}$









3. a) $f(x) = a(x-1)(x-2)(x+1)(x+2)$

Since $f(x)$ has a y-intercept of 4, $f(0) = 4$.

$4 = a(-1)(-2)(1)(2)$

$4 = 4a$

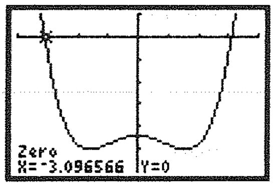
$1 = a$

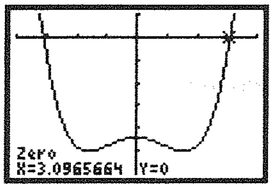
So, $f(x) = (x-1)(x-2)(x+1)(x+2)$ or $f(x) = x^4 - 5x^2 + 4$.

b) $x^4 - 5x^2 + 4 = 48$

$x^4 - 5x^2 - 44 = 0$

Graph $y = x^4 - 5x^2 - 44$ on a graphing calculator and use the calculator to determine the zeros.





The function has a value of 48 when $x \doteq -3.10 \doteq 3.10$.

4. $x(24 - 2x)(30 - 2x) = 1040$

$x(720 - 108x + 4x^2) = 1040$

$720x - 108x^2 + 4x^3 = 1040$

$4x^3 - 108x^2 + 720x - 1040 = 0$

$f(2) = 0$

$$\begin{array}{r|rrrr} 2 & 4 & -108 & 720 & -1040 \\ & \downarrow & 8 & -200 & 1040 \\ \hline & 4 & -100 & 520 & 0 \end{array}$$

$(x-2)(4x^2 - 100x + 520) = 0$

$x = 2$

For $4x^2 - 100x + 520 = 0$

$$x = \frac{-(-100) \pm \sqrt{(-100)^2 - 4(4)(520)}}{2(4)}$$

$$= \frac{25}{2} \pm \frac{\sqrt{1680}}{8}$$

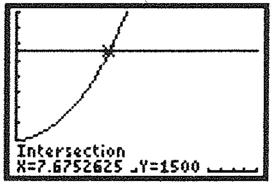
$= 7.4, 17.6$ (17.6 is too large)

2 cm by 2 cm or 7.4 cm by 7.4 cm

5. a) The given information states that the model is valid between 1985 and 1995, so it can be used for 1993, but not 2005.

b) Set $C(t) = 1500$ (since the units are in thousands) and solve using a graphing calculator.

c)



y = 1500

Sales reach 1.5 million in the 8th year after 1985, so in 1993.

6. a) Answers may vary. For example:
$2x + 1 > 17$
$2x > 16$
$x > 8$

b) Answers may vary. For example:
$3x - 4 \geq -16$
$3x \geq -12$
$x \geq -4$

c) Answers may vary. For example:
$2x + 3 \leq -21$
$2x \leq -24$
$x \leq -12$

d) Answers may vary. For example:
$-19 < 2x - 1 < -3$
$-18 < 2x < -2$
$-9 < x < -1$

7. a) $2(4x - 7) > 4(x + 9)$
$8x - 14 > 4x + 36$
$4x > 50$
$$x > \frac{25}{2}$$
$$x \in \left(\frac{25}{2}, \infty\right)$$

b) $\dfrac{x-4}{5} \geq \dfrac{2x+3}{2}$
$2(x - 4) \geq 5(2x + 3)$
$2x - 8 \geq 10x + 15$
$-8x \geq 23$
$$x \leq -\frac{23}{8}$$
$$x \in \left[-\frac{23}{8}, \infty\right)$$

c) $-x + 2 > x - 2$
$-2x > -4$
$x < 2$
$x \in (-\infty, 2)$

d) $5x - 7 \leq 2x + 2$
$3x \leq 9$
$x \leq 3$
$x \in (-\infty, 3]$

8. a) $-3 < 2x + 1 < 9$
$-3 - 1 < 2x + 1 - 1 < 9 - 1$
$-4 < 2x < 8$
$-2 < x < 4$
$\{x \in \mathbf{R} \mid -2 < x < 4\}$

b) $8 \leq -x + 8 \leq 9$
$8 - 8 \leq -x + 8 - 8 \leq 9 - 8$
$0 \leq -x \leq 1$
$-1 \leq x \leq 0$
$\{x \in \mathbf{R} \mid -1 \leq x \leq 0\}$

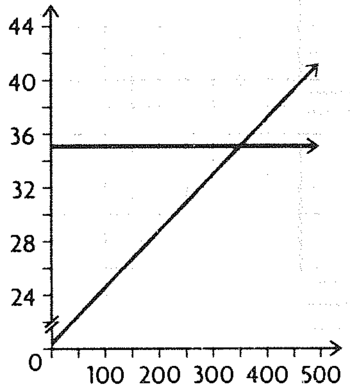
c) $6 + 2x \geq 0 \geq -10 + 2x$
$6 + 2x - 2x \geq 0 - 2x \geq -10 + 2x - 2x$
$6 \geq -2x \geq -10$
$-3 \leq x \leq 5$
$\{x \in \mathbf{R} \mid -3 \leq x \leq 5\}$

d) $x + 1 < 2x + 7 < x + 5$
$x + 1 - x < 2x + 7 - x < x + 5 - x$
$1 < x + 7 < 5$
$-6 < x < -2$
$\{x \in \mathbf{R} \mid -6 < x < -2\}$

9. a) $0.04x + 20.95 > 34.95$
$0.04x > 14$
$x > 350$

The second plan is better if one calls more than 350 minutes per month.

b)



10. a) $(x + 1)(x - 2)(x + 3)^2 < 0$
$x + 1 = 0$ and $x - 2 = 0$ and $x + 3 = 0$
$x = -3, -1, 2$

This divides the domain of real numbers into 4 intervals:
$x \leq -3; -3 < x < -1; -1 < x < 2; x \geq 2$

Test for each interval:

$\boldsymbol{x \leq -3}$**:**
$f(-4) = (-4 + 1)(-4 - 2)(-4 + 3)^2 < 0$
$= (-3)(-6)(-1)^2 < 0$
$= 18 < 0$ No

$\boldsymbol{-3 < x < -1}$**:**
$f(-2) = (-2 + 1)(-2 - 2)(-2 + 3)^2 < 0$
$= (-1)(-4)(1)^2 < 0$
$= 4 < 0$ No

$\boldsymbol{-1 < x < 2}$**:**
$f(0) = (0 + 1)(0 - 2)(0 + 3)^2 < 0$
$= (1)(-2)(3)^2 < 0$
$= (-2)(9) < 0$
$= -18 < 0$ Yes

$\boldsymbol{x \geq 2}$**:**
$f(3) = (3 + 1)(3 - 2)(3 + 3)^2 < 0$
$= (4)(1)(6)^2 < 0$

$= (4)(36) < 0$

$= 144 < 0$ No

The interval is $-1 < x < 2$.

b) $\frac{(x-4)(2x+3)}{5} \geq \frac{2x+3}{5}$

$(x-4)(2x+3) \geq 2x+3$

$2x^2 - 5x - 12 \geq 2x + 3$

$2x^2 - 7x - 15 \geq 0$

$2x^2 - 10x + 3x - 15 \geq 0$

$2x(x-5) + 3(x-5) \geq 0$

$(2x+3)(x-5) \geq 0$

$2x + 3 = 0$ and $x - 5 = 0$

$x = -\frac{3}{2}, 5$

This divides the domain of real numbers into 3 intervals:

$x \leq -\frac{3}{2}; -\frac{3}{2} < x <; x \geq 5$

Test for each interval:

$x \leq -\frac{3}{2}$:

$f(-2) = (2(-2)+3)((-2)-5) \geq 0$

$= (-1)(-7) \geq 0$

$= 7 \geq 0$ Yes

$-\frac{3}{2} < x < 5$:

$f(0) = (2(0)+3)((0)-5) \geq 0$

$= (3)(-5) \geq 0$

$= -15 \geq 0$ No

$x \geq 5$:

$f(6) = (2(6)+3)((6)-5) \geq 0$

$= (15)(1) \geq 0$

$= 15 \geq 0$ Yes

The intervals are $x \leq -\frac{3}{2}$ or $x \geq 5$.

c) $-2(x-1)(2x+5)(x-7) > 0$

$x - 1 = 0$ and $2x + 5 = 0$ and $x - 7 = 0$

$x = -\frac{5}{2}, 1, 7$

This divides the domain of real numbers into 4 intervals:

$x < -\frac{5}{2}; -\frac{5}{2} < x < 1; 1 < x < 7; x > 7$

Test for each interval:

$x < -\frac{5}{2}$:

$f(-3)$

$= -2((-3)-1)(2(-3)+5)((-3)-7) > 0$

$= -2(-4)(-1)(-10) < 0$

$= 80 > 0$ Yes

$-\frac{5}{2} < x < 1$:

$f(0) = -2((0)-1)(2(0)+5)((0)-7)$

$= -2(-1)(5)(-7)$

$= -70 > 0$ No

$1 < x < 7$:

$f(2) = -2((2)-1)(2(2)+5)((2)-7)$

$= -2(1)(9)(-5)$

$= 90 > 0$ Yes

$x > 7$:

$f(8) = (-2(8)-1)(2(8)+5)((8)-7)$

$= -2(7)(21)(1)$

$= -294 > 0$ No

The intervals are $x < -\frac{5}{2}$ or $1 < x < 7$.

d) $x^3 + x^2 - 21x + 21 \leq 3x^2 - 2x + 1$

$x^3 - 2x^2 - 19x + 20 \leq 0$

$f(1) = 0$

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -19 & 20 \\ & \downarrow & 1 & -1 & -20 \\ \hline & 1 & -1 & -20 & 0 \end{array}$$

$(x-1)(x^2 - x - 20) \leq 0$

$(x-1)(x-5)(x+4) \leq 0$

$x - 1 = 0$ and $x - 5 = 0$ and $x + 4 = 0$

$x = -4, 1, 5$

This divides the domain of real numbers into 4 intervals:

$x \leq -4; -4 < x < 1; 1 \leq x \leq 5; x > 5$

Test for each interval:

$x \leq -4$:

$f(-5) = ((-5)-1)((-5)-5)((-5)+4) \leq 0$

$= (-6)(-10)(-1) \leq 0$

$= -60 \leq 0$ Yes

$-4 < x < 1$:

$f(0) = ((0)-1)((0)-5)((0)+4) \leq 0$

$= (-1)(-5)(4) \leq 0$

$= 20 \leq 0$ No

$1 \leq x \leq 5$:

$f(2) = ((2)-1)((2)-5)((2)+4) \leq 0$

$= (1)(-3)(6) \leq 0$

$= -18 \leq 0$ Yes

$x > 5$:

$f(6) = ((6)-1)((6)-5)((6)+4) \leq 0$

$= (5)(1)(10) \leq 0$

$= 50 \leq 0$ No

The intervals are $x \leq -4$ or $1 \leq x \leq 5$.

11. $f(x) = 2x^4 - 2x^3 - 32x^2 - 40x$
$x(2x^3 - 2x^2 - 32x - 40)$
For $2x^3 - 2x^2 - 32x - 40, f(-2) = 0$

$$\begin{array}{r|rrrr} -2 & 2 & -2 & -32 & -40 \\ & \downarrow & -4 & 12 & 40 \\ \hline & 2 & -6 & -20 & 0 \end{array}$$

$x(x + 2)(2x^2 - 6x - 20)$
$x(x + 2)(2x^2 - 10x + 4x - 20)$
$x(x + 2)(2x(x - 5) + 4(x - 5))$
$x(x + 2)(2x + 4)(x - 5)$
Set each of these equal to zero to find the intervals and how each interval relates to zero (either positive or negative).
$x = 0$ and $x + 2 = 0$ and $2x + 4 = 0$ and $x - 5 = 0$
$x = -2, 0, 5$
This divides the domain of real numbers into 4 intervals:
$x \leq -2; -2 < x < 0; 0 \leq x \leq 5; x > 5$
Test for each interval:

$\boldsymbol{x \leq -2}$**:**
$f(-3) = -3((-3) + 2)(2(-3) + 4)((-3) - 5)$
$= -3(-1)(-2)(-8)$
$= 48$ Positive

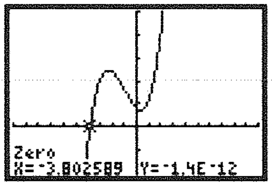
$\boldsymbol{-2 < x < 0}$**:**
$f(-1) = -1((-1) + 2)(2(-1) + 4)((-1) - 5)$
$= -1(1)(2)(-6)$
$= 12$ Positive

$\boldsymbol{0 \leq x \leq 5}$**:**
$f(2) = 2((2) + 2)(2(2) + 4)((2) - 5)$
$= 2(4)(8)(-3)$
$= -192$ Negative

$\boldsymbol{x > 5}$**:**
$f(6) = 6((6) + 2)(2(6) + 4)((6) - 5)$
$= 6(8)(16)(1)$
$= 768$ Positive

The intervals of the function are negative when $x \in (0, 5)$, positive when $x \in (-\infty, -2), (-2, 0), (5, \infty)$.

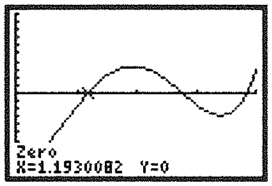
12. Rewrite as an equivalent inequality with 0 on one side; graph and determine the zeros of the polynomial function that is graphed.
$x^3 - 2x^2 + x - 3 \geq 2x^3 + x^2 - x + 1$
$0 \geq x^3 + 3x^2 - 2x + 4$



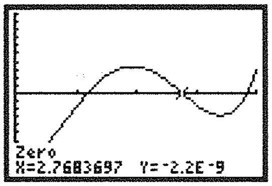
```
Zero
X=⁻3.802589  Y=⁻1.4E⁻12
```

$x \leq -3.81$

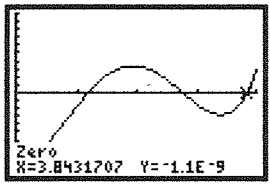
13. $f(x) = 1135x^4 - 8197x^3 + 15\,868x^2 - 2157x + 176\,608, 0 \leq x \leq 4$
To determine in which years the harvest was less than 185 000 m^3, write an inequality, rewrite an equivalent inequality with 0 on one side, and use a graphing calculator to solve.
$1135x^4 - 8197x^3 + 15\,868x^2 - 2157x + 176\,608 \leq 185\,000$
$1135x^4 - 8197x^3 + 15\,868x^2 - 2157x - 8392 \leq 0$



```
Zero
X=1.1930082  Y=0
```



```
Zero
X=2.7683697  Y=⁻2.2E⁻9
```



```
Zero
X=3.8431707  Y=⁻1.1E⁻9
```

Convert the zeros to years and months after January 1993. So the harvest is less than 185 000 m^3 between January 1993 and March 1994 and between October 1995 and October 1996.

14. a) $f(x) = x^2 - 2x + 3$
Average Rate (from $x = 2$ to $x = 7$):

$$\text{Average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
$$= \frac{f(7) - f(2)}{7 - 2}$$
$$= \frac{(7^2 - 2(7) + 3) - (2^2 - 2(2) + 3)}{5}$$

$$= \frac{38 - 3}{5}$$

$$= 7$$

Instantaneous Rate (at $x = 5$).

$$\text{Slope} = \frac{f(a + h) - f(a)}{h}, \text{ where } a = 5; h = 0.001$$

$$= \frac{f(5.001) - f(5)}{0.001}$$

$$= \frac{(5.001^2 - 2(5.001) + 3) - (5^2 - 2(5) + 3)}{0.001}$$

$$= \frac{18.008\,001 - 18}{0.001}$$

$$\doteq 8$$

b) $h(x) = (x - 3)(2x + 1)$

Average Rate (from $x = 2$ to $x = 7$):

$$\text{Average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(7) - f(2)}{7 - 2}$$

$$h = \frac{((7 - 3)(2(7) + 1))}{5} - \frac{((2 - 3)(2(2) + 1))}{5}$$

$$= \frac{60 - (-5)}{5}$$

$$= 13$$

Instantaneous Rate (at $x = 5$).

$$\text{Slope} = \frac{f(a + h) - f(a)}{h}, \text{ where } a = 5; h = 0.001$$

$$= \frac{f(5.001) - f(5)}{0.001}$$

$$= \frac{((5.001 - 3)(2(5.001) + 1))}{0.001} - \frac{((5 - 3)(2(5) + 1))}{0.001}$$

$$= \frac{22.015\,002 - 22}{0.001}$$

$$\doteq 15$$

c) $g(x) = 2x^3 - 5x$

Average Rate (from $x = 2$ to $x = 7$):

$$\text{Average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(7) - f(2)}{7 - 2}$$

$$= \frac{(2(7)^3 - 5(7)) - (2(2)^3 - 5(2))}{5}$$

$$= \frac{651 - 6}{5}$$

$$= 129$$

Instantaneous Rate (at $x = 5$).

$$\text{Slope} = \frac{f(a + h) - f(a)}{h}, \text{ where } a = 5; h = 0.001$$

$$= \frac{f(5.001) - f(5)}{0.001}$$

$$= \frac{(2(5.001)^3 - 5(5.001)) - (2(5)^3 - 5(5))}{0.001}$$

$$= \frac{225.145\,03 - 225}{0.001}$$

$$\doteq 145$$

d) $v(x) = -x^4 + 2x^2 - 5x + 1$

Average Rate (from $x = 2$ to $x = 7$):

$$\text{Average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(7) - f(2)}{7 - 2}$$

$$= \frac{(-(7)^4 + 2(7)^2 - 5(7) + 1)}{5} - \frac{(-(2)^4 + 2(2)^2 - 5(2) + 1)}{5}$$

$$= \frac{-2337 - (-17)}{5}$$

$$= -464$$

Instantaneous Rate (at $x = 5$).

$$\text{Slope} = \frac{f(a + h) - f(a)}{h}, \text{ where } a = 5; h = 0.001$$

$$= \frac{f(5.001) - f(5)}{0.001}$$

$$= \frac{(-(5.001)^4 + 2(5.001)^2 - 5(5.001) + 1)}{0.001} - \frac{(-(5)^4 + 2(5)^2 - 5(5) + 1)}{0.001}$$

$$= \frac{-599.485\,148 - (-599)}{0.001}$$

$$\doteq -485$$

15. By examining the graph, the instantaneous rate of change is positive when $-1 < x < 1$, negative when $x < -1$ or $x > 1$, and zero at $x = -1, 1$.

16. $h(t) = -5t^2 + 25$

a) $-5t^2 + 25 = 0$

$$-5t^2 = -25$$

$$t^2 = 5$$

$$t \doteq 2.2$$

The object hits the ground at 2.2 seconds.

b) Average rate from $t = 0$ to $t = 2.2$:

$$\text{Average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(2.2) - f(0)}{2.2 - 0}$$

$$= \frac{(-5(2.2)^2 + 25) - (-5(0)^2 + 25)}{2.2}$$

$$= \frac{0.8 - 25}{2.2}$$

$$= -11 \text{ m/s}$$

c) Instantaneous rate at $t = 2.2$

$\text{Slope} = \frac{f(a + h) - f(a)}{h}$, where $a = 2.2$;

$h = 0.001$

$$= \frac{f(2.201) - f(2.2)}{0.001}$$

$$= \frac{(-5(2.201)^2 + 25) - (-5(2.2)^2 + 25)}{0.001}$$

$$= \frac{0.777\,995 - 0.8}{0.001}$$

$$\doteq -22 \text{ m/s}$$

17. $f(x) = 2x^3 + 3x - 1$

a) $\text{Average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

$$= \frac{f(3.0001) - f(3)}{3.001 - 3}$$

$$= \frac{(2(3.0001)^3 + 3(3.0001) - 1)}{0.0001}$$

$$- \frac{(2(3)^3 + 3(3) - 1)}{0.0001}$$

$$= \frac{62.005\,700\,18 - 62}{0.0001}$$

$$\doteq 57.002$$

b) $\text{Average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

$$= \frac{f(3) - f(2.9999)}{3 - 2.9999}$$

$$= \frac{(2(3)^3 + (3) - 1) - (2(2.9999)^3}{0.0001}$$

$$+ \frac{3(2.9999) - 1)}{0.0001}$$

$$= \frac{62 - 61.994\,300\,18}{0.0001}$$

$$\doteq 56.988$$

c) Both approximate the instantaneous rate of change at $x = 3$.

Instantaneous rate at $x = 3$

$\text{Slope} = \frac{f(a + h) - f(a)}{h}$, where $a = 3$; $h = 0.001$

$$= \frac{f(3.001) - f(3)}{0.001}$$

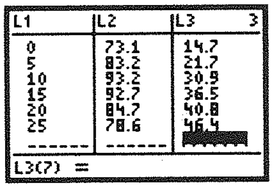
$$= \frac{(2(3.001)^3 + 3(3.001) - 1)}{0.001}$$

$$- \frac{(2(3)^3 + 3(3) - 1)}{0.001}$$

$$= \frac{62.057018 - 62}{0.001}$$

$$\doteq 57$$

18. a) Enter the data into a graphing calculator and then use the calculator to determine a cubic function for both sets of data. Let the independent variable represent the number of years since 1975, so $x = 0$ corresponds to 1975.

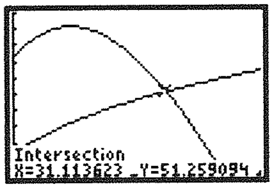


L1	L2	L3 3
0	73.1	14.7
5	83.2	21.7
10	93.2	30.9
15	92.7	36.5
20	84.7	40.8
25	78.6	46.4
------	------	

L3(7) =

The cubic function for the male data is:
$f(x) = 0.001x^3 - 0.162x^2 + 3.394x + 72.365$.
The cubic function for the female data is:
$g(x) = 0.0002x^3 - 0.026x^2 + 1.801x + 14.369$.

b) Use a graphing calculator to determine when $g(x) > f(x)$.



The cubic functions intersect at approximately 31.11, so more females will have lung cancer in 2006.

c) The rate was changing faster for females, on average. Looking only at 1975 and 2000, the incidence among males increased only 5.5 per 100 000, while the incidence among females increased by 31.7.

d) Between 1995 and 2000, the incidence among males decreased by 6.1 while the incidence among females increased by 5.6. Since 1998 is about halfway between 1995 and 2000, an estimate for the instantaneous rate of change in 1998 is the average rate of change from 1995 to 2000. The two rates of change are about the same in magnitude, but the rate for females is positive, while the rate for males is negative.

Chapter Self-Test, p. 242

1. $3x^3 - 3x^2 - 7x + 5 = x^3 - 2x^2 - 1$

$2x^3 - x^2 - 7x + 6 = 0$

$f(1) = 0$

$$\begin{array}{r|rrrr} 1 & 2 & -1 & -7 & 6 \\ & \downarrow & 2 & 1 & -6 \\ \hline & 2 & 1 & -6 & 0 \end{array}$$

$(x - 1)(2x^2 + x - 6) = 0$

$(x - 1)(2x^2 + 4x - 3x - 6) = 0$

$(x - 1)(2x(x + 2) - 3(x + 2)) = 0$

$(x - 1)(2x - 3)(x + 2) = 0$

$x - 1 = 0$ and $2x - 3 = 0$ and $x + 2 = 0$

$x = 1, \frac{3}{2}, -2$

2. a) By examining the graph, the function is positive when $x < -2$ and $0 < x < 2$, negative when $-2 < x < 0$, and $x > 2$ and zero at $x = -2, 0,$ and 2.

b) By examining the graph, the instantaneous rate of change is positive when $-1 < x < 1$, negative when $x < -1$ or $1 < x$, and zero at $x = -1, 1$.

$\text{m} = \frac{0 - 1}{2 - 1} = \frac{-1}{1} = -1$

3. a) Cost with card: $50 + 5n$;

Cost without card: $12n$

b) $12n > 50 + 5n$

$7n > 50$

$n > 7.14$

They must buy at least 8 pizzas to make the card worthwhile.

4. a) $4x - 5 < -2(x + 1)$

$4x - 5 < -2x - 2$

$6x < 3$

$x < \frac{1}{2}$

b) $-4 \le -(3x + 1) \le 5$

$-4 \le -3x - 1 \le 5$

$-4 + 1 \le -3x - 1 + 1 \le 5 + 1$

$-3 \le -3x \le 6$

$-2 \le x \le 1$

c) $(x + 1)(x - 5)(x + 2) > 0$

$(x + 1)(x - 5)(x + 2) = 0$

$x + 1 = 0$ and $x - 5 = 0$ and $x + 2 = 0$

$x = -2, -1, 5$

This divides the domain of real numbers into 4 intervals:

$x < -2;\ -2 < x < -1;\ -1 < x < 5;\ x > 5$

Test for each interval:

$\boldsymbol{x < -2:}$

$f(-3) = (-3 + 1)(-3 - 5)(-3 + 2) > 0$

$= (-2)(-8)(-1) > 0$

$= -16 > 0$ No

$\boldsymbol{-2 < x < -1:}$

$f(-1.5) = (-1.5 + 1)(-1.5 - 5)(-1.5 + 2) > 0$

$= (-0.5)(-6.5)(0.5) > 0$

$= 1.625 > 0$ Yes

$\boldsymbol{-1 < x < 5:}$

$f(0) = (0 + 1)(0 - 5)(0 + 2) > 0$

$= (1)(-5)(2) > 0$

$= -10 > 0$ No

$\boldsymbol{x > 5:}$

$f(6) = (6 + 1)(6 - 5)(6 + 2) > 0$

$= (7)(1)(8) > 0$

$= 56 > 0$ Yes

The intervals are $-2 < x < -1$ or $x > 5$.

d) $(2x - 4)^2(x + 3) \ge 0$

$2x - 4 = 0$ and $x + 3 = 0$

$x = -3, 2$

This divides the domain of real numbers into 3 intervals:

$x < -3;\ -3 \le x < 2;\ x \ge 2$

Test for each interval:

$\boldsymbol{x > -3:}$

$f(-4) = (2(-4) - 4)^2((-4) + 3) \ge 0$

$= (-12)^2(-1) \ge 0$

$= -144 \ge 0$ No

$\boldsymbol{-3 \le x < 2:}$

$f(0) = (2(0) - 4)^2((0) + 3) \ge 0$

$= (-4)^2(3) \ge 0$

$= 48 \ge 0$ Yes

$\boldsymbol{x \ge 2:}$

$f(3) = (2(3) - 4)^2((3) + 3) \ge 0$

$= (2)^2(6) \ge 0$

$= 24 \ge 0$ Yes

The interval is $x \ge -3$.

5. $h(t) = -5t^2 + 20t + 15$

a) $h(0) = -5(0)^2 + 20(0) + 15 = 15$

15 metres

b) $-5t^2 + 20t + 15 = 0$

$t = \frac{-20 \pm \sqrt{(-20)^2 - 4(-5)(15)}}{2(-5)}$

$= \frac{-20}{-10} \pm \frac{\sqrt{400 - -300}}{-10}$

$= 2 \pm 2.6$

$= 4.6, -0.6$

Since time cannot be negative, it will take 4.6 seconds.

c) Average rate of change from $t = 0$ to $t = 4.6$

$$\text{Average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(4.6) - f(0)}{4.6 - 0}$$

$$= \frac{(-5(4.6)^2 + 20(4.6) + 15)}{4.6}$$

$$- \frac{(-5(0)^2 + 20(0) + 15)}{4.6}$$

$$= \frac{1.2 - 15}{4.6}$$

$$= -3 \text{ m/s}$$

6. $f(x) = x^3 + x^2 + 1$

a) The slope at $x = 1$:

$$\text{Slope} = \frac{f(a + h) - f(a)}{h}, \text{ where } a = 1; h = 0.001$$

$$= \frac{f(1.001) - f(1)}{0.001}$$

$$= \frac{(1.001^3 + 1.001^2 + 1) - (1^3 + 1^2 + 1)}{0.001}$$

$$= \frac{3.005\,004\,001 - 3}{0.001}$$

$$\doteq 5$$

b) $f(1) = 1^3 + 1^2 + 1 = 1 + 1 + 1 = 3$

The coordinates are $(1, 3)$.

c) $y = 5x + b$

$3 = 5(1) + b$

$-2 = b$

$y = 5x - 2$

7. Since all the exponents are even and all the coefficients are positive, all values of the function are positive and greater than or equal to 4 for all real numbers x.

8. a) $\{x \in \mathbf{R} \mid -2 < x < 7\}$

b) $-5 < 2x - 1 < 13$

$-5 + 1 < 2x - 1 + 1 < 13 + 1$

$-4 < 2x < 14$

$-2 < x < 7$

9. $(x)(x)(x + 13) = 60$

$x^2(x + 13) = 60$

$x^3 + 13x^2 - 60 = 0$

$f(2) = 0$

$$\begin{array}{r|rrrr} 2 & 1 & 13 & 0 & -60 \\ & \downarrow & 2 & 30 & 60 \\ \hline & 1 & 15 & 30 & 0 \end{array}$$

$(x - 2)(x^2 + 15x + 30) = 0$

The roots of $x^2 + 15x + 30$ are both negative. Since the dimensions cannot be negative, $x - 2$ is the root that we use.

$x - 2 = 0$

$x = 2$

The dimensions are 2 cm by 2 cm by 15 cm.

b) Answers may vary. For example:

$$3x + 1 \le 4 + x$$
$$2x \le 3$$
$$x \le \frac{3}{2}$$

9. a) $\{x \in \mathbf{R} \mid -6 \le x \le 4\}$

b)
$$-13 \le 2x - 1 \le 7$$
$$-13 + 1 \le 2x - 1 + 1 \le 7 + 1$$
$$-12 \le 2x \le 8$$
$$-6 \le x \le 4$$

10. Attempting to solve $x - 3 < 3 - x < x - 5$ yields $3 > x > 4$, which has no solution. Solving $x - 3 > 3 - x > x - 5$ yields $3 < x < 4$.

11. a) $\frac{1}{2}x + 1 < 3$

b) $x < 4$

c)
$$\frac{1}{2}x + 1 < 3$$
$$\frac{1}{2}x < 2$$
$$x < 4$$

12. a) $18 \le \frac{5}{9}(F - 32) \le 22$

b)
$$18 \le \frac{5}{9}(F - 32) \le 22$$
$$9(18) \le 9\left(\frac{5}{9}(F - 32)\right) \le 9(22)$$
$$162 \le 5(F - 32) \le 198$$
$$\frac{162}{5} \le \frac{5(F - 32)}{5} \le \frac{198}{5}$$
$$32.4 \le F - 32 \le 39.6$$
$$32.4 + 32 \le F - 32 + 32 \le 39.6 + 32$$
$$64.4 \le F \le 71.6$$

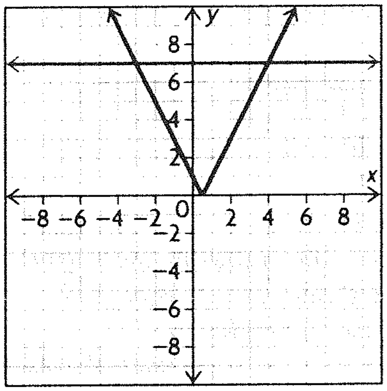
13.
$$0.50 + 0.10x \le 2.00$$
$$0.10x \le 1.50$$
$$x \le 15$$

The volunteers can talk for the initial 3 minutes plus an additional 15 minutes, or a total of 18 minutes.

14. a)
$$C = \frac{5}{9}(F - 32)$$
$$\frac{9}{5}C = F - 32$$
$$\frac{9}{5}C + 32 = F$$

b)
$$\frac{9}{5}C + 32 > C$$
$$\frac{4}{5}C > -32$$
$$C > -32\left(\frac{5}{4}\right)$$
$$C > -40$$

15. a)

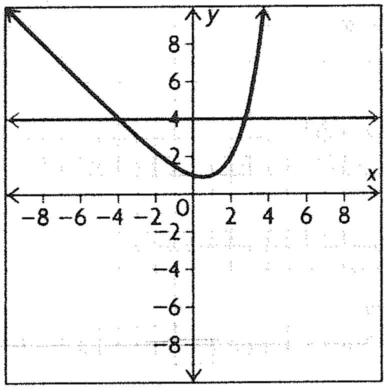


b) $-3 < x < 4$

16. The solution will always have an upper and lower bound due to the manner in which the inequality is solved. The only exception to this is when there is no solution set.

17. a) Isolating x is very hard.

b) A graphical approach as described in the lesson yields a solution of $x > 2.75$ (rounded to two places).



18. a) maintained

b) Maintained if both positive; switched if both negative; varies if one positive and one negative.

c) maintained

d) switched

e) Switched unless one is positive and the other is negative, in which case it is maintained. (If either side is zero, it becomes undefined.)

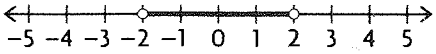
f) Maintained, except that < and > become ≤ and ≥, respectively.

g) Maintained, but it is undefined for negative numbers.

19. a) $x^2 < 4$

The solutions to this inequality are numbers that have a square less than 4.

The solution can be written as $\{x \in \mathbf{R} | -2 < x < 2\}$ or $(-2, 2)$.



b) $4x^2 + 5 \geq 41$

$4x^2 \geq 36$

$x^2 \geq 9$

The solutions to this inequality are numbers that have a square greater than or equal to 9.

The solution can be written as $\{x \in \mathbf{R} | x \leq -3 \text{ or } x \geq 3\}$. In interval notation the solution is $(-\infty, -3]$ or $[3, \infty)$.

c) $|2x + 2| < 8$

Consider two cases.

If $2x + 2 \geq 0$, then $|2x + 2| = 2x + 2$.

$2x + 2 < 8$

$2x < 6$

$x < 3$

If $2x + 2 < 0$, the $|2x + 2| = -(2x + 2)$.

$-(2x + 2) < 8$

$2x + 2 > -8$

$2x > -10$

$x > -5$

The solution can be written as $\{x \in \mathbf{R} | -5 < x < 3\}$ or $(-5, 3)$.

d) $-3x^2 \geq 81$

Divide by -3 and switch the direction of the inequality.

$x^3 \leq 27$

$x \leq 3$

The solution can be written as $\{x \in \mathbf{R} | x \leq 3\}$ or $(-\infty, 3]$.

Mid-Chapter Review, p. 218

1. a) $0 = -2x^3(2x - 5)(x - 4)^2$

$-2x^3 = 0$ and $2x - 5 = 0$ and $(x - 4)^2 = 0$

$x = 0, \frac{5}{2}, 4$

b) $0 = (x^2 + 1)(2x + 4)(x + 2)$

$x^2 + 1 = 0$ and $2x + 4 = 0$ and $x + 2 = 0$

$x = -2$

c) $x^3 - 4x^2 = 7x - 10$

$x^3 - 4x^2 - 7x + 10 = 0$

$f(1) = 0$

$$\begin{array}{r|rrrr} 1 & 1 & -4 & -7 & 10 \\ & \downarrow & 1 & -3 & -10 \\ \hline & 1 & -3 & -10 & 0 \end{array}$$

$(x - 1)(x^2 - 3x - 10) = 0$

$(x - 1)(x + 2)(x - 5) = 0$

$x - 1 = 0$ and $x + 2 = 0$ and $x - 5 = 0$

$x = 1, -2, 5$

d) $0 = (x^2 - 2x - 24)(x^2 - 25)$

$x^2 - 2x - 24 = 0$ and $x^2 - 25 = 0$

$(x + 4)(x - 6) = 0$ and $x^2 = 25$

$x + 4 = 0$ and $x - 6 = 0$ and $x = \pm\sqrt{25}$

$x = -4, 6, 5, -5$

e) $0 = (x^3 + 2x^2)(x + 9)$

$0 = x^2(x + 2)(x + 9)$

$x^2 = 0$ and $x + 2 = 0$ and $x + 9 = 0$

$x = 0, -2, -9$

f) $-x^4 = -13x^2 + 36$

$0 = x^4 - 13x^2 + 36$

$(x^2 - 9)(x^2 - 4) = 0$

$x^2 - 9 = 0$ and $x^2 - 4 = 0$

$x^2 = 9$ and $x^2 = 4$

$x = \pm\sqrt{9}$ and $x = \pm\sqrt{4}$

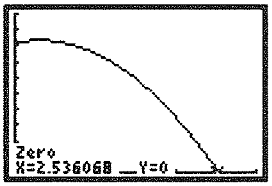
$x = 3, -3, 2, -2$

2. a)
$$\begin{aligned} h(t) &= -5(t - 0.3)^2 + 25 \\ &= -5(t - 0.3)(t - 0.3) + 25 \\ &= -5(t^2 - 0.6t + 0.09) + 25 \\ &= -5t^2 + 3t - 0.45 + 25 \\ &= -5t^2 + 3t + 24.55 \end{aligned}$$

b)
$$\begin{aligned} h(0) &= -5(0)^2 + 3(0) + 24.55 \\ &= 24.55 \end{aligned}$$

The cliff is 24.55 metres high.

c) Graph $h(t)$ using a graphing calculator. Determine when $h(t) = 0$.



Jude hits the water after about 2.5 s.

CHAPTER 5

Rational Functions, Equations, and Inequalities

Getting Started, pp. 246–247

1. Examine the coefficient for each term. To factor each expression, you will need two factors of the coefficient of the third term whose sum is the coefficient of the second term. If the coefficient of the first term is not 1, then you will have to consider the factors of that coefficient too. If the expression doesn't factor easily you can use the quadratic formula.

a) $x^2 - 3x - 10$

$-5 \times 2 = -10; -5 + 2 = -3$

$(x - 5)(x + 2)$

b) $3x^2 + 12x - 15$

$3(x^2 + 4x - 5)$

$5 \times -1 = -5; 5 + -1 = 4$

$3(x + 5)(x - 1)$

c) $16x^2 - 49$

Notice that this is the difference of two squares. This means that the factored form of the expression will be $(4x - 7)(4x + 7)$.

d) $9x^2 - 12x + 4$

Notice that the coefficients of the first and third terms are squares. Since the coefficient of the second term is negative, the factors of the coefficient of the third term will also probably be negative. The factored form of the expression is $(3x - 2)(3x - 2)$.

e) $3a^2 + a - 30$

This is a trinomial of the form $ax^2 + bx + c$ where $a \neq 0$, and it has no common factor. The expression can be factored using decomposition by finding two numbers whose sum is 1 and whose product is $(3)(-30) = -90$. The numbers are -9 and 10. These numbers are used to decompose the middle term.

$3a^2 - 9a + 10a - 30$

$= 3a(a - 3) + 10(a - 3)$

$= (a - 3)(3a + 10)$

f) $6x^2 - 5xy - 21y^2$

This trinomial can again be factored by decomposition. Find two numbers whose sum is -5 and whose product is $(6)(-21) = -126$. The numbers are -14 and 9. These numbers are used to decompose the middle term.

$6x^2 - 14xy + 9xy - 21y^2$

$= 2x(3x - 7y) + 3y(3x - 7y)$

$= (2x + 3y)(3x - 7y)$

2. a) $\frac{12 - 8s}{4}$

$= \frac{4(3 - 2s)}{4}$

$= 3 - 2s$

b) $\frac{6m^2n^4}{18m^3n}$

$= \frac{6m^2n(n)^3}{6m^2n(3m)}$

$= \frac{n^3}{3m}$, m and $n \neq 0$

c) $\frac{9x^3 - 12x^2 - 3x}{3x}$

$= \frac{3x(3x^2 - 4x - 1)}{3x}$

$= 3x^2 - 4x - 1$

$x \neq 0$

d) $\frac{25x - 10}{5(5x - 2)^2}$

$= \frac{5(5x - 2)}{5(5x - 2)(5x - 2)}$

$= \frac{1}{5x - 2}$

$x \neq \frac{2}{5}$

e) $\frac{x^2 + 3x - 18}{9 - x^2}$

$= \frac{(x - 3)(x + 6)}{(3 - x)(3 + x)}$

$= \frac{-(3 - x)(x + 6)}{(3 - x)(3 + x)}$

$= -\frac{x + 6}{3 + x}$, $x \neq -3, 3$

f) $\dfrac{a^2 + 4ab - 5b^2}{2a^2 + 7ab - 15b^2}$

$= \dfrac{(a + 5b)(a - 1b)}{(2a - 3b)(a + 5b)}$

$= \dfrac{a - b}{a - 3b}, a \neq -5b, \dfrac{3b}{2}$

3. a) $\dfrac{3}{5} \times \dfrac{7}{9}$

$= \dfrac{1}{5} \times \dfrac{7}{3}$

$= \dfrac{7}{15}$

b) $\dfrac{2x}{5} \div \dfrac{x^2}{15}$

$= \dfrac{2x}{5} \times \dfrac{15}{x^2}$

$= \dfrac{2x}{1} \times \dfrac{3}{x^2}$

$= \dfrac{6x}{x^2}$

$= \dfrac{6}{x}$

$x \neq 0$

c) $\dfrac{x^2 - 4}{x - 3} \div \dfrac{x + 2}{12 - 4x}$

$= \dfrac{(x - 2)(x + 2)}{x - 3} \times \dfrac{4(3 - x)}{(x + 2)}$

$= \dfrac{(x - 2)}{x - 3} \times \dfrac{4(3 - x)}{1}$

$= \dfrac{4(3 - x)(x - 2)}{x - 3}$

$= \dfrac{-4x^2 + 20x - 6}{x - 3}$

$x \neq -2, 3$

d) $\dfrac{x^3 + 4x^2}{x^2 - 1} \times \dfrac{x^2 - 5x + 6}{x^2 - 3x}$

$= \dfrac{x^2(x + 4)}{(x - 1)(x + 1)} \times \dfrac{(x - 3)(x - 2)}{x(x - 3)}$

$= \dfrac{x^2(x + 4)}{(x - 1)(x + 1)} \times \dfrac{(x - 2)}{x}$

$= \dfrac{x^2(x + 4)(x - 2)}{x(x - 1)(x + 1)}$

$= \dfrac{x(x + 4)(x - 2)}{(x - 1)(x + 1)}$

$= \dfrac{x^3 + 2x - 8x}{x^2 - 1}, x \neq -1, 0, 1, 3$

4. a) $\dfrac{2}{3} + \dfrac{6}{7}$

$= \dfrac{7}{7} \times \dfrac{2}{3} + \dfrac{3}{3} \times \dfrac{6}{7}$

$= \dfrac{14}{21} + \dfrac{18}{21}$

$= \dfrac{32}{21}$

$= 1\dfrac{11}{21}$

b) $\dfrac{3x}{4} + \dfrac{5x}{6}$

$= \left(\dfrac{3}{3}\right)\left(\dfrac{3x}{4}\right) + \left(\dfrac{2}{2}\right)\left(\dfrac{5x}{6}\right)$

$= \dfrac{9x}{12} + \dfrac{10x}{12}$

$= \dfrac{19x}{12}$

c) $\dfrac{1}{x} + \dfrac{4}{x^2}$

$= \dfrac{x}{x} \times \dfrac{1}{x} + \dfrac{4}{x^2}$

$= \dfrac{x}{x^2} + \dfrac{4}{x^2}$

$= \dfrac{4 + x}{x^2}$

$x \neq 0$

d) $\dfrac{5}{x - 3} - \dfrac{2}{x}$

$= \dfrac{x}{x} \times \dfrac{5}{x - 3} - \dfrac{x - 3}{x - 3} \times \dfrac{2}{x}$

$= \dfrac{5x}{x^2 - 3x} - \dfrac{2x - 6}{x^2 - 3x}$

$= \dfrac{3x - 6}{x^2 - 3x}$

$x \neq 0, 3$

e) $\dfrac{2}{x - 5} + \dfrac{y}{x^2 - 25}$

$= \dfrac{2}{x - 5} + \dfrac{y}{(x - 5)(x + 5)}$

$= \left(\dfrac{x + 5}{x + 5}\right)\left(\dfrac{2}{x - 5}\right) + \dfrac{y}{(x - 5)(x + 5)}$

$= \dfrac{2x + 10}{(x + 5)(x - 5)} + \dfrac{y}{(x - 5)(x + 5)}$

$= \dfrac{2x + 10 + y}{(x + 5)(x - 5)}$

$$= \frac{2x + 10 + y}{(x^2 - 25)}$$

$x \neq 5, -5$

f) $\dfrac{6}{a^2 - 9a + 20} - \dfrac{8}{a^2 - 2a - 15}$

$$= \frac{6}{(a - 5)(a - 4)} - \frac{8}{(a - 5)(a + 3)}$$

$$= \left(\frac{a + 3}{a + 3}\right)\left(\frac{6}{(a - 5)(a - 4)}\right) - \left(\frac{a - 4}{a - 4}\right)\left(\frac{8}{(a - 5)(a + 3)}\right)$$

$$= \frac{6a + 18}{(a + 3)(a - 5)(a - 4)} - \frac{8a - 32}{(a + 3)(a - 5)(a + 3)}$$

$$= \frac{-2a + 50}{(a + 3)(a - 5)(a + 3)}$$

$x \neq -3, 4, 5$

5. a) $\dfrac{5x}{8} = \dfrac{15}{4}$

$$\begin{aligned} 4(5x) &= 8(15) \\ 20x &= 120 \\ \frac{20x}{20} &= \frac{120}{20} \\ x &= 6 \end{aligned}$$

b) $\dfrac{x}{4} + \dfrac{1}{3} = \dfrac{5}{6}$

$$\begin{aligned} \left(\frac{3}{3}\right)\left(\frac{x}{4}\right) + \left(\frac{4}{4}\right)\left(\frac{1}{3}\right) &= \frac{5}{6} \\ \frac{3x}{12} + \frac{4}{12} &= \frac{5}{6} \\ \frac{3x + 4}{12} &= \frac{5}{6} \\ 6(3x + 4) &= 12(5) \\ 18x + 24 &= 60 \\ 18x + 24 - 24 &= 60 - 24 \\ 18x &= 36 \\ \frac{18x}{18} &= \frac{36}{18} \\ x &= 2 \end{aligned}$$

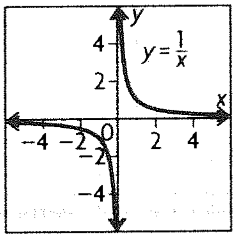
c) $\dfrac{4x}{5} - \dfrac{3x}{10} = \dfrac{3}{2}$

$$\begin{aligned} \left(\frac{2}{2}\right)\left(\frac{4x}{5}\right) - \frac{3x}{10} &= \frac{3}{2} \\ \frac{8x}{10} - \frac{3x}{10} &= \frac{3}{2} \\ \frac{5x}{10} &= \frac{3}{2} \\ 2(5x) &= 3(10) \\ 10x &= 30 \\ \frac{10x}{10} &= \frac{30}{10} \\ x &= 3 \end{aligned}$$

d) $\dfrac{x + 1}{2} + \dfrac{2x - 1}{3} = -1$

$$\begin{aligned} \left(\frac{3}{3}\right)\left(\frac{x + 1}{2}\right) + \left(\frac{2}{2}\right)\left(\frac{2x - 1}{3}\right) &= -1 \\ \frac{3x + 3}{6} + \frac{4x - 2}{6} &= -1 \\ \frac{7x + 1}{6} &= -1 \\ 6\left(\frac{7x + 1}{6}\right) &= 6(-1) \\ 7x + 6 &= -6 \\ 7x + 6 - 6 &= -6 - 6 \\ 7x &= -12 \\ \frac{7x}{7} &= \frac{-12}{7} \\ x &= -\frac{12}{7} \end{aligned}$$

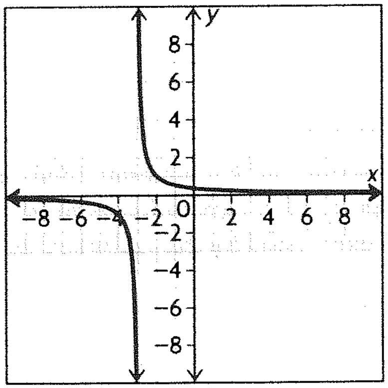
6.



The graph has vertical and horizontal asymptotes that follow the corresponding axis. The domain of the function is $\{x \in \mathbf{R} \mid x \neq 0\}$; the range of the function is $\{y \in \mathbf{R} \mid y \neq 0\}$.

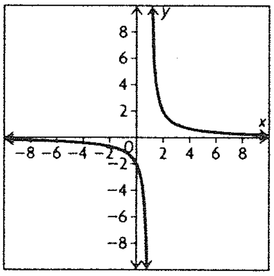
7. a) $f(x) = \dfrac{1}{x + 3}$

The function is translated 3 units to the left.



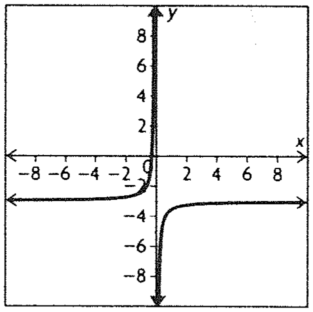
b) $f(x) = \dfrac{2}{x - 1}$

The graph has a vertical stretch by a factor of 2 and a horizontal translation 1 unit to the right.



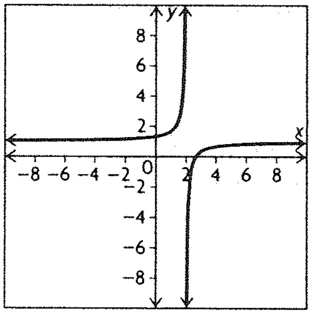
c) $f(x) = -\dfrac{1}{2x} - 3$

The graph has a reflection in the x-axis, vertical compression by a factor of $\frac{1}{2}$, and a vertical translation 3 units down.



d) $f(x) = \dfrac{2}{-3(x - 2)} + 1$

The graph has a reflection in the x-axis, a vertical compression by a factor of $\frac{2}{3}$, horizontal translation 2 units right, and a vertical translation 1 unit up.



8. Factor the expressions in the numerator and the denominator. Simplify each expression as necessary. Multiply the first expression by the reciprocal of the second.

$$\frac{9y^2 - 4}{4y - 12} \div \frac{9y^2 + 12y + 4}{18 - 6y}$$

$$= \frac{(3y + 2)(3y - 2)}{4(y - 3)} \div \frac{(3y + 2)(3y + 2)}{6(3 - y)}$$

$$= \frac{(3y + 2)(3y - 2)}{4(y - 3)} \times \frac{6(3 - y)}{(3y + 2)(3y + 2)}$$

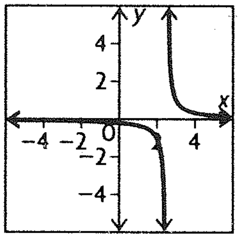
$$= -\frac{(3y - 2)(y - 3)}{2(y - 3)(3y + 2)}$$

$$= -\frac{3(3y - 2)}{2(3y + 2)}$$

5.1 Graphs of Reciprocal Functions, pp. 254–257

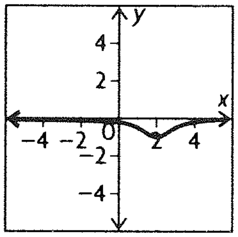
1. Graph each function and compare to the given graphs.

a)



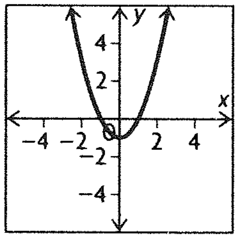
This is the graph of $y = \dfrac{1}{2x - 5}$. **C**

b)



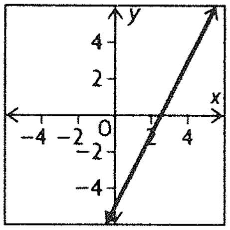
This is the graph of $y = \dfrac{1}{-(x - 2)^2 - 1}$. **A**

c)



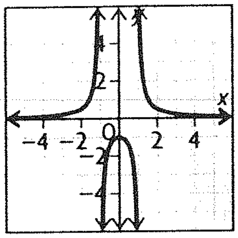
This is the graph of $y = x^2 - 1$. **D**

d)



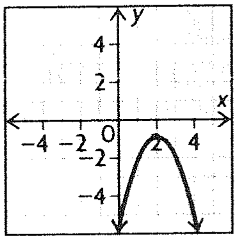
This is the graph of $y = 2x - 5$. **F**

e)



This is the graph of $y = \frac{1}{x^2 - 1}$. **B**

f)



This is the graph of $y = -(x - 2)^2 - 1$. **E**
A and E are reciprocals. B and D are reciprocals.
C and F are reciprocals.

2. The zeros of a function occur when $f(x) = 0$.

a) $f(x) = x - 6$
$0 = x - 6$
$6 = x$

The vertical asymptote of $g(x) = \frac{1}{(x - 6)}$ occurs at $x = 6$.

b) $f(x) = 3x + 4$
$0 = 3x + 4$
$0 - 4 = 3x$
$-\frac{4}{3} = x$

The vertical asymptote of $g(x) = \frac{1}{(3x + 4)}$ occurs at $x = -\frac{4}{3}$.

c) $f(x) = x^2 - 2x - 15$
$0 = x^2 - 2x - 15$
$0 = (x - 5)(x + 3)$
$0 = (x - 5)$ and $0 = (x + 3)$
$0 + 5 = x - 5 + 5$ and $0 - 3 = x + 3 - 3$
$5 = x$ and $-3 = x$. The vertical asymptotes of $g(x) = \frac{1}{x^2 - 2x - 15}$ occur at $x = 5$ and $x = -3$.

d) $f(x) = 4x^2 - 25$
$0 = 4x^2 - 25$
$0 = (2x + 5)(2x - 5)$
$0 = (2x + 5)$ and $0 = (2x - 5)$
$0 - 5 = 2x + 5 - 5$ and
$0 + 5 = 2x - 5 + 5$
$\frac{-5}{2} = \frac{2x}{2}$ and $\frac{5}{2} = \frac{2x}{2}$
$\frac{-5}{2} = x$ and $\frac{5}{2} = x$

The vertical asymptotes of $g(x) = \frac{1}{4x^2 - 25}$ occur at $x = -\frac{5}{2}$ and $x = \frac{5}{2}$.

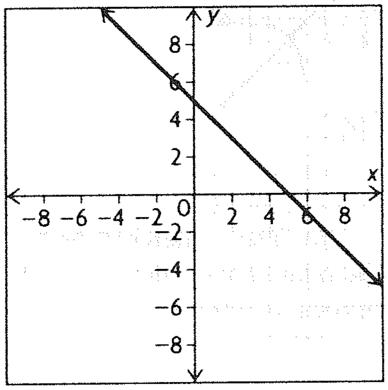
e) $f(x) = x^2 + 4$
$0 = x^2 + 4$
$-4 = x^2$

There are no real solutions to $-4 = x^2$. Therefore, there are no asymptotes for the function $\frac{1}{x^2 + 4}$.

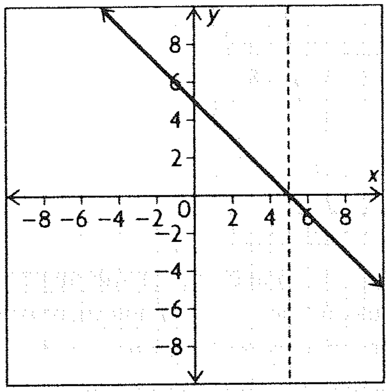
f) $f(x) = 2x^2 + 5x + 3$
$0 = 2x^2 + 5x + 3$
$0 = (2x + 3)(x + 1)$
$0 = (2x + 3)$ and $0 = (x + 1)$
$x = -1.5$ and $x = -1$

The asymptotes for $g(x) = \frac{1}{(2x^2 + 5x + 3)}$ are at $x = -1.5$ and -1.

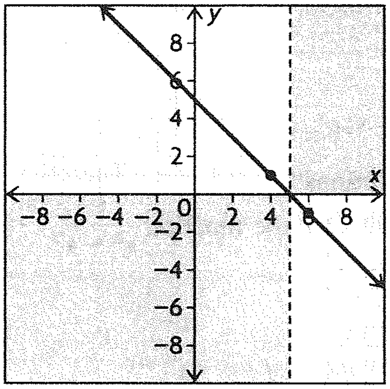
3. a) Graph the function $f(x) = 5 - x$. The y-intercept is $(0, 5)$ and the x-intercept is $(5, 0)$. Let $g(x)$ be the reciprocal function.



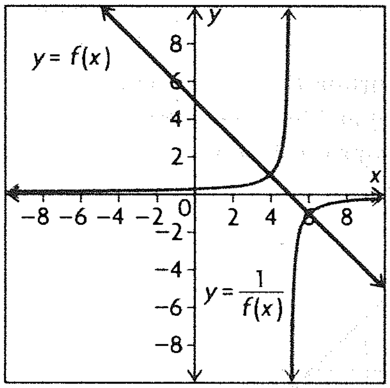
The zero of $f(x)$ is $x = 5$ and so that is where $g(x)$ will have a vertical asymptote.



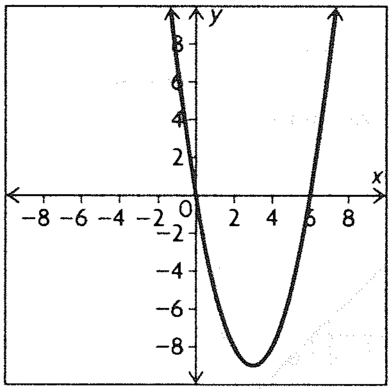
$f(x)$ is positive on $(-\infty, 5)$, negative on $(5, \infty)$, and always decreasing. Therefore, $g(x)$ positive on $(-\infty, 5)$, negative on $(5, \infty)$, and always increasing. $f(x) = 1$ at 4 and -1 at 6. The points of intersection for $f(x)$ and $g(x)$ will be at $(1, 4)$ and $(-1, 6)$.



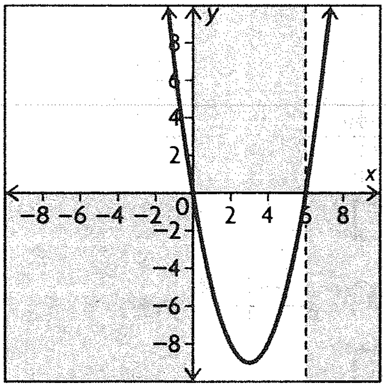
Use this information to draw the graph of $g(x)$.



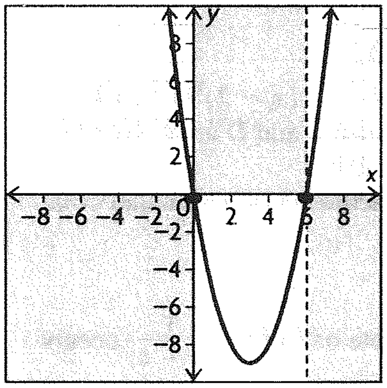
b) Graph $f(x) = x^2 - 6x$. The x-intercepts are at $x = 0$ and $x = 6$. The minimum occurs at $(3, -9)$. Let $g(x)$ be the reciprocal function.



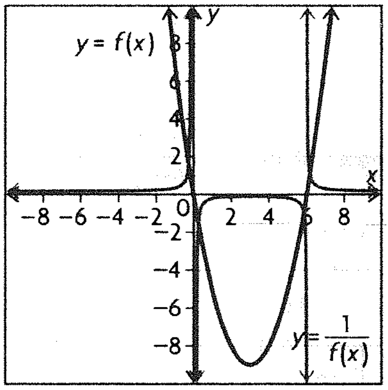
$f(x)$ is positive on $(-\infty, 0)$ and $(6, \infty)$ and negative on $(0, 6)$. It is decreasing on $(-\infty, 3)$ and increasing on $(3, \infty)$. Therefore, $g(x)$ is positive on $(-\infty, 0)$ and $(6, \infty)$ and negative on $(0, 6)$. $g(x)$ is increasing on $(-\infty, 3)$ and decreasing on $(3, \infty)$. There are vertical asymptotes at $x = 0$ and 6.



$f(x) = 1$ at $x = 6.1$ and -0.2. $f(x) = -1$ at $x = 5.8$ and 0.2. These will be points of intersection for $f(x)$ and $g(x)$.



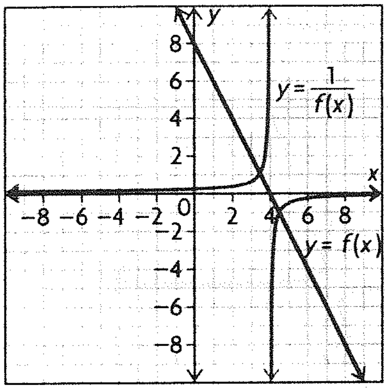
Use this information to graph $g(x)$.



4. a) Complete the table with the reciprocals of the values of $f(x)$.

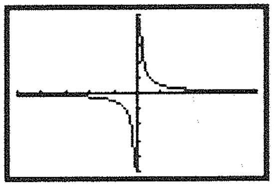
x	−4	−3	−2	−1	0	1	2	3	4	5	6	7
f(x)	16	14	12	10	8	6	4	2	0	−2	−4	−6
$\frac{\mathbf{1}}{\mathbf{f(x)}}$	$\frac{1}{16}$	$\frac{1}{14}$	$\frac{1}{12}$	$\frac{1}{10}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$	undefined	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{6}$

b)

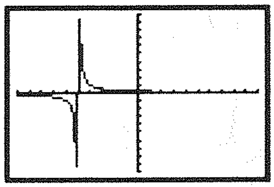


c) The first differences in the values of $f(x)$ are a constant, -2, so the function is linear. Since the y-intercept is 8, the equation for the function is $y = -2x + 8$. The equation for the reciprocal function is $y = \frac{1}{-2x + 8}$.

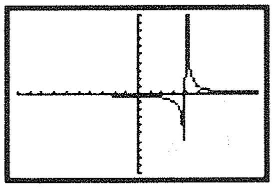
5. a) $y = \frac{1}{2x}$; vertical asymptote at $x = 0$



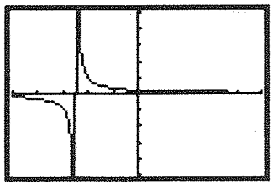
b) $y = \frac{1}{x + 5}$; vertical asymptote at $x = -5$



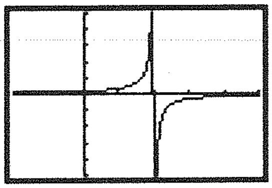
c) $y = \frac{1}{x - 4}$; vertical asymptote at $x = 4$



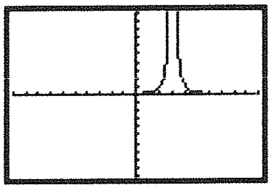
d) $y = \frac{1}{2x + 5}$; vertical asymptote at $x = -\frac{5}{2}$



e) $y = \frac{1}{-3x + 6}$; vertical asymptote at $x = 2$

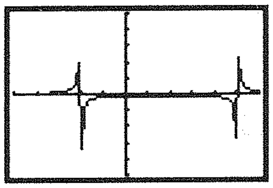


f) $y = \frac{1}{(x - 3)^2}$; vertical asymptote at $x = 3$



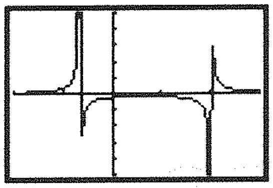
g) $y = \frac{1}{x^2 - 3x - 10}$

$x^2 - 3x - 10 = (x - 5)(x + 2)$, so the vertical asymptotes are at $x = -2$ and $x = 5$.

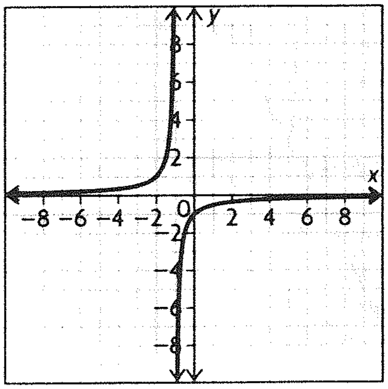


h) $y = \frac{1}{3x^2 - 4x - 4}$

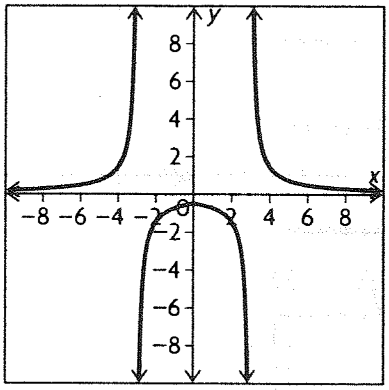
$3x^2 - 4x - 4 = (x - 2)(3x + 2)$, so the vertical asymptotes are at $x = -\frac{2}{3}$ and $x = 2$.



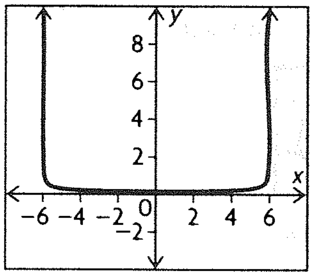
6. a) $f(x) = -x - 1$, so the reciprocal function is $y = -\frac{1}{x + 1}$.



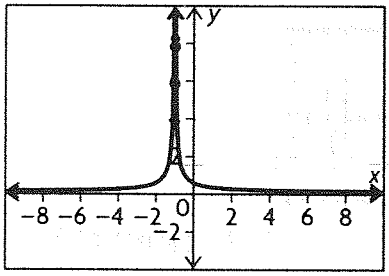
b) $f(x) = \frac{2}{3}|x| - 2$, so the reciprocal function is $y = \dfrac{1}{\frac{2}{3}|x| - 2}$.



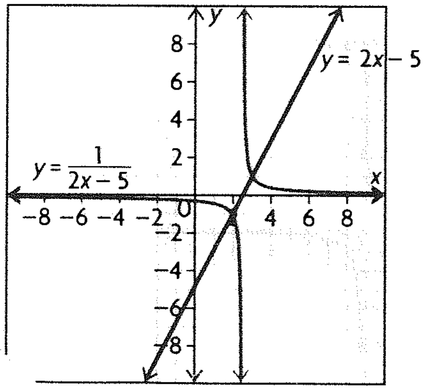
c) $f(x) = \sqrt{36 - x^2}$, so the reciprocal function is $y = \dfrac{1}{\sqrt{36 - x^2}}$.



d) $f(x) = 2|x + 1|$, so the reciprocal function is $y = \dfrac{1}{2|x + 1|}$.

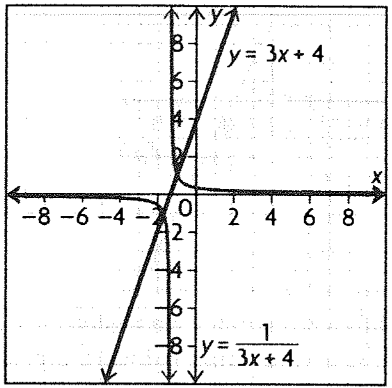


7. a)



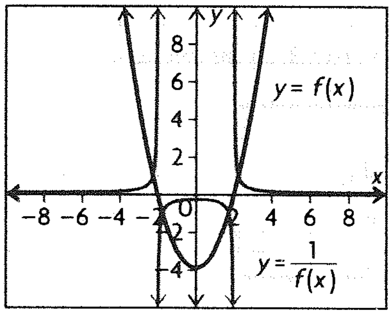
For the reciprocal function: D = $\{x \in \mathbf{R} \mid x \neq \frac{5}{2}\}$, R = $\{y \in \mathbf{R} \mid y \neq 0\}$

b)

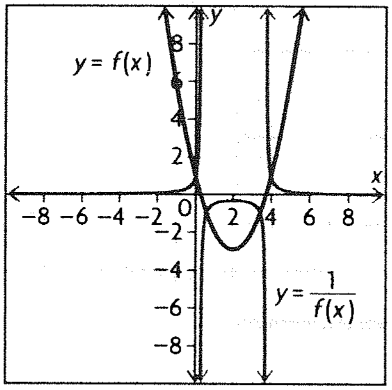


For the reciprocal function: D = $\{x \in \mathbf{R} \mid x \neq -\frac{4}{3}\}$, R = $\{y \in \mathbf{R} \mid y \neq 0\}$

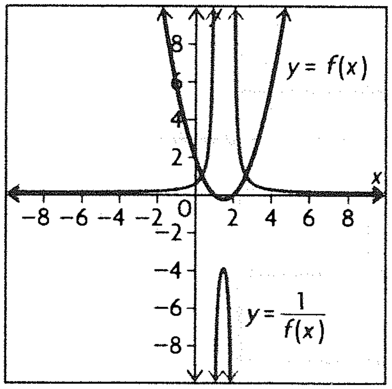
8. a)



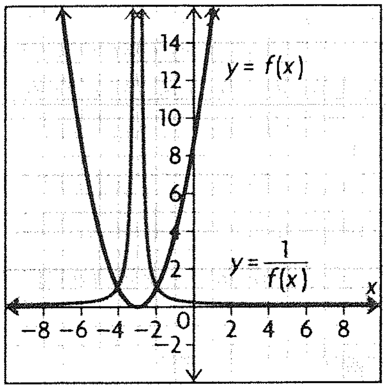
b)



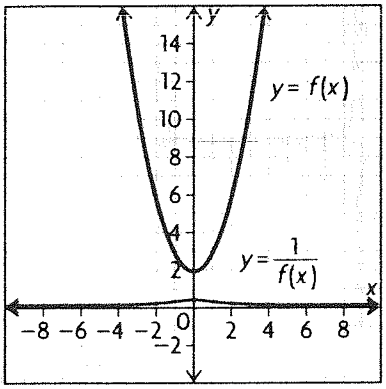
c)



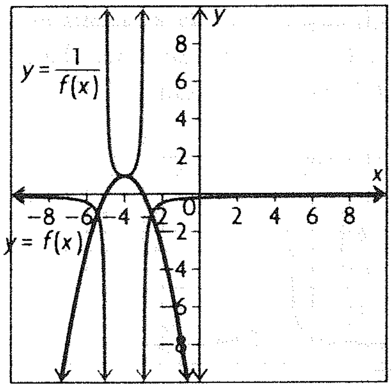
d)



e)



f)

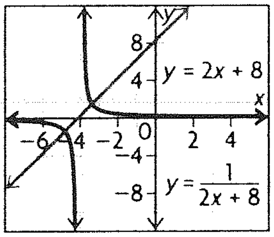


9. a) The domain and range of $f(x)$ are all real numbers because x is a linear polynomial. Also, because $f(x)$ is a linear polynomial with a positive slope, it is always increasing. The y-intercept is $2(0) + 8 = 8$. The x-intercept is

$$0 = 2x + 8$$
$$-8 = 2x$$
$$-4 = x$$

The y-intercept is $(0, 8)$ and x-intercept is $(-4, 0)$. Use the x-intercept to determine the intervals upon which the function is negative and positive. Because the function is always increasing, the function is negative on $(-\infty, -4)$ and positive on $(-4, -\infty)$.

$g(x) = \frac{1}{f(x)}$ so the reciprocal of $f(x)$ is $\frac{1}{2x + 8}$.

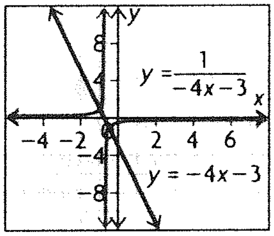


b) The domain and range of $f(x)$ are all real numbers because x is a linear polynomial. Also, because $f(x)$ is a linear polynomial with a negative slope, it is always decreasing. The y-intercept is $-4(0) - 3 = -3$. The x-intercept is

$$0 = -4x - 3$$
$$3 = -4x$$
$$-\frac{3}{4} = x$$

The y-intercept is $(0, -3)$ and x-intercept is $(-\frac{3}{4}, 0)$. Use the x-intercept to determine the intervals upon which the function is negative and positive. Because the function is always decreasing, the function is positive on $(-\infty, -\frac{3}{4})$ and negative on $(-\frac{3}{4}, \infty)$.

$g(x) = \frac{1}{f(x)}$ so the reciprocal of $f(x)$ is $\frac{1}{-4x - 3}$.



c) The y-intercept is $f(0) = 0^2 - 0 - 12 = 12$. The x-intercepts are:

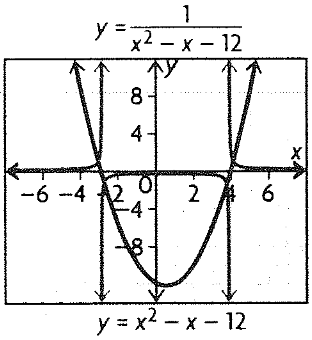
$$f(x) = x^2 - x - 12$$
$$0 = x^2 - x - 12$$
$$0 = (x - 4)(x + 3)$$
$$0 = (x - 4) \text{ and } 0 = (x + 3)$$

The x-intercepts are 4 and -3. The vertex is at 0.5.

$$f(0.5) = (0.5)^2 - (0.5) - 12$$
$$= 0.25 - 0.5 - 12$$
$$= -12.25$$

The leading coefficient of the function is positive and so the function opens up; the vertex $(0.5, -12.25)$ is a local minimum. The domain is all real numbers and the range is all real numbers greater than -12.25. The function is decreasing on $(-\infty, 0.5)$ and increasing on $(0.5, \infty)$. The function is positive on $(-\infty, -3)$ and on $(4, \infty)$. The function is negative on $(-3, 4)$.

$g(x) = \frac{1}{f(x)}$ so the reciprocal of $f(x)$ is $\frac{1}{x^2 - x - 12}$.



d) The y-intercept is $-2(0)^2 + 10(0) - 12$. The x-intercepts are:

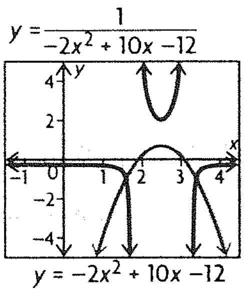
$$\begin{aligned}
f(x) &= -2x^2 + 10x - 12\\
0 &= -2x^2 + 10x - 12\\
0 &= -2(x^2 - 5x + 6)\\
0 &= -2(x - 3)(x - 2)\\
\frac{0}{-2} &= \frac{-2(x - 3)(x - 2)}{-2}\\
0 &= (x - 3)(x - 2)\\
0 &= (x - 3) \text{ and } 0 = (x - 2)\\
x &= 3 \text{ and } x = 2
\end{aligned}$$

The vertex occurs at $x = 2.5$.

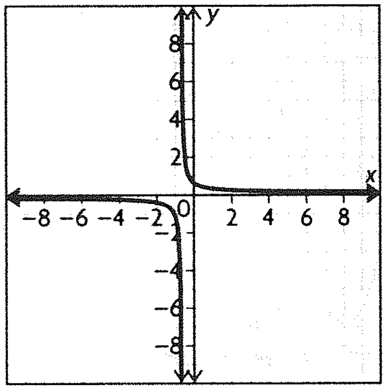
$$\begin{aligned}
f(2.5) &= -2(2.5)^2 + 10(2.5) - 12\\
&= -12.5 + 25 - 12\\
&= 0.5
\end{aligned}$$

The vertex is at $(2.5, 0.5)$. Because the leading coefficient is negative, the vertex is a local maximum. This means that $f(x)$ is increasing on $(-\infty, 2.5)$ and decreasing on $(2.5, \infty)$. The domain of $f(x)$ is all real numbers and the range is all numbers less than 2.5. The function is negative on $(-\infty, 2)$ and $(3, \infty)$ and positive on $(2, 3)$.

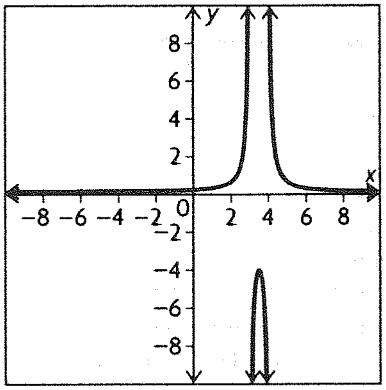
$g(x)\frac{1}{f(x)}$ so the reciprocal of $f(x)$ is $\frac{1}{-2x^2 + 10x - 12}$.



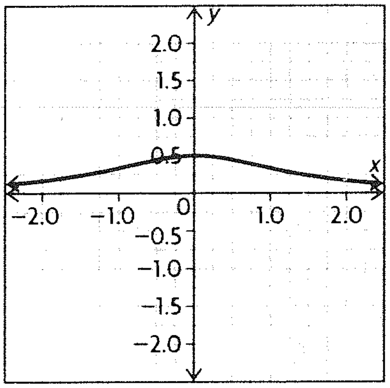
10. Answers may vary. For example: A reciprocal function creates a vertical asymptote when the denominator is equal to 0 for a specific value of x. Consider $\frac{1}{ax + b}$. For this expression, there is always some value of x that is $\frac{-b}{a}$ that will result in a vertical asymptote for the function. This is a graph of $y = \frac{1}{3x + 2}$ and the vertical asymptote is at $x = -\frac{2}{3}$.



Consider the function $\frac{1}{(x - 3)(x - 4)}$. The graph of the quadratic function in the denominator crosses the x-axis at 3 and 4 and therefore will have vertical asymptotes at 3 and 4 in the graph of the reciprocal function.



However, a quadratic function, such as $x^2 + c$, which has no real zeros, will not have a vertical asymptote in the graph of its reciprocal function. For example, this is the graph of $y = \frac{1}{x^2 + 2}$.



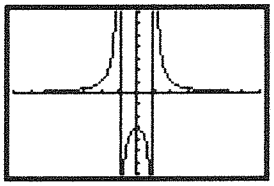
11. There are vertical asymptotes at $x = -1$ and $x = 1$, so the denominator is $(x + 1)(x - 1)$ or $x^2 - 1$.

$$y = \frac{k}{x^2 - 1}$$

Since $(0, -3)$ is on the graph,

$$-3 = \frac{k}{0 - 1}, \text{ or } k = 3.$$

The equation is $y = \frac{3}{x^2 - 1}$.



12. a) Substitute 20 into the formula to find the number of bacteria that will be left after 20 seconds.

$$b(t) = 10\,000\frac{1}{t}$$

$$b(20) = 10\,000\frac{1}{20}$$

$$b(20) = 500$$

There will be 500 bacteria after 20 seconds.

b) Find the time t for which there will be 5000 bacteria left.

$$5000 = 10\,000\frac{1}{t}$$

$$\frac{5000}{10\,000} = \frac{1}{10\,000} \times 10\,000 \times \frac{1}{t}$$

$$\frac{1}{2} = \frac{1}{t}$$

$$2 = t$$

c) Determine the time at which there will be only 1 bacteria left.

$$1 = 10\,000\frac{1}{t}$$

$$\frac{1}{10\,000} = \frac{1}{10\,000} \times 10\,000 \times \frac{1}{t}$$

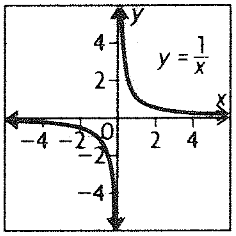
$$\frac{1}{10\,000} = \frac{1}{t}$$

$$10\,000 = t$$

d) If you were to use a value of t that was less than one, the equation would tell you that the number of bacteria was increasing as opposed to decreasing. Also, after time $t = 10\,000$ the formula indicates that there is a smaller and smaller fraction of 1 bacteria left.

e) Because the time less than 1 second and greater than 10 000 is inaccurate, the domain should be $\{x \in \mathbf{R} \mid 1 < x < 10\,000\}$. Because you cannot have a negative number of bacteria and because there are 10 000 bacteria at the beginning of the trial, the range should be $\{y \in \mathbf{R} \mid 1 < y < 10\,000\}$.

13. a) Think of the general shape of the reciprocal of a linear function—it will have a horizontal and a vertical asymptote.



The vertical asymptote occurs when the original function is equal to 0.

$$0 = x + n - n$$

$$0 - n = x + n - n$$

$$-n = x$$

Because there is a vertical asymptote at $x = -n$, the domain does not include $-n$.

Domain $= \{x \in \mathbf{R} \mid x \neq -n\}$.

Examine the end behaviours of the function to find the vertical asymptote. As $x \to \infty$, $g(x)$ will get closer and closer to 0 because no matter how small n is $x + n$ will always get closer to ∞. The same holds true for $g(x)$ as $x \to -\infty$. Range $= \{y \in \mathbf{R} \mid y \neq 0\}$.

b) The vertical asymptote occurs at $x = -n$. Changes in n in the $f(x)$ family cause changes in the y-intercept—an increase in n causes the intercept to move up the y-axis and a decrease causes it to move down the y-axis. Changes in n in the $g(x)$ family cause changes in the vertical asymptote of the function—an increase in n causes the asymptote to move down the x-axis and a decrease in n causes it to move up the x-axis.

c) The functions will intersect when $f(x) = g(x)$.

$$x + n = \frac{1}{x + n}$$

$$(x + n)(x + n) = \frac{1}{x + n}(x + n)$$

$$(x + n)^2 = 1$$

$$\sqrt[2]{(x + n)^2} = \sqrt{1}$$

$$x + n = 1 \text{ or } x + n = -1$$

$$x = 1 - n \text{ or } x = -1 - n$$

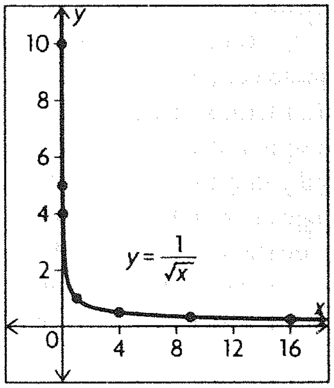
The functions intersect at $x = 1 - n$ and $x = -1-n$.

14. Answers may vary. For example: 1) Determine the zero(s) of the function $f(x)$—these will be the asymptote(s) for the reciprocal function $g(x)$.
2) Determine where the function $f(x)$ is positive and where it is negative—the reciprocal function $g(x)$ will have the same characteristics. 3) Determine where the function $f(x)$ is increasing and where it is decreasing—the reciprocal function $g(x)$ will have opposite characteristics.

15. Use a series of tables to help you graph the functions.

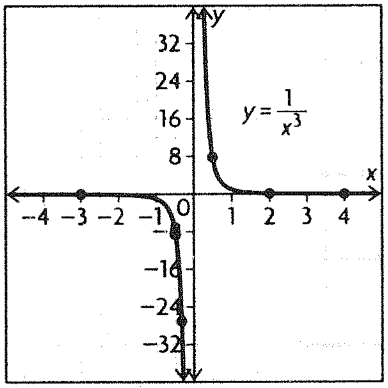
a)

$y = \frac{1}{\sqrt{x}}$	
x	y
$\frac{1}{100}$	10
$\frac{1}{25}$	5
$\frac{1}{16}$	4
1	1
4	$\frac{1}{2}$
9	$\frac{1}{3}$
16	$\frac{1}{4}$



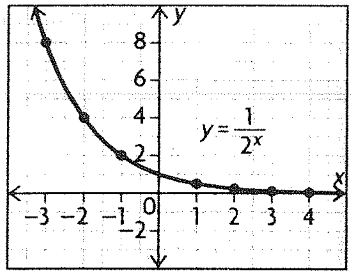
b)

$y = \frac{1}{x^3}$	
x	y
-3	$-\frac{1}{27}$
$-\frac{1}{2}$	-8
$-\frac{1}{3}$	-27
$\frac{1}{2}$	8
1	1
2	$\frac{1}{8}$
4	$\frac{1}{64}$



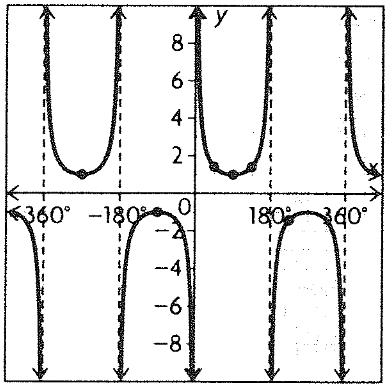
c)

$y = \frac{1}{2^x}$	
x	y
-3	8
-2	4
-1	-2
1	$\frac{1}{2}$
2	$\frac{1}{14}$
3	$\frac{1}{8}$
4	$\frac{1}{16}$



d)

$y = \frac{1}{\sin x}$	
x	y
−360°	undefined
−270°	1
−180°	undefined
−90°	−1
0°	undefined
45°	$\sqrt{2}$
90°	1
135°	$\sqrt{2}$
180°	undefined
225°	$-\sqrt{2}$



16. The graph looks like the graph of $f(x) = \frac{1}{x}$, but translated 4 units to the left and 1 unit down. Therefore, the equation of the function shown in the graph is $y = \frac{1}{x + 4} - 1$.

5.2 Exploring Quotients of Polynomial Functions, p. 262

1. a) A; $y = \frac{-1}{x - 3}$; The function has a zero at 3 and the reciprocal function has a vertical asymptote at $x = 3$. The function is positive for $x < 3$ and negative for $x > 3$.

b) C; $y = \frac{x^2 - 9}{x - 3}$; The function in the numerator factors to $(x + 3)(x - 3)$. $(x - 3)$ factors out of both the numerator and the denominator. The equation simplifies to $y = (x + 3)$, but has a hole at $x = 3$.

c) F; $y = \frac{1}{(x + 3)^2}$; The function in the denominator has a zero at $x = -3$, so there is a vertical asymptote at $x = -3$. The function is always positive.

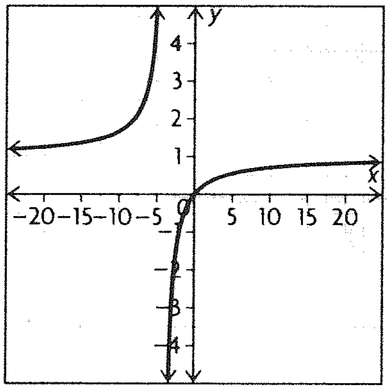
d) D; $y = \frac{x}{(x - 1)(x + 3)}$; The function in the denominator has zeros at $y = 1$ and $y = -3$. The rational function has vertical asymptotes at $x = 1$ and $x = -3$.

e) B; $y = \frac{1}{x^2 + 5}$; The function has no zeros and no vertical asymptotes or holes.

f) E; $y = \frac{x^2}{x - 3}$; The function in the denominator has a zero at $x = 3$ and the rational function has a vertical asymptote at $x = 3$. The degree of the numerator is exactly 1 more than the degree of the denominator, so the graph has an oblique asymptote.

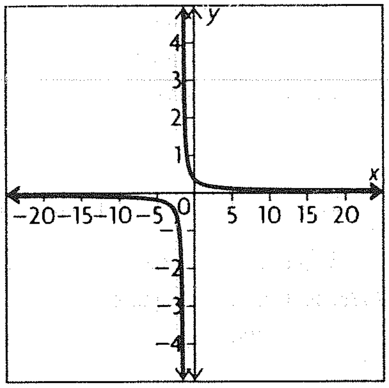
2. Use the graph of each equation to find the equations of any vertical asymptotes, the location of any holes, and the existence of any horizontal or oblique asymptotes.

a)



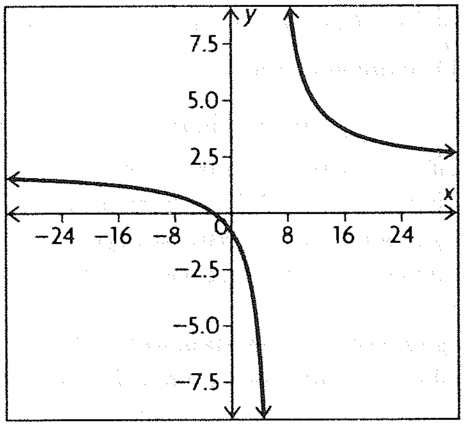
vertical asymptote at $x = -4$; horizontal asymptote at $y = 1$

b)



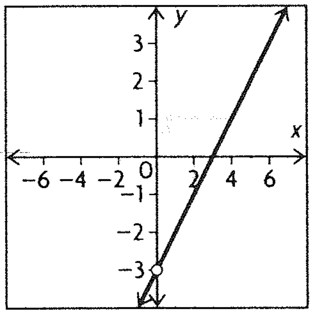
vertical asymptote at $x = -\frac{3}{2}$; horizontal asymptote at $y = 0$

c)



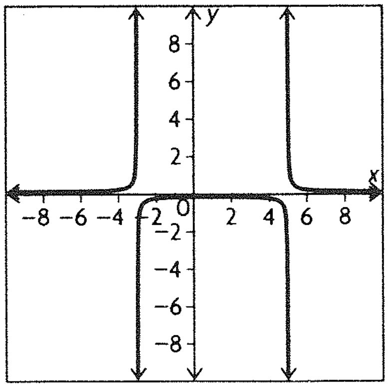
vertical asymptote at $x = 6$; horizontal asymptote at $y = 2$

d)



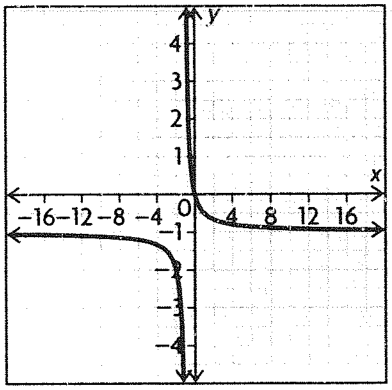
hole at $x = -3$

e)



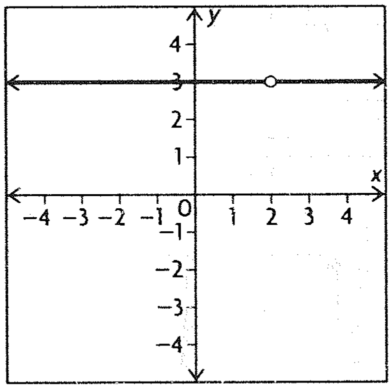
vertical asymptotes at $x = -3$ and 5; horizontal asymptote at $y = 0$

f)



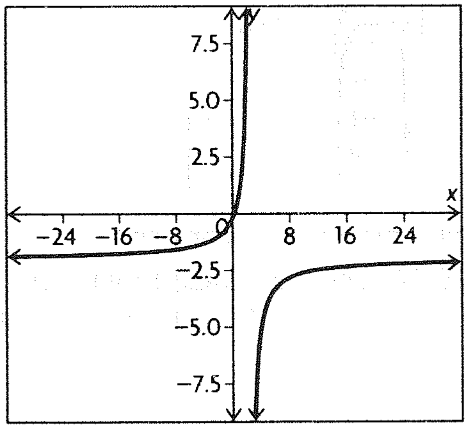
vertical asymptote at $x = -1$; horizontal asymptote at $y = -1$

g)



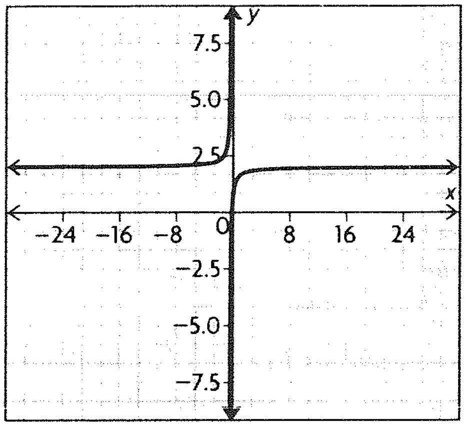
hole at $x = 2$

h)



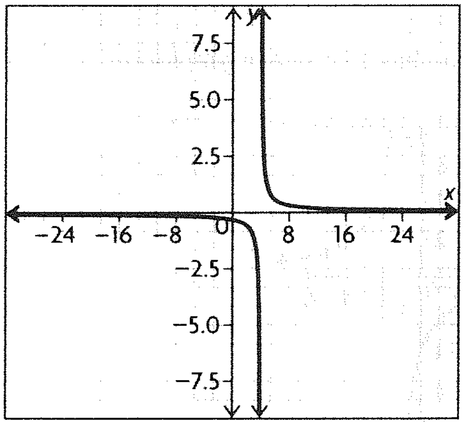
vertical asymptote at $x = \frac{5}{2}$; horizontal asymptote at $y = -2$

i)



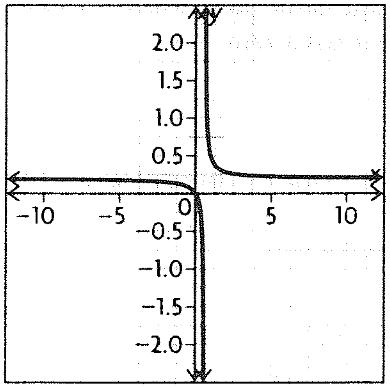
vertical asymptote at $x = -\frac{1}{4}$; horizontal asymptote at $y = 1$

j)



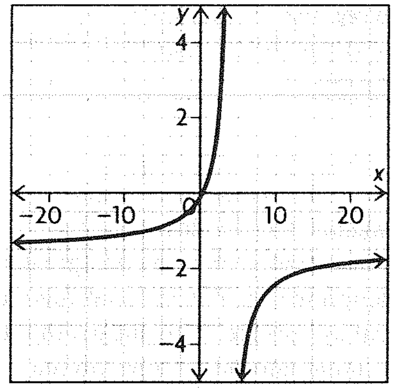
vertical asymptote at $x = 4$; hole at $x = -4$; horizontal asymptote at $y = 0$

k)



vertical asymptote at $x = \frac{3}{5}$; horizontal asymptote at $y = \frac{1}{5}$

l)



vertical asymptote at $x = 4$; horizontal asymptote at $y = -\frac{3}{2}$

3. a) The graph of the rational function has a hole at $x = 1$. This means that $(x - 1)$ must be a factor in both the numerator and the denominator of the function. Answers may vary. For example:

$$y = \frac{x - 1}{x^2 + x - 2}$$

b) The graph of the rational function has a vertical asymptote anywhere. This means that the polynomial that makes up the denominator must have a zero. The graph of the function also has a horizontal asymptote along the x-axis. This means that the numerator of the function must be a constant and the denominator must be a polynomial. Answers may vary. For example:

$$y = \frac{1}{x^2 - 4}$$

c) The graph of the rational function has a hole at $x = -2$. This means that $(x + 2)$ is a factor in both the numerator and the denominator. The graph also has a vertical asymptote at $x = 1$. This means that the ratio between the leading terms of the polynomials in the numerator and denominator must be 1. Answers may vary. For example:

$$y = \frac{x^2 - 4}{x^2 + 3x + 2}$$

d) The graph has a vertical asymptote at $x = -1$. This means that $(x + 1)$ must be a factor in the denominator of the rational function. The graph also has a horizontal asymptote at $y = 2$. This means that the ratio between the leading coefficients of the numerator and the denominator must be 2. Answers may vary. For example:

$$y = \frac{2x}{x + 1}$$

e) The graph of the function has an oblique asymptote. This means that the degree of the polynomial in the numerator must be one greater than that of the denominator. The rational function also has no vertical asymptote. This means that the polynomial in the numerator must have no real zeros. Answers may vary. For example:

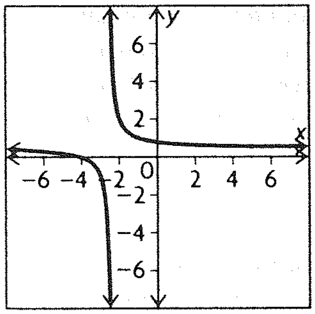
$$y = \frac{x^3}{x^2 + 5}$$

5.3 Graphs of Rational Functions of the Form $f(x) = \frac{ax + b}{cx + d}$, pp. 272–274

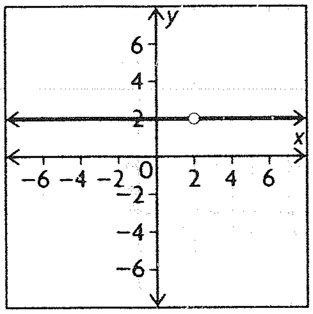
1. a) The rational function $h(x) = \frac{x + 4}{2x + 5}$ would have a vertical asymptote at the point where the function in the denominator has a zero.

$$\begin{aligned} 2x + 5 &= 0 \\ 2x + 5 - 5 &= 0 - 5 \\ 2x &= -5 \\ \frac{2x}{2} &= \frac{-5}{2} \\ x &= -2.5 \end{aligned}$$

The function has a vertical asymptote at $x = -2.5$. This is graph **A**.



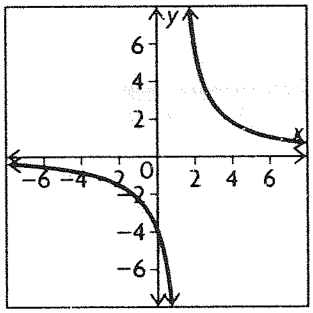
b) The rational function $h(x) = \frac{2x - 4}{x - 2}$ has $x - 2$ as a factor of both the numerator and the denominator. This means that the function has a hole at $x = 2$. This is graph **C**.



c) The rational function $h(x) = \frac{3}{x - 1}$ would have a vertical asymptote at the point where the function in the denominator has a zero.

$$\begin{aligned} x - 1 &= 0 \\ x - 1 + 1 &= 0 + 1 \\ x &= 1 \end{aligned}$$

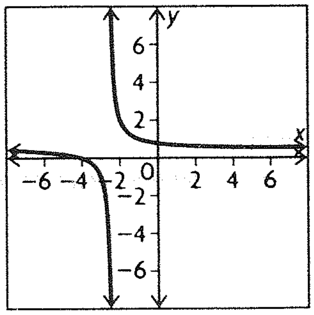
The function has a vertical asymptote at $x = 1$. This is graph **D**.



d) The rational function $h(x) = \frac{2x - 3}{x + 2}$ would have a vertical asymptote at the point where the function in the denominator has a zero.

$$\begin{aligned} x + 2 &= 0 \\ x + 2 - 2 &= 0 - 2 \\ x &= -2 \end{aligned}$$

The function has a vertical asymptote at $x = -2$. This is graph **B**.



2. a) The rational function $f(x) = \frac{3}{x - 2}$ will have a vertical asymptote at the point where the function in the denominator has a zero.

$$x - 2 = 0$$
$$x - 2 + 2 = 0 + 2$$
$$x = 2$$

The equation has a vertical asymptote at $x = 2$.

b) Use a table to examine the values of $f(x)$ as $x \to 2$. Use small increments between values of x.

x	f(x)
2.25	12
2.10	30
1.90	−30
1.75	−12
1.5	−6
1	−3

As x approaches 2 from the right, the values of $f(x)$ get larger. As x approaches 2 from the left, the values become larger in magnitude but are negative.

c) As $x \to \pm\infty$, the value of $f(x)$ approaches $\frac{3}{\infty}$ or 0.

d) Use a table to examine the values of $f(x)$ as x approaches $\pm\infty$. Use large increments between values of x.

x	f(x)
5000	0.000 600 24
2500	0.001 200 961
250	0.012 096 774
−250	−0.011 904 762
−2500	−0.001 199 041
−5000	−0.000 599 76

As x grows larger, $f(x)$ gets closer and closer to 0.

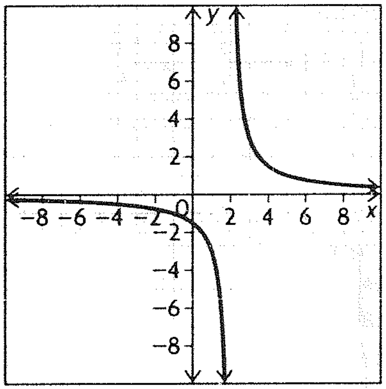
e) The domain is all real numbers except for 3 and the range is all real numbers except for 0.

f) Examine the equation $y = \frac{3}{x - 2}$. The numerator is always positive. A positive number divided by a positive number is always positive. When $x - 2$ is positive, $y = \frac{3}{x - 2}$ will be positive. A positive number divided by a negative number is always negative. When $x - 2$ is negative, $y = \frac{3}{x - 2}$ will be negative.

$$x - 2 > 0$$
$$x - 2 + 2 > 0 + 2$$
$$x > 2$$

When $x > 2$, $y = \frac{3}{x - 2}$ is positive. When $x < 2$, $y = \frac{3}{x - 2}$ is negative.

g)



3. a) The rational function $f(x) = \frac{4x - 3}{x + 1}$ will have a vertical asymptote at the point where the function in the denominator has a zero.

$$x + 1 = 0$$
$$x + 1 - 1 = 0 - 1$$
$$x = -1$$

The equation has a vertical asymptote at $x = -1$.

b) Use a table to examine the values of $f(x)$ as x approaches 2. Use small increments between values of x.

x	f(x)
−0.1	−3.777 777 778
−0.5	−10
−0.75	−24
−1.25	32
−1.5	18

As $x \to -1$ from the left, $y \to \infty$.
As $x \to -1$ from the right, $y \to -\infty$.

c) The equation of the horizontal asymptote can be found by dividing the leading coefficients of the equation in the numerator and the denominator.

$$y = \frac{4x}{x} = 4$$

The equation of the horizontal asymptote is $y = 4$.

d) Use a table to examine the values of $f(x)$ as x approaches ∞ and $-\infty$. Use large increments between values of x.

x	f(x)
5000	3.998 600 28
2500	3.997 201 12
250	3.972 111 554
−250	4.028 112 45
−2500	4.002 801 12
−5000	4.001 400 28

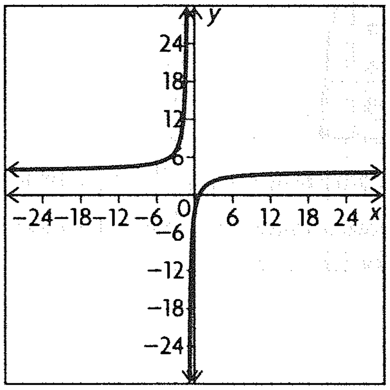
As $x \to \pm\infty$, $f(x)$ gets closer and closer to 4.

e) $D = \{x \in \mathbf{R} \mid x \neq -1\}$
$R = \{y \in \mathbf{R} \mid y \neq 4\}$

f) Use a table to help you determine the positive and negative intervals.

	$x < -1$	$-1 < x < \frac{3}{4}$	$x > \frac{3}{4}$
$4x - 3$	−	−	+
$x + 1$	−	+	+
$\frac{4x - 3}{x + 1}$	+	−	+

g)



4. The vertical asymptote of a rational function can be found by finding the zero(s) of the denominator.

a) $y = \dfrac{2}{x + 3}$

$x + 3 = 0$
$x + 3 - 3 = 0 - 3$
$x = -3$

The equation of the vertical asymptote of $y = \dfrac{2}{x + 3}$ is $x = -3$.
When $x = -3.1$, $y = -20$. So as $x \rightarrow -3$, $y \rightarrow -\infty$ on the left.
When $x = -2.9$, $y = 20$. So as $x \rightarrow -3$, $y \rightarrow \infty$ on the right.

b) $y = \dfrac{x - 1}{x - 5}$

$x - 5 = 0$
$x = 5$

The equation of the vertical asymptote of $y = \dfrac{x - 1}{x - 5}$ is $x = 5$.
When $x = 4.9$, $y = -39$. So as $x \rightarrow 5$, $y \rightarrow -\infty$ on the left.
When $x = 5.1$, $y = 41$. So as $x \rightarrow 5$, $y \rightarrow \infty$ on the right.

c) $y = \dfrac{2x + 1}{2x - 1}$

$2x - 1 + 1 = 0 + 1$
$2x = 1$
$x = \dfrac{1}{2}$

The equation of the vertical asymptote of $y = \dfrac{2x + 1}{2x - 1}$ is $x = \frac{1}{2}$.
When $x = 0.49$, $y = -99$. So as $x \rightarrow \frac{1}{2}$, $y \rightarrow -\infty$ on the left.
When $x = 0.51$, $y = 101$. So as $x \rightarrow \frac{1}{2}$, $y \rightarrow \infty$ on the right.

d) $y = \dfrac{3x + 9}{4x + 1}$

$4x + 1 - 1 = 0 - 1$
$4x = -1$
$x = -\dfrac{1}{4}$

The equation of the vertical asymptote of $y = \dfrac{3x + 9}{4x + 1}$ is $x = -\frac{1}{4}$.
When $x = -0.26$, $y = -205.5$. So as $x \rightarrow -\frac{1}{4}$, $y \rightarrow -\infty$ on the left.
When $x = -0.24$, $y = 207$. So as $x \rightarrow -\frac{1}{4}$, $y \rightarrow \infty$ on the right.

5. a) The function is $f(x) = \dfrac{3}{x + 5}$.

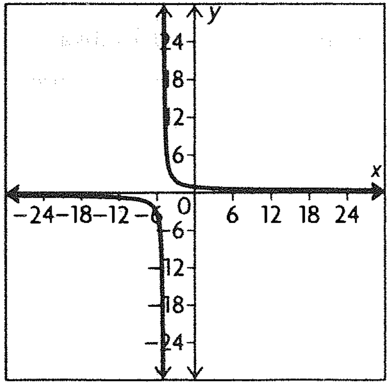
$f(x) = \dfrac{3}{x + 5}$ will have a vertical asymptote at $x = -5$. The horizontal asymptote will be $y = 0$. Therefore, the domain will be $\{x \in \mathbf{R} \mid x \neq -5\}$. The range will be $\{y \in \mathbf{R} \mid y \neq 0\}$. Because the horizontal asymptote is $y = 0$, there is no x-intercept. Substitute 0 for x to find the y-intercept.

$\dfrac{3}{0 + 5} = \dfrac{3}{5}$

The y-intercept is $\frac{3}{5}$. Use a table to determine the positive and negative intervals.

	$x < -5$	$x > -5$
3	+	+
$x + 5$	−	+
$\frac{3}{x + 5}$	−	+

$f(x)$ is negative on $(-\infty, -5)$ and positive on $(-5, \infty)$.
The graph of the function is:



Examine the graph to determine where the function is increasing or decreasing. The function is decreasing on $(-\infty, -5)$ and on $(-5, \infty)$. The function is never increasing.

b) The function is $\frac{10}{2x - 5}$.

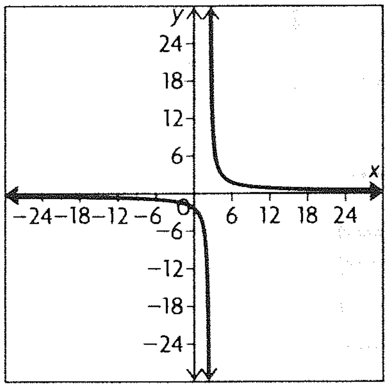
$f(x) = \frac{10}{2x - 5}$ will have a vertical asymptote at $x = \frac{5}{2}$. The horizontal asymptote will be $y = 0$. Therefore, the domain will be $\{x \in \mathbf{R} \mid x \neq \frac{5}{2}\}$ and the range will be $\{y \in \mathbf{R} \mid y \neq 0\}$. Because the horizontal asymptote is $y = 0$, there is no x-intercept. Substitute 0 for x to find the y-intercept.

$$\frac{10}{2(0) - 5} = -2$$

The y intercept is -2. Use a table to determine the positive and negative intervals.

	$x < \frac{5}{2}$	$x > \frac{5}{2}$
10	+	+
$2x - 5$	−	+
$\frac{10}{2x - 5}$	−	+

$f(x)$ is negative on $(-\infty, \frac{5}{2})$ and positive on $(\frac{5}{2}, \infty)$. The graph of the function is:



Examine the graph to determine where the function is increasing or decreasing. The function is decreasing on $(-\infty, \frac{5}{2})$ and on $(\frac{5}{2}, \infty)$. The function is never increasing.

c) The function is $f(x) = \frac{x + 5}{4x - 1}$.

$f(x) = \frac{x + 5}{4x - 1}$ will have a vertical asymptote at $x = \frac{1}{4}$. The horizontal asymptote will be $y = \frac{1}{4}$. Therefore, the domain will be $\{x \in \mathbf{R} \mid x \neq \frac{1}{4}\}$ and the range will be $\{y \in \mathbf{R} \mid y \neq \frac{1}{4}\}$.

Substitute 0 for y to find the x-intercept.

$$0 = \frac{x + 5}{4x - 1}$$

$$(4x - 1) \times 0 = (4x - 1) \times \frac{x + 5}{4x - 1}$$

$$0 = x + 5$$

$$-5 = x$$

The x-intercept is $x = -5$.

Substitute 0 for x to find the y-intercept.

$$\frac{0 + 5}{4(0) - 1} = -5$$

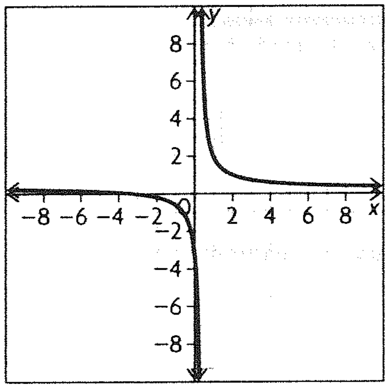
The y-intercept is -1.

Use a table to determine the positive and negative intervals.

	$x < -5$	$-5 < x < \frac{1}{4}$	$x > \frac{1}{4}$
$x + 5$	−	+	+
$4x - 1$	−	−	+
$\frac{x + 5}{4x - 1}$	+	−	+

$f(x)$ is positive on $(-\infty, -5)$ and $(\frac{1}{4}, \infty)$, and negative on $(-5, \frac{1}{4})$.

The graph of the function is:

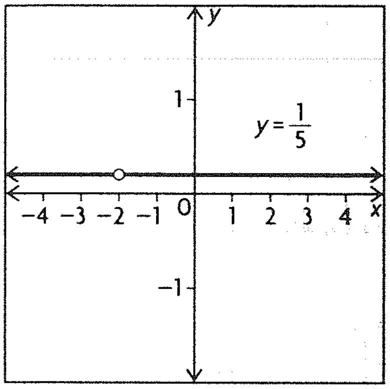


Examine the graph to determine where the function is increasing or decreasing. The function is decreasing on $(-\infty, \frac{1}{4})$ and on $(\frac{1}{4}, \infty)$. The function is never increasing.

d) The function is $f(x) = \frac{x + 2}{5(x + 2)}$. The function has the factor $(x + 2)$ in both the numerator and the denominator.

Examine the function. For any value of x, $f(x)$ will always be $\frac{1}{5}$. Because the function in the denominator will have zero at $x = -2$, $f(x)$ will have a hole at $x = -2$. The domain is $\{x \in \mathbf{R} \mid x \neq -2\}$. The range is $\{y = \frac{1}{5}\}$. The y-intercept is $y = \frac{1}{5}$. There is no x-intercept. The function will always be positive.

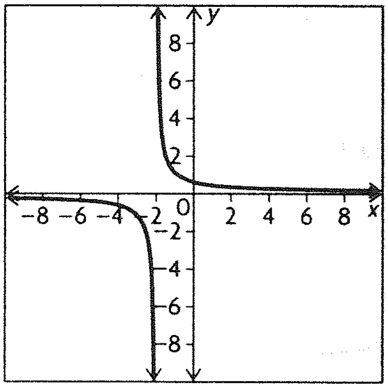
The graph of the function is:



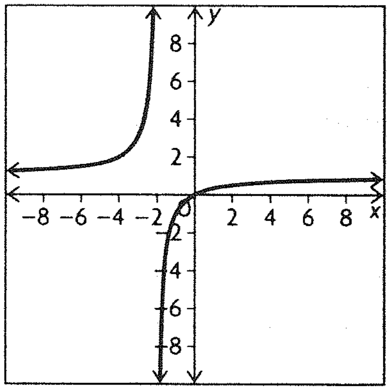
The function is neither increasing nor decreasing; it is constant.

6. a) Answers may vary. For example:

$$f(x) = \frac{1}{x + 2}$$



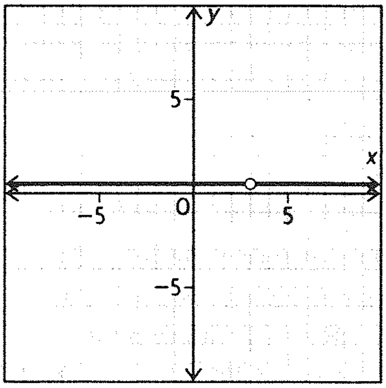
b) Answers may vary. For example: $f(x) = \frac{x}{x + 2}$



c) Answers may vary. For example:

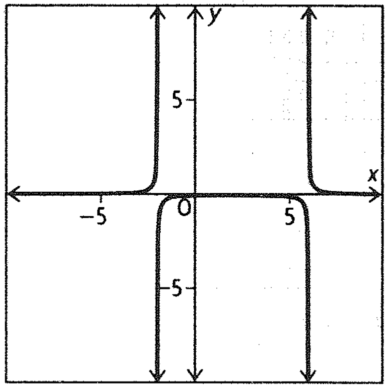
$$f(x) = \frac{x - 3}{2x - 6}$$

This has a hole at $x = 3$ and a y-intercept of 0.5.

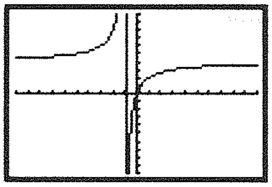


d) Answers may vary. For example:

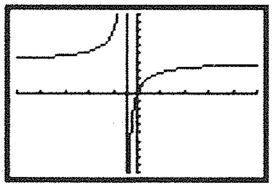
$$f(x) = \frac{1}{x^2 - 4x - 12}$$



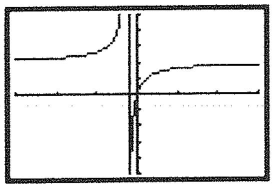
7. a) Graph the function $f(x) = \frac{8x}{nx + 1}$ for each value of n, $n = 1, 2, 4$, and 8.



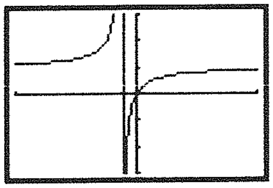
The graph is shown in a window from −10 to 10 by 1's on the horizontal axis and from −20 to 20 by 2's on the vertical axis.



The graph is shown in a window from −5 to 5 by 1's on the horizontal axis and from −10 to 10 by 1's on the vertical axis.



The graph is shown in a window from -3 to 3 by 1's on the horizontal axis and from -5 to 5 by 1's on the vertical axis.



The graph is shown in a window from -1 to 1 by 1's on the horizontal axis and from -3 to 3 by 1's on the vertical axis.

Use this information to discuss the differences between the graphs. The equation has a general vertical asymptote at $x = -\frac{1}{n}$. The function has a general horizontal asymptote $y = \frac{8}{n}$. The vertical asymptotes are $-\frac{1}{8}$, $-\frac{1}{4}$, $-\frac{1}{2}$, and -1. The horizontal asymptotes are 8, 4, 2, and 1. The function contracts as n increases. The function is always increasing. The function is positive on $\left(-\infty, -\frac{17}{n}\right)$ and $\left(\frac{3}{10}, \infty\right)$ The function is negative on $\left(-\frac{17}{n}, \frac{3}{10}\right)$.

b) The horizontal and vertical asymptotes both approach 0 as the value of n increases; the x- and y-intercepts do not change nor do the positive and negative characteristics or the increasing and decreasing characteristics.

c) The vertical asymptote becomes $x = \frac{17}{n}$ and the horizontal becomes $x = -\frac{10}{n}$. The function is always increasing. The function is positive on $\left(-\infty, \frac{3}{10}\right)$ and $\left(\frac{17}{n}, \infty\right)$. The function is negative on $\left(\frac{3}{10}, \frac{17}{n}\right)$. The rest of the characteristics do not change.

8. $f(x)$ will have a vertical asymptote at $x = 1$; $g(x)$ will have a vertical asymptote at $x = -\frac{3}{2}$. $f(x)$ will have a horizontal asymptote at $x = 3$; $g(x)$ will have a vertical asymptote at $x = \frac{1}{2}$.

9. Substitute the values of t to find the value of the investment over a given period of time. The function is $I(t) = \frac{15t + 25}{t}$.

a) $I(2) = \frac{15(2) + 25}{2} = \frac{55}{2} = 27.5$

The investment will be worth \$27 500 after 2 years.

b) $I(1) = \frac{15(1) + 25}{1} = 40$

The investment will be worth \$40 000 after 1 year.

c) $I(0.5) = \frac{15(0.5) + 25}{0.5} = 65$

The investment will be worth \$65 000 after 0.5 years.

d) No, the value of the investment at $t = 0$ should be the original value invested.

e) The function is probably not accurate at very small values of t because as $t \rightarrow 0$ from the right, $x \rightarrow \infty$.

f) The horizontal asymptote will indicate where the value of the investment will settle over time. Divide the leading terms to find the equation of the horizontal asymptote.

$y = \frac{15t}{t} = 15$

The value of the investment will settle at around \$15 000.

10. Use a table to help you examine the concentration of the chlorine in the pool over the 24-hour period.

t	$c(t)$
1	0.666 666 667
2	1
3	1.2
4	1.333 333 333
8	1.6
12	1.714 285 714
16	1.777 777 778
20	1.818 181 818
24	1.846 153 846

The concentration increases over the 24 hour period and approaches approximately 1.89 mg/L.

11. Answers may vary. For example: The rational functions will all have vertical asymptotes at $x = -\frac{d}{c}$. They will all have horizontal asymptotes at $y = \frac{a}{c}$. They will intersect the y-axis at $y = \frac{b}{d}$.

The rational functions will have an x-intercept at $x = -\frac{b}{a}$. You can use $a = 1$, $b = 2$, $c = 3$, and $d = 4$ to illustrate this. The function is

$$f(x) = \frac{1x + 2}{3x + 4}$$

Vertical Asymptote:

$$\begin{aligned} 0 &= 3x + 4 \\ 0 - 4 &= 3x + 4 - 4 \\ -4 &= 3x \\ \frac{-4}{3} &= \frac{3x}{3} \\ \frac{-4}{3} &= x = \frac{-d}{c} \end{aligned}$$

Horizontal Asymptote:

$$y = \frac{1x}{3x} = \frac{1}{3} = \frac{a}{c}$$

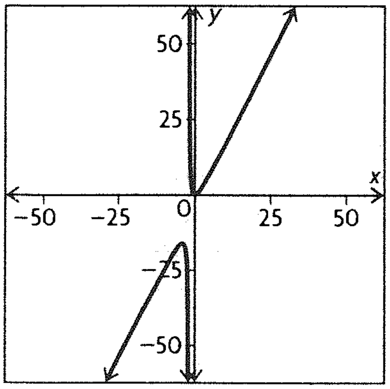
y-intercept:

$$y = \frac{1(0) + 2}{3(0) + 4} = \frac{2}{4} = \frac{b}{d}$$

x-intercept:

$$\begin{aligned} 0 &= \frac{1x + 2}{3x + 4} \\ (3x + 4) \times 0 &= (3x + 4) \times \frac{1x + 2}{3x + 4} \\ 0 &= 1x + 2 \\ 0 - 2 &= 1x + 2 - 2 \\ -2 &= 1x \\ \frac{-2}{1} &= \frac{1x}{1} \\ \frac{-2}{1} &= x = \frac{-b}{a} \end{aligned}$$

12. Answers may vary. For example: The function $f(x) = \frac{2x^2}{2 + x}$ will have an oblique asymptote. Examine the graph below to see the oblique asymptote.

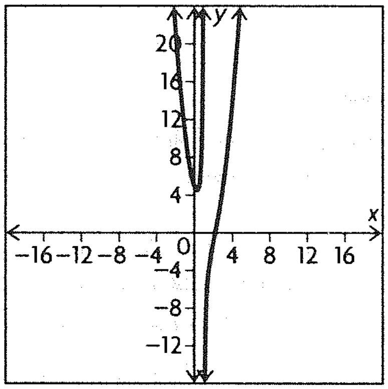


13. Use synthetic division to help you rewrite $f(x) = \frac{2x^3 - 7x^2 + 8x - 5}{x - 1}$.

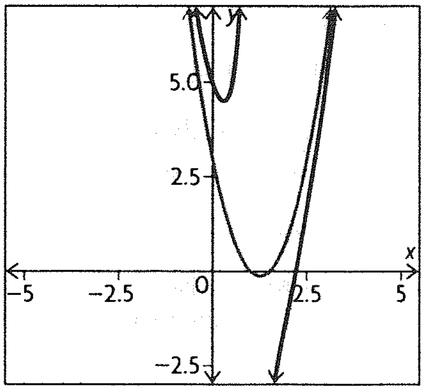
$$\begin{array}{r|rrrr} 1 & 2 & -7 & 8 & -5 \\ & \downarrow & 2 & -5 & 3 \\ \hline & 2 & -5 & 3 & -2 \end{array}$$

$$\begin{aligned} f(x) &= \frac{2x^3 - 7x^2 + 8x - 5}{x - 1} \\ &= 2x^2 - 5x + 3 - \frac{2}{x - 1} \end{aligned}$$

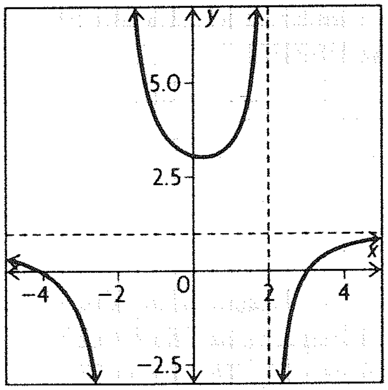
As $x \to \pm\infty$, $f(x) \to \infty$.



The equation of the vertical asymptote would be $x = 1$. When you find the equation of a horizontal asymptote of a rational function $f(x)$, you find a constant value that the equation approaches as $x \to \infty$. In this case, $f(x)$ isn't approaching a constant value, but rather an equation or a parabola. For $f(x)$, the equation of the oblique asymptote would be $y = 2x^2 - 5x + 3$, which is the quotient without the remainder because the remainder is what keeps the rational function from reaching the parabola. Graph both equations to verify this.



14. a) $f(x)$ would have a horizontal asymptote because the degree of the numerator is less than that of the denominator.
b) $g(x)$ and $h(x)$ would have an oblique asymptote because the degree of the numerator is greater than that of the denominator.
c) $g(x)$ has no vertical asymptote because the numerator has no real zero.
d) Use the horizontal and vertical asymptotes to help you draw the graph. The denominator has zeros at $x = 2$ and -2—these will be the vertical asymptotes. Divide the leading coefficients to find the horizontal asymptote of $y = 1$.



Mid-Chapter Review, p. 277

1. The reciprocal of a function, $f(x)$, is equal to $\frac{1}{f(x)}$. Vertical asymptotes can be found using the zeros of $f(x)$.

a) $f(x) = x - 3$

$$\frac{1}{f(x)} = \frac{1}{x-3}$$
$$0 = x - 3$$
$$0 + 3 = x - 3 + 3$$
$$3 = x$$

The equation of the vertical asymptote is $x = 3$.

b) $f(q) = -4q + 6$

$$\frac{1}{f(q)} = \frac{1}{-4q+6}$$
$$0 = -4q + 6$$
$$0 - 6 = -4q + 6 - 6$$
$$-6 = -4q$$
$$\frac{-6}{-4} = \frac{-4q}{-4}$$
$$\frac{3}{2} = q$$

The equation of the vertical asymptote is $q = \frac{3}{2}$.

c) $f(z) = z^2 + 4z - 5$

$$\frac{1}{f(z)} = \frac{1}{z^2 + 4z - 5}$$
$$0 = z^2 + 4z - 5$$
$$0 = (z + 5)(z - 1)$$
$$0 = z + 5$$
$$0 - 5 = z + 5 - 5$$
$$-5 = z$$
$$0 + 1 = z - 1 + 1$$
$$1 = z$$

The equations of the vertical asymptotes are $z = -5$ and 1.

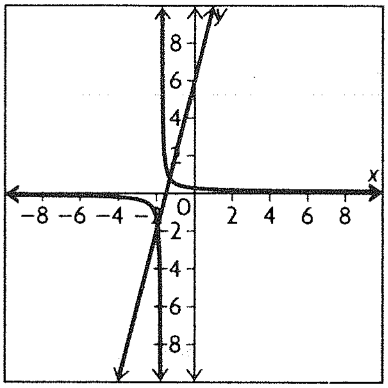
d) $f(d) = 6d^2 + 7d - 3$

$$\frac{1}{f(d)} = \frac{1}{6d^2 + 7d - 3}$$
$$0 = 6d^2 + 7d - 3$$
$$0 = (3d - 1)(2d + 3)$$
$$0 = 3d - 1$$
$$0 + 1 = 3d - 1 + 1$$
$$1 = 3d$$
$$\frac{1}{3} = \frac{3d}{3}$$
$$\frac{1}{3} = d$$
$$0 = 2d + 3$$
$$0 - 3 = 2d + 3 - 3$$
$$-3 = 2d$$
$$\frac{-3}{2} = \frac{2d}{2}$$
$$-\frac{3}{2} = d$$

2. a) The function is $f(x) = 4x + 6$. The function is a straight line. The domain is $\{x \in \mathbf{R}\}$ and the range is $\{y \in \mathbf{R}\}$. Find the x-intercept to determine the positive and negative intervals.

$$0 = 4x + 6$$
$$0 - 6 = 4x + 6 - 6$$
$$-6 = 4x$$
$$\frac{-6}{4} = \frac{4x}{4}$$
$$-\frac{3}{2} = x$$

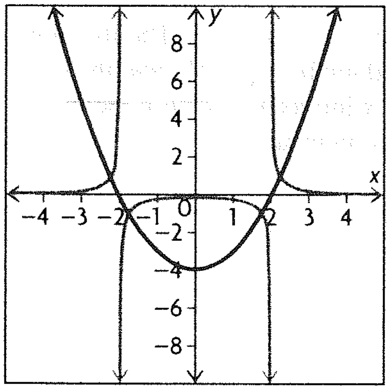
Choose a number on either side of $-\frac{3}{2}$ and substitute them into the function to find the positive. The function is negative on $\left(-\infty, -\frac{3}{2}\right)$ and positive on $\left(-\frac{3}{2}, \infty\right)$. Because the function is linear, and because the slope of the function is positive, the function is always positive. Use this information to graph the function and its reciprocal.



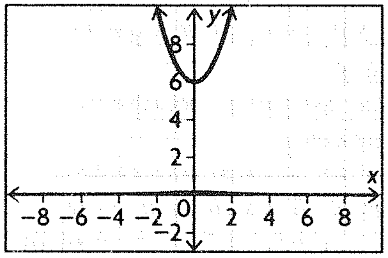
b) The function is $f(x) = x^2 - 4$. The function is a parabola. The coefficient of the first term is positive and so the parabola points up. This means that the domain is $\{x \in \mathbf{R}\}$ and the range is $\{y \in \mathbf{R} \mid y > -4\}$. Find the x-intercepts to help you determine the positive and negative intervals.

$$0 = x^2 - 4$$
$$0 = (x + 2)(x - 2)$$
$$0 - 2 = x + 2 - 2$$
$$-2 = x$$
$$0 = x - 2$$
$$0 + 2 = x - 2 + 2$$
$$2 = x$$

The x-intercepts are 2 and -2. This means that the vertex is at 0. The function is decreasing on $(-\infty, 0)$ and increasing $(0, \infty)$. That means that the function is positive on $(-\infty, -2)$ and $(2, \infty)$. The function is negative on $(-2, 2)$. Use this information to help you graph the function and its reciprocal.



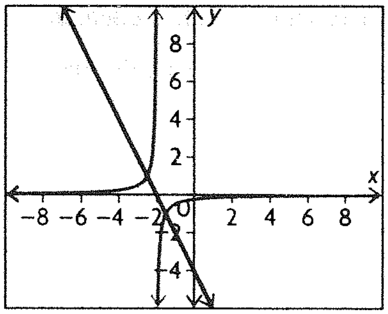
c) The function is $f(x) = x^2 + 6$. The function is a parabola. The coefficient of the first term is positive and so the parabola points up. This means that the range is $\{x \in \mathbf{R}\}$ and the range is $\{y \in \mathbf{R} \mid y > 6\}$. Because the range is $\{y \in \mathbf{R} \mid y > 6\}$, there are no x-intercepts and the function will never be negative. The vertex of the parabola will be $(0, 6)$. This means that the function will be decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. Use this information to help you sketch the graph of $f(x) = x^2 + 6$ and its reciprocal.



d) The function is $f(x) = -2x - 4$. The function is a straight line. The domain is $\{x \in \mathbf{R}\}$ and the range is $\{y \in \mathbf{R}\}$. Find the x-intercept to determine the positive and negative intervals.

$$0 = -2x - 4$$
$$0 + 4 = -2x - 4 + 4$$
$$4 = -2x$$
$$\frac{4}{-2} = \frac{-2x}{-2}$$
$$-2 = x$$

The x-intercept is $x = -2$. Examine the slope of the function. The slope is negative and therefore the function is always decreasing. This means that the function is positive on $(-\infty, -2)$ and negative on $(-2, \infty)$. Use this information to help you graph the function and the reciprocal.



3. Answers may vary. For example: (1) Hole: Both the numerator and the denominator contain a common factor, resulting in $\frac{0}{0}$ for a specific value of x.
(2) Vertical asymptote: A value of x causes the denominator of a rational function to be 0.
(3) Horizontal asymptote: A horizontal asymptote is created by the ratio between the numerator and the denominator of a rational function as the function approaches ∞ and $-\infty$. A continuous rational function is created when the denominator of the rational function has no zeros.

4. Vertical asymptotes can be found by finding the zero of the denominator. An equation will have a horizontal asymptote if the degree of the expression in the numerator is less than or equal to the degree of the expression in the denominator. An equation will have an oblique asymptote if the degree of the numerator is 1 greater than the degree of the denominator. A hole occurs when there is a common linear factor in the numerator and denominator.

a) The function is $y = \frac{x}{x - 2}$.

$$\begin{aligned} 0 &= x - 2 \\ 0 + 2 &= x - 2 + 2 \\ 2 &= x \end{aligned}$$

The equation of the vertical asymptote is $x = 2$.

b) The function is $y = \frac{x - 1}{3x - 3}$. In both the numerator and the denominator, $x - 1$ is a factor. This means that the function will have a straight line at $y = \frac{1}{3}$ and a hole at $x - 1$.

c) The function is $y = \frac{-7x}{4x + 2}$.

$$\begin{aligned} 0 &= 4x + 2 \\ 0 - 2 &= 4x + 2 - 2 \\ -2 &= 4x \\ -\frac{2}{4} &= x \\ -\frac{1}{2} &= x \end{aligned}$$

The equation of the vertical asymptote is $x = -\frac{1}{2}$. The degrees of the numerator and denominator are equal. This means that there is a horizontal asymptote.

d) The function is $y = \frac{x^2 + 2}{x - 6}$.

$$\begin{aligned} 0 &= x - 6 \\ 0 + 6 &= x - 6 + 6 \\ 6 &= x \end{aligned}$$

The equation of the vertical asymptote is $x = 6$. The degree of the expression in the numerator is 1 larger than the expression in the denominator. This means that the function will have an oblique asymptote.

e) The equation is $y = \frac{1}{x^2 + 2x - 15}$.

$$\begin{aligned} 0 &= x^2 + 2x - 15 \\ 0 &= (x + 5)(x - 3) \\ 0 &= x + 5 \\ 0 - 5 &= x + 5 - 5 \\ -5 &= x \\ 0 &= x - 3 \\ 0 + 3 &= x - 3 + 3 \\ 3 &= x \end{aligned}$$

The equations of the vertical asymptotes are $x = -5$ and $x = 3$. Since the numerator is a constant, the horizontal asymptote will be $x = 0$.

5. The functions that had horizontal asymptotes were: $y = \frac{x}{x - 2}$, $y = \frac{-7x}{4x + 2}$, and $y = \frac{1}{x^2 + 2x - 15}$. Begin with $y = \frac{1}{x^2 + 2x - 15}$. Because there is a constant in the numerator, the horizontal asymptote for this equation will be $x = 0$. For the other two equations, divide the first terms of the expressions in the numerator and denominator to find the equation of the horizontal asymptotes.

$$y = \frac{x}{x} = 1$$

$$y = \frac{-7x}{4x} = -\frac{7}{4}$$

6. a) The function is $f(x) = \frac{5}{x - 6}$.

Vertical Asymptote:

$$\begin{aligned} 0 &= x - 6 \\ 0 + 6 &= x - 6 + 6 \\ 6 &= x \end{aligned}$$

The horizontal asymptote will be $y = 0$ because the numerator is a constant.

x-intercept:

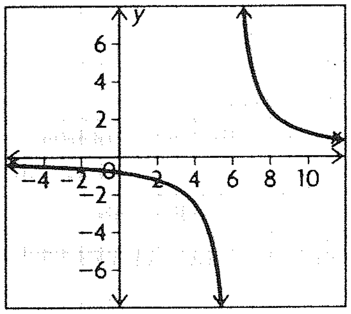
$$\begin{aligned} 0 &= \frac{5}{x - 6} \\ (x - 6) \times 0 &= \frac{5}{x - 6}(x - 6) \\ 0 &= 5 \end{aligned}$$

The equation will not have an x-intercept.

y-intercept:

$$y = \frac{5}{0 - 6} = -\frac{5}{6}$$

Because the numerator is a constant, the function will be negative when the denominator is negative and positive when the numerator is positive. $x - 6$ is negative on $x < 6$. $f(x)$ is negative on $(-\infty, 6)$ and positive on $(6, \infty)$. Use this information to graph the function.



From the graph, it can be seen that the function is always decreasing.

b) The function is $f(x) = \frac{3x}{x + 4}$.

Vertical Asymptote:

$$0 = x + 4$$
$$0 - 4 = x + 4 - 4$$
$$-4 = x$$

Horizontal Asymptote:

$$y = \frac{3x}{x} = 3$$

x-intercept:

$$0 = \frac{3x}{x + 4}$$
$$(x + 4) \times 0 = \frac{3x}{x + 4} \times (x + 4)$$
$$0 = 3x$$
$$\frac{0}{3} = \frac{3x}{3}$$
$$0 = x$$

The x-intersect is $x = 0$.

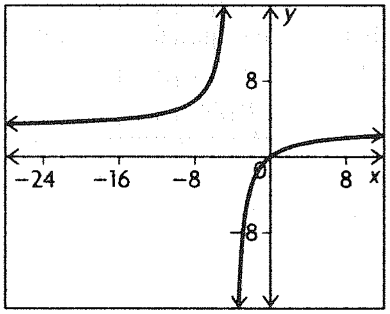
y-intercept:

$$f(0) = \frac{3(0)}{(0) + 4}$$
$$f(0) = 0$$

Use a table to determine when the equation is positive and negative.

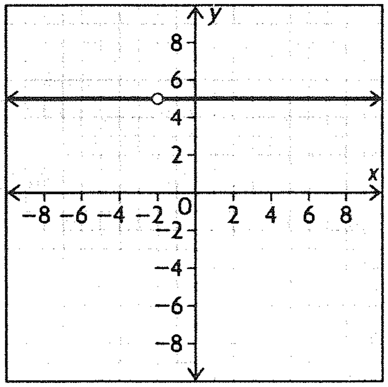
	$x < -4$	$-4 < x < 0$	$x > 0$
$3x$	$-$	$-$	$+$
$x + 4$	$-$	$+$	$+$
$\frac{3x}{x + 4}$	$+$	$-$	$+$

Use this information to help you graph the function.



From the graph it can be seen that the function is always increasing.

c) The function is $f(x) = \frac{5x + 10}{x + 2}$. Both the numerator and the denominator have a common factor, $x + 2$. $f(x) = \frac{5(x + 2)}{x + 2} = 5$. The function is a straight, horizontal line with a hole at $x = -2$. It will always be positive and will never increase or decrease. Use this information to graph the function.



d) The function is $f(x) = \frac{x - 2}{2x - 1}$.

Vertical Asymptote:

$$0 = 2x - 1$$
$$0 + 1 = 2x - 1 + 1$$
$$1 = 2x$$
$$\frac{1}{2} = \frac{2x}{2}$$
$$\frac{1}{2} = x$$

Horizontal Asymptote:

$$y = \frac{x}{2x} = \frac{1}{2}$$

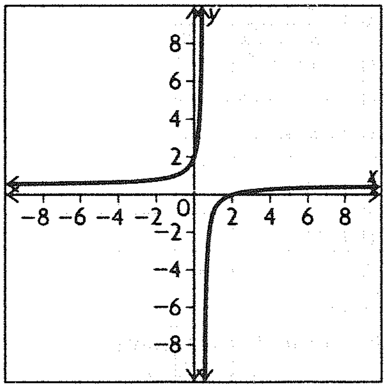
x-intercept:

$$0 = \frac{x - 2}{2x - 1}$$
$$(2x - 1) \times 0 = \frac{x - 2}{2x - 1} \times (2x - 1)$$
$$0 = x - 2$$
$$0 + 2 = x - 2 + 2$$
$$2 = x$$

y-intercept:

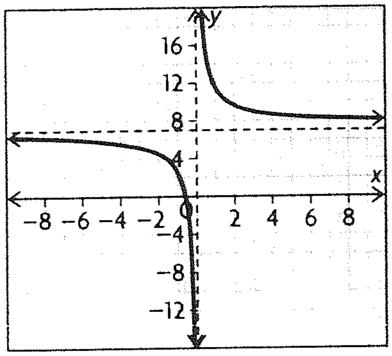
$$f(0) = \frac{5(0) + 10}{(0) + 2}$$
$$f(0) = 5$$

Use this information to help you graph the function.

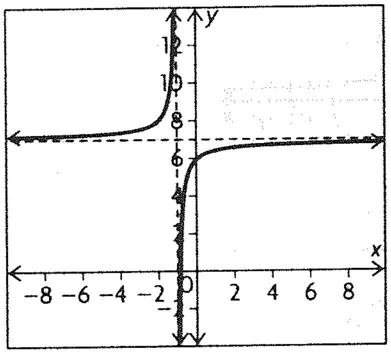


The function is always increasing.

7. Answers may vary. For example: Examine the function $y = \frac{7x + 6}{x}$.



The horizontal asymptote is $y = 7$ and the vertical asymptote is $x = 0$. Changing the function to $y = \frac{7x + 6}{x + 1}$ changes the graph.



The function now has a vertical asymptote at $x = -1$ and still has a horizontal asymptote at $y = 7$. However, the function is now constantly increasing instead of decreasing. The new function still has an x-intercept at $x = -\frac{6}{7}$, but now has a y-intercept at $y = 6$. To explain why the new function is always increasing instead of decreasing, examine the function near their respective asymptotes.

For $\frac{7x + 6}{x}$, when you have values of x that are less than 1, you have a number greater than 1 being divided by a fraction—which leads to larger and larger values of y as x gets closer to 0. But with $y = \frac{7x + 6}{x + 1}$ you don't have that. When $0 < x < 1$, you're dividing a number that is greater than 1 by another number that is greater than 1—giving you a fraction. When $-1 < x < 0$, then you have a number that is greater than 1 being divided by fraction. Additionally, when $-1 < x < -\frac{6}{7}$ the fraction is negative and approaching ∞ as x gets closer to -1. This is why the function is always decreasing instead of increasing.

8. Vertical asymptotes can be found by finding the zeros of the denominator of the rational function. In this case, $x = 6$.

$$\begin{aligned} 0 &= 2 - n(6) \\ -2 + 0 &= -2 + 2 - n(6) \\ -2 &= -n(6) \\ \frac{-2}{6} &= \frac{-n(6)}{6} \\ \frac{-1}{3} &= -n \\ n &= \frac{1}{3} \end{aligned}$$

The x-intercept of a function occurs when $y = 0$. In this case, the y-intercept will be $(5, 0)$.

$$\begin{aligned} 0 &= \frac{7(5) - m}{2 - n(5)} \\ (2 - n(5)) \times 0 &= \frac{7(5) - m}{2 - n(5)} \times 2 - n(5) \\ 0 &= 35 - m \\ 0 + m &= 35 - m + m \\ m &= 35 \end{aligned}$$

9. A graph that has a factor of $(x + 2)$ in both the numerator and denominator will have a domain of $\{x \in \mathbf{R} \mid x \neq -2\}$. If the only other factor in the denominator is a constant, there will also be no vertical asymptote. Answers may vary. For example:

$$f(x) = \frac{4x + 8}{x + 2}$$

The graph of the function will be a horizontal line at $y = 4$ with a hone at $x = -2$.

5.4 Solving Rational Equations, pp. 285–287

1. Substitute both values into the equation. If both sides are equal, then the two values are solutions. The equation is

$$\frac{2}{x} = \frac{x - 1}{3}$$

$$\frac{2}{x} = \frac{x - 1}{3}$$

$$\frac{2}{3} = \frac{3 - 1}{3}$$

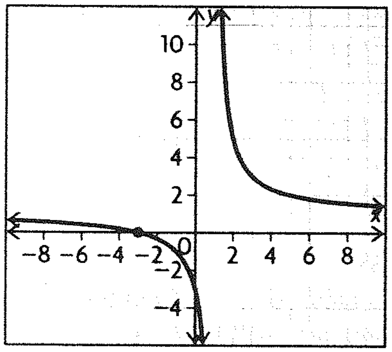
$$\frac{2}{3} = \frac{2}{3}$$

3 is a solution of the equation.

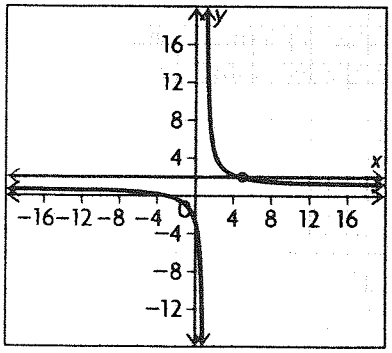
$$\frac{2}{x} = \frac{x-1}{3}$$
$$\frac{2}{-2} = \frac{-2-1}{3}$$
$$\frac{2}{-2} = \frac{-3}{3}$$
$$-1 = -1$$

−2 is a solution of the equation.

2. a)
$$\begin{aligned}\frac{x+3}{x-1} &= 0\\ (x-1)\times\frac{x+3}{x-1} &= 0\times(x-1)\\ x+3 &= 0\\ x &= -3\end{aligned}$$

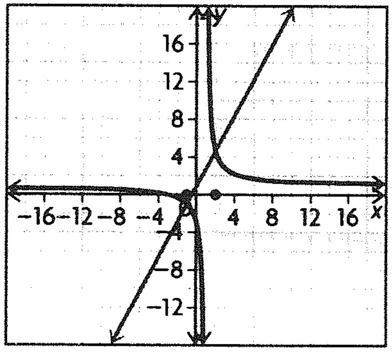


b)
$$\begin{aligned}\frac{x+3}{x-1} &= 2\\ (x-1)\times\frac{x+3}{x-1} &= 2\times(x-1)\\ x+3 &= 2x-2\\ x+3-3 &= 2x-2-3\\ x &= 2x-5\\ x-2x &= 2x-2x-5\\ -x &= -5\\ x &= 5\end{aligned}$$

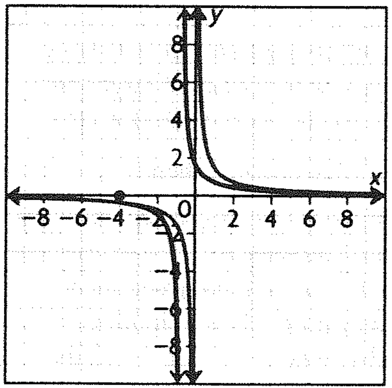


c)
$$\begin{aligned}\frac{x+3}{x-1} &= 2x+1\\ (x-1)\times\frac{x+3}{x-1} &= (2x+1)\times(x-1)\\ x+3 &= (2x+1)(x-1)\\ x+3 &= 2x^2+x-2x-1\\ x+3 &= 2x^2-x-1\\ 0 &= 2x^2-2x-4\\ 0 &= 2(x^2-x-2)\\ 0 &= (x^2-x-2)\\ 0 &= (x-2)(x+1)\end{aligned}$$
$$0 = x+1 \text{ or } 0 = x-2$$
$$-1 = x \qquad 2 = x$$

The solutions are $x = -1$ and 2.



d)
$$\begin{aligned}\frac{3}{3x+2} &= \frac{6}{5x}\\ \frac{3}{3x+2} &= \frac{6}{5x}\\ 6(3x+2) &= 3(5x)\\ 18x+12 &= 15x\\ 18x-18x+12 &= 15x-18x\\ 12 &= -3x\\ \frac{12}{-3} &= \frac{-3x}{-3}\\ -4 &= x\end{aligned}$$



3. Move all expressions to one side, so that one side of the equation is 0.

a) $\frac{x-3}{x+3}-2=0$

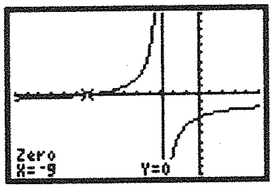
b) $\frac{3x-1}{x}-\frac{5}{2}=0$

c) $\frac{x-1}{x}-\frac{x+1}{x+3}=0$

d) $\frac{x-2}{x+3}-\frac{x-4}{x+5}=0$

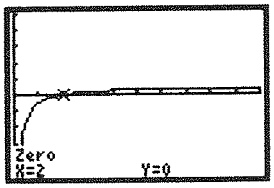
4. a)

$$\begin{aligned}\frac{x-3}{x+3}&=2\\(x+3)\times\frac{x-3}{x+3}&=2\times(x+3)\\x-3&=2x+6\\-9&=x\end{aligned}$$



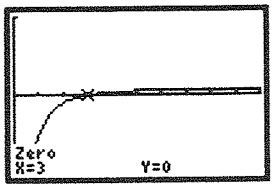
b)

$$\begin{aligned}\frac{3x-1}{x}&=\frac{5}{2}\\(3x-1)2&=5(x)\\6x-2&=5x\\x-2&=0\\x&=2\end{aligned}$$



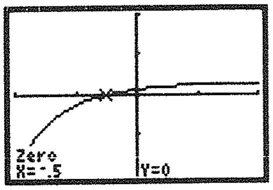
c)

$$\begin{aligned}\frac{x-1}{x}&=\frac{x+1}{x+3}\\(x-1)(x+3)&=x(x+1)\\x^2+3x-x-3&=x^2+x\\x^2+2x-3&=x^2+x\\2x-3&=x\\x&=3\end{aligned}$$



d)

$$\begin{aligned}\frac{x-2}{x+3}&=\frac{x-4}{x+5}\\(x-2)(x+5)&=(x+3)(x-4)\\x^2+3x-10&=x^2-x-12\\4x&=-2\\x&=-\frac{1}{2}\end{aligned}$$



5. a)

$$\begin{aligned}\frac{2}{x}+\frac{5}{3}&=\frac{7}{x}\\(3x)\left(\frac{2}{x}+\frac{5}{3}=\frac{7}{x}\right)\\6+5x&=21\\5x&=15\\x&=3\end{aligned}$$

b)

$$\begin{aligned}\frac{10}{x+3}+\frac{10}{3}&=6\\3(x+3)\left(\frac{10}{x+3}+\frac{10}{3}=6\right)\\30+10(x+3)&=18(x+3)\\30+10x+30&=18x+54\\60+10x&=18x+54\\-8x&=-6\\x&=\frac{3}{4}\end{aligned}$$

c)

$$\begin{aligned}\frac{2x}{x-3}&=1-\frac{6}{x-3}\\(x-3)\left(\frac{2x}{x-3}=1-\frac{6}{x-3}\right)\\2x&=1(x-3)-6\\2x&=(x-3)-6\\2x&=x-9\\x&=-9\end{aligned}$$

d)

$$\begin{aligned}\frac{2}{x+1}+\frac{1}{x+1}&=3\\(x+1)\left(\frac{2}{x+1}+\frac{1}{x+1}=3\right)\\2+1&=3x+3\\3&=3x+3\\0&=3x\\x&=0\end{aligned}$$

e) $$\frac{2}{2x+1} = \frac{5}{4-x}$$
$$2(4-x) = 5(2x+1)$$
$$8 - 2x = 10x + 5$$
$$8 - 5 - 2x = 10x + 5 - 5$$
$$3 - 2x = 10x$$
$$3 - 2x + 2x = 10x + 2x$$
$$3 = 12x$$
$$\frac{3}{12} = \frac{12x}{12}$$
$$\frac{1}{4} = x$$

f) $$\frac{5}{x-2} = \frac{4}{x+3}$$
$$5(x+3) = 4(x-2)$$
$$5x + 15 = 4x - 8$$
$$5x + 15 - 15 = 4x - 8 - 15$$
$$5x + 15 - 15 = 4x - 8 - 15$$
$$5x = 4x - 23$$
$$5x - 4x = 4x - 4x - 23$$
$$x = -23$$

6. a) $$\frac{2x}{2x+1} = \frac{5}{4-x}$$
$$2x(4-x) = 5(2x+1)$$
$$8x - 2x^2 = 10x + 5$$
$$8x - 2x^2 + 2x^2 = 2x^2 + 10x + 5$$
$$8x = 2x^2 + 10x + 5$$
$$8x - 8x = 2x^2 + 10x - 8x + 5$$
$$0 = 2x^2 + 2x + 5$$

Examine the equation. Notice that it will not have any real zeros. Therefore, the function will have no real solutions.

b) $$\frac{3}{x} + \frac{4}{x+1} = 2$$
$$x(x+1)\frac{3}{x} + x(x+1)\frac{4}{x+1} = x(x+1)2$$
$$3 + 7x = 2x^2 + 2x$$
$$3 - 3 + 7x - 7x = 2x^2 + 2x - 7x - 3$$
$$0 = 2x^2 - 5x - 3$$

Use the quadratic formula to solve the quadratic equation.

$x = 3$ and $x = -0.5$

c) $$\frac{2x}{5} = \frac{x^2 - 5x}{5x}$$
$$2x \times 5x = 5(x^2 - 5x)$$
$$10x^2 = 5x^2 - 25x$$
$$10x^2 - 5x^2 = 5x^2 - 5x^2 - 25x$$
$$5x^2 + 25x = -25x + 25x$$
$$5x^2 + 25x = 0$$
$$5x(x+5) = 0$$
$$5x = 0 \text{ or } \quad x + 5 = 0$$
$$x + 5 - 5 = 0 - 5$$
$$x = 0 \qquad x = -5$$

d) $$x + \frac{x}{x-2} = 0$$
$$(x-2)x + (x-2)\frac{x}{x-2} = (x-2)0$$
$$x^2 - 2x + x = 0$$
$$x^2 + x = 0$$
$$x(x+1) = 0$$
$$x = 0 \text{ or } x + 1 = 0$$
$$x + 1 = 0$$
$$x + 1 - 1 = 0 - 1$$
$$x = -1$$

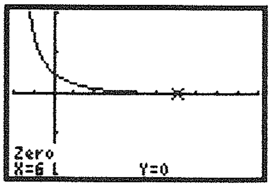
e) $$\frac{1}{x+2} + \frac{24}{x+3} = 13$$
$$(x+2)(x+3)\frac{1}{x+2} + (x+2)(x+3)\frac{24}{x+3}$$
$$= 13(x+2)(x+3)$$
$$x + 3 + 24(x+2) = 13(x^2 + 5x + 6)$$
$$25x \overset{48}{+} 5 = 13x^2 + 65x + 78$$
$$25x - 25x + 5 - 5 = 13x^2 + 65x - 25x + 78 - 5$$
$$0 = 13x^2 + 65x - 25x + 78 - 5$$
$$0 = 13x^2 + 40x + \cancel{73}\ 23$$

Examine the quadratic equation. There are no real zeros and that means that the original equation has no real solutions.

f) $$\frac{-2}{x-1} = \frac{x-8}{x+1}$$
$$-2(x+1) = (x-8)(x-1)$$
$$-2x - 2 = x^2 - 9x + 8$$
$$0 = x^2 - 7x + 10$$
$$0 = (x-5)(x-2)$$
$$0 = x - 5 \text{ or } 0 = x - 2$$
$$x = 5 \qquad x = 2$$

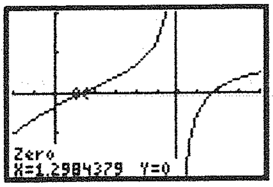
7. Move all terms to one side of the equation so that one side of the equation is 0. Graph the expression on the other side and use the zero function of the calculator to solve.

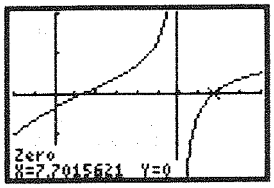
a)



$x = 6$

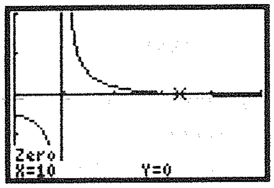
b)


Zero
X=1.2984379 Y=0


Zero
X=7.7015621 Y=0

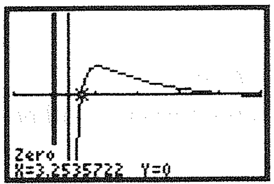
$x = 1.30, 7.70$

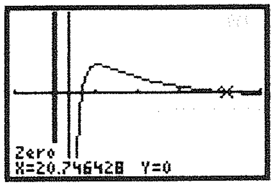
c)


Zero
X=10 Y=0

$x = 10$

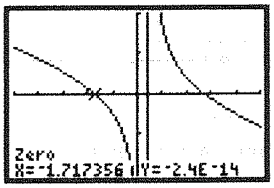
d)


Zero
X=3.2535722 Y=0

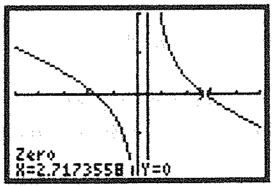

Zero
X=20.746428 Y=0

$x = 3.25, 20.75$

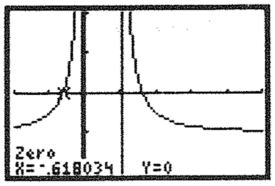
e)

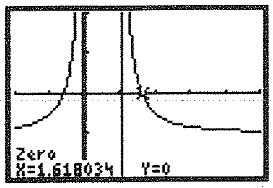

Zero
X=⁻1.717356 Y=⁻2.4E⁻14

$x = -1.71, 2.71$


Zero
X=2.7173558 Y=0

f)


Zero
X=⁻.618034 Y=0


Zero
X=1.618034 Y=0

$x = -0.62, 1.62$

8. a) $\dfrac{x + 1}{x - 2} = \dfrac{x + 3}{x - 4}$

Multiply both sides of the equation by the LCD, $(x - 2)(x - 4)$.

$$(x - 2)(x - 4)\left(\frac{x + 1}{x - 2}\right) = (x - 2)(x - 4)\left(\frac{x + 3}{x - 4}\right)$$

$(x - 4)(x + 1) = (x - 2)(x + 3)$

Simplify.

$x^2 - 3x - 4 = x^2 + x - 6$

Simplify the equation so that 0 is on one side of the equation.

$$x^2 - x^2 - 3x - x - 4 + 6 = x^2 - x^2 + x - x - 6 + 6$$

$$-4x + 2 = 0$$

$$-2(2x - 1) = 0$$

Since the product is equal to 0 one of the factors must be equal to 0. It must be $2x - 1$ because 2 is a constant.

$$2x - 1 = 0$$

$$2x - 1 + 1 = 0 + 1$$

$$2x = 1$$

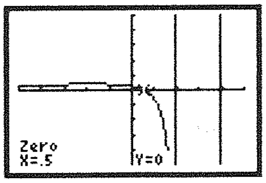
$$\frac{2x}{2} = \frac{1}{2}$$

$$x = \frac{1}{2}$$

b) Substitute $x = \frac{1}{2}$ to verify the solution.

$\dfrac{\frac{1}{2} + 1}{\frac{1}{2} - 2} = -1$ and $\dfrac{\frac{1}{2} + 3}{\frac{1}{2} - 4} = -1$

c) Graph the equation $\dfrac{(x + 1)}{(x - 2)} - \dfrac{(x + 3)}{(x - 4)}$ and determine the zeros to verify the solution.


Zero
X=.5 Y=0

9. Multiply both sides by the LCD, $w(15 - w)$

$$w(15 - w)\left(\frac{15}{w}\right) = w(15 - w)\left(\frac{w}{15 - w}\right)$$

$$(15 - w)(15) = w^2$$

$$225 - 15w = w^2$$

$$\begin{aligned}225 - 225 - 15w + 15w &= w^2 - 15w - 225\\ 0 &= w^2 - 15w - 225\\ w^2 - 15w - 225 &= 0\end{aligned}$$

Use the quadratic equation to help you solve the quadratic formula.
$w = 9.271$ and $w = -24.27$
Since a width has to be positive, $w = 9.271$.

10. Machine A has a rate of $\frac{1}{s}$ boxes/minute.
Machine B has a rate of $\frac{1}{s + 10}$ boxes/minute.
Their combined rate is $\frac{1}{s} + \frac{1}{s + 10} = 15$. Solve this equation for s.

$$\begin{aligned}\frac{1}{s} + \frac{1}{s + 10} &= \frac{1}{15}\\ 15s(s + 10)\left(\frac{1}{s} + \frac{1}{s + 10} = \frac{1}{15}\right)&\\ 15(s + 10) + 15s &= s(s + 10)\\ 30s + 150 &= s^2 + 10s\\ 30s - 30s + 150 - 150 &= s^2 + 10s - 30s - 150\\ 0 &= s^2 - 20s - 150\end{aligned}$$

Use the quadratic formula to help you solve the quadratic equation.
$s = 25.8$
Machine A takes 25.8 and Machine B takes 35.8 minutes.

11. The price per comic in the box that Tayla purchase is $\frac{300}{s}$, where s is the number of comics in the box. She gave 15 away, and so the number of comics in the box becomes $s - 15$. The price per comic in the box when she resold the box on the Internet then is $\frac{330}{s - 15}$. Tayla made a profit of \$1.50 on each comic, which is the sale price per comic minus the original purchase price per comic. Solve the equation $\frac{330}{s - 15} - \frac{300}{s} = 1.50$ to find the original number of comics.

$$\begin{aligned}\frac{330}{s - 15} - \frac{300}{s} &= 1.50\\ s(s - 15)\left(\frac{330}{s - 15} - \frac{300}{s} = 1.50\right)&\\ 330s - 300(s - 15) &= 1.5s(s - 15)\\ 330s - 300s + 4500 &= 1.5s^2 - 22.5s\\ 30s + 4500 &= 1.5s^2 - 22.5s\\ 30s - 30s + 4500 - 4500 &= 1.5s^2 - 30s\\ &\quad - 22.5s - 4500\\ 0 &= 1.5s^2 - 52.5s - 4500\end{aligned}$$

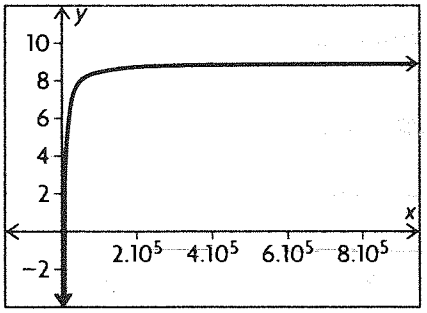
The roots are 75.00 and -40. Since you can't have a negative number of comics, the correct answer would be 75. The original price per comic would be $\frac{300}{75} = \$4$. The resale price per comic would be $\frac{300}{60} = \$5.50$.

12. a) Substitute 6 into the formula for $c(t)$ and solve for t.

$$\begin{aligned}6 &= 9 - 90\,000\left(\frac{1}{10\,000 + 3t}\right)\\ 6 - 9 &= 9 - 9 - 90\,000\left(\frac{1}{10\,000 + 3t}\right)\\ -3(10\,000 + 3t) &= -90\,000\left(\frac{1}{10\,000 + 3t}\right)\\ &\quad(10\,000 + 3t)\\ -30\,000 - 9t &= -90\,000\\ -30\,000 + 30\,000 - 9t &= -90\,000 + 30\,000\\ -9t &= -60\,000\\ \frac{-9t}{-9} &= \frac{-60\,000}{-9}\\ t &= 6666.67\end{aligned}$$

After 6666.67 seconds the concentration will be 6 kg/m^3.

b) Graph the function to help you understand the function's behaviour over time.



The function appears to approach 9 kg/m^3 as time increases.

13. a) Tom can fill $\frac{1}{s}$ of an order in 1 minute. Paco and Carl's rates are similar: $\frac{1}{s - 2}$ and $\frac{1}{s + 1}$.
Working together Tom and Paco can fill an order in about 1 minute and 20 seconds, or about 1.33 minutes. Solve $\frac{1}{s - 2} + \frac{1}{s} = \frac{1}{1.33}$ to find how long it takes each person to fill an order.

$$\begin{aligned}\frac{1}{s - 2} + \frac{1}{s} &= \frac{1}{1.33}\\ \left(\frac{s}{s}\right)\frac{1}{s - 2} + \frac{1}{s}\left(\frac{s - 2}{s - 2}\right) &= \frac{1}{1.33}\end{aligned}$$

$$\frac{s}{(s)(s-2)} + \frac{s-2}{(s)(s-2)} = \frac{1}{1.33}$$
$$\frac{2s-2}{(s)(s-2)} = \frac{1}{1.33}$$
$$1.33(2s-2) = s^2 - 2s$$
$$0 = s^2 - 4.66 + 2.66$$

Use the quadratic formula to solve this equation. The roots of the function are $s = 3.994$ or 0.66. Because Paco can fill the order in 2 minutes less than Tom, and because you can't have a negative amount of time, Tom's time must be 3.994, or about 4 minutes. So, Paco can fill the order in about 2 minutes and Carl can fill the order in about 5.

b) Add their rates together to determine how long it would take them to fill the order working together.

$$\frac{1}{5} + \frac{1}{4} + \frac{1}{2} = 0.95$$
$$\frac{1}{0.95} = 1.05$$

Working together they can fill the order in about 1.05 minutes.

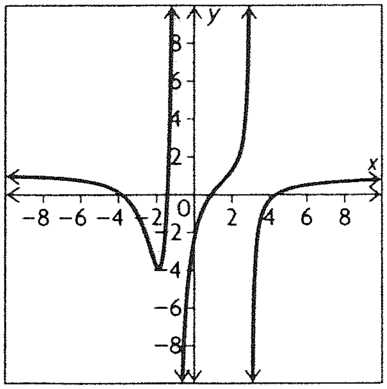
14. Answers may vary. For example: You can use either algebra or graphing technology to solve a rational equation. With algebra, solving the equation takes more time, but you get an exact answer. With graphing technology, you can solve the equation quickly, but you do not always get an exact answer.

15. $\dfrac{x^2 - 6x + 5}{x^2 - 2x - 3} = \dfrac{2 - 3x}{x^2 + 3x + 3}$

Turn this into an equation that you can graph to find the solutions.

$$y = \frac{x^2 - 6x + 5}{x^2 - 2x - 3} - \frac{2 - 3x}{x^2 + 3x + 3}$$

Graph the equation.

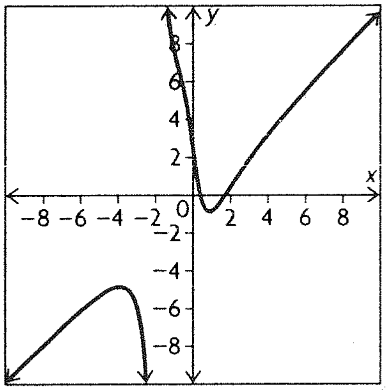


The solutions are approximately $x = -3.80, -1.42, 0.90$, and 4.33.

16. a) The graphs will have the same position when their equations are equal. Set the two equations equal to each other and then solve for t. Graph the equation to help you.

$$\frac{7t}{t^2 + 1} = t + \frac{5}{t + 2}$$
$$y = t + \frac{5}{t+2} - \frac{7t}{t^2 + 1}$$

Graph the equation.

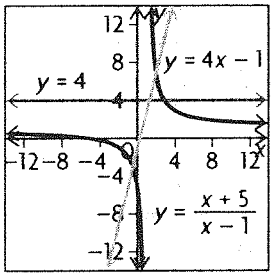


Examine the graph. The zeros are $x = 0.438$ and 1.712.

b) Object A is closer to the origin than object B when $\frac{7t}{t^2 + 1} < t + \frac{5}{t + 2}$ or when $0 < t + \frac{5}{t + 2} - \frac{7t}{t^2 + 1}$. Examine the graph and find when the function is positive to solve the inequality. The graph shows that the inequality is true on (0, 0.438) and (1.712, ∞).

5.5 Solving Rational Inequalities, pp. 295–297

1. Use the graph given to help you solve the inequalities.



a) $\dfrac{x + 5}{x - 1} < 4$

Examine the graph. To determine when $\frac{x + 5}{x - 1} < 4$, determine when the green curve is below the blue line. This is true on the intervals (∞, 1) and (3, ∞).

b) $4x - 1 > \frac{x + 5}{x - 1}$

Examine the graph. To determine when $4x - 1 > \frac{x + 5}{x - 1}$, determine when the red line is above the green curve. This is true on the intervals $(-0.5, 1)$ and $(2, \infty)$.

2. a) Solve the inequality for x.

$$\frac{6x}{x + 3} \le 4$$

$$\frac{6x}{x + 3} - 4 \le 0$$

$$\frac{6x}{x + 3} - 4\frac{x + 3}{x + 3} \le 0$$

$$\frac{6x - 4x - 12}{x + 3} \le 0$$

$$\frac{2x - 12}{x + 3} \le 0$$

$$\frac{2(x - 6)}{x + 3} \le 0$$

b)

c) The solution is $(-3, 6]$.

3. a)

$$x + 2 > \frac{15}{x}$$

$$x + 2 - \frac{15}{x} > 0$$

$$\frac{x^2}{x} + \frac{2x}{x} - \frac{15}{x} > 0$$

$$\frac{x^2 + 2x - 15}{x} > 0$$

$$\frac{(x + 5)(x - 3)}{x} > 0$$

b)

	$x < -5$	$-5 < x < 0$	$0 < x < 3$	$x > 3$
$x + 5$	−	+	+	+
$x - 3$	−	−	−	+
x	−	−	+	+
$\frac{(x + 5)(x - 3)}{x}$	−	+	−	+

The equation is negative on $x < -5$ and $0 < x < 3$ and positive on $-5 < x < 0$ and $x > 3$.

c) The solution to the equation is $\{x \in \mathbf{R} \mid x > -5\}$. This can also be written as $(-5, \infty)$.

4. a)

$$\frac{1}{x + 5} > 2$$

$$\frac{1}{x + 5} - 2 > 0$$

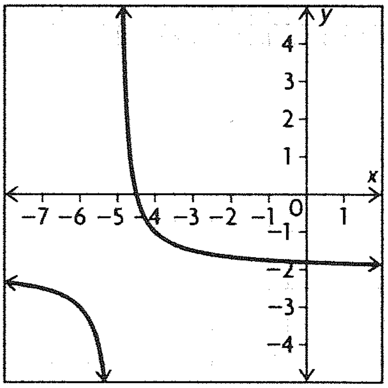
$$\frac{1}{x + 5} - 2\left(\frac{x + 5}{x + 5}\right) > 0$$

$$\frac{1}{x + 5} + \frac{-2x - 10}{x + 5} > 0$$

$$\frac{-2x - 9}{x + 5} > 0$$

	$x < -5$	$-5 < x < -4.5$	$x > -4.5$
$-2x - 9$	+	+	−
$x + 5$	−	+	+
$\frac{-2x - 9}{x + 5}$	−	+	−

The inequality is true on $-5 < x < -4.5$.



b)

$$\frac{1}{2x + 10} < \frac{1}{x + 3}$$

$$\frac{1}{x + 5} - \frac{1}{x + 3} < 0$$

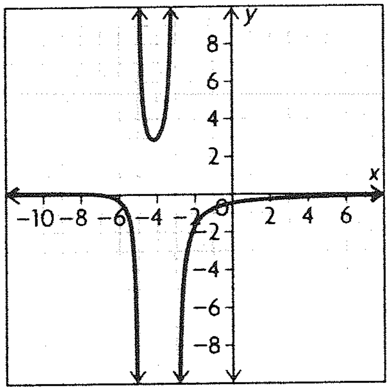
$$\left(\frac{x + 3}{x + 3}\right)\frac{1}{x + 5} - \frac{1}{x + 3}\left(\frac{x + 5}{x + 5}\right) < 0$$

$$\frac{x + 3}{(x + 5)(x + 3)} + \frac{-2x - 10}{(x + 3)(x + 5)} < 0$$

$$\frac{-x - 7}{(x + 5)(x + 3)} < 0$$

	$x < -7$	$-7 < x < -5$	$-5 < x < -3$	$x > -3$
$-x - 7$	+	−	−	−
$x + 5$	−	−	+	+
$x + 3$	−	−	−	+
$\frac{-x - 7}{(x + 5)(x + 3)}$	+	−	+	−

The inequality is true on $-7 < x < -5$ and $x > -3$.



c)

$$\frac{3}{x-2} < \frac{4}{x}$$

$$\frac{3}{x-2} - \frac{4}{x} < 0$$

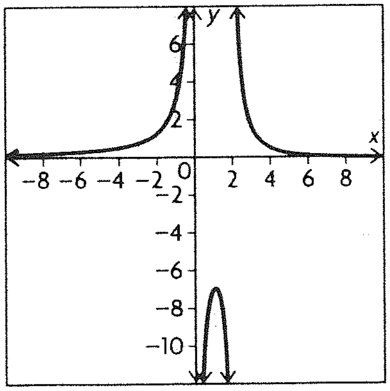
$$\left(\frac{x}{x}\right)\frac{3}{x-2} - \frac{4}{x}\left(\frac{x-2}{x-2}\right) < 0$$

$$\frac{3x}{x(x-2)} + \frac{-4x+8}{x(x-2)} < 0$$

$$\frac{-x+8}{x(x-2)} < 0$$

	$x < 0$	$0 < x < 2$	$2 < x < 8$	$x > 8$
$-x + 8$	+	+	+	−
x	−	+	+	+
$x - 2$	−	−	+	+
$\frac{-x+8}{x(x-2)}$	+	−	+	−

The inequality is true on $0 < x < 2$ and $x > 8$.



d)

$$\frac{7}{x-3} \geq \frac{2}{x+4}$$

$$\frac{7}{x-3} - \frac{2}{x+4} \geq 0$$

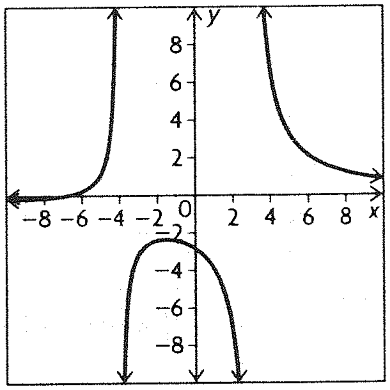
$$\left(\frac{x+4}{x+4}\right)\frac{7}{x-3} - \frac{2}{x+4}\left(\frac{x-3}{x-3}\right) \geq 0$$

$$\frac{7x+28}{(x+4)(x-3)} + \frac{-2x+6}{(x+4)(x-3)} \geq 0$$

	$x < -6.8$	$-6.8 < x < -4$	$-4 < x < 3$	$x > 3$
$5x + 34$	−	+	+	+
$x + 4$	−	−	+	+
$x - 3$	−	−	−	+
$\frac{5x+34}{(x+4)(x-3)}$	−	+	−	+

$$\frac{5x+34}{(x+4)(x-3)} \geq 0$$

The inequality is true on $-6.8 \leq x < -4$ and $x > 3$.



e)

$$\frac{-6}{x+1} > \frac{1}{x}$$

$$\frac{-6}{x+1} - \frac{1}{x} > 0$$

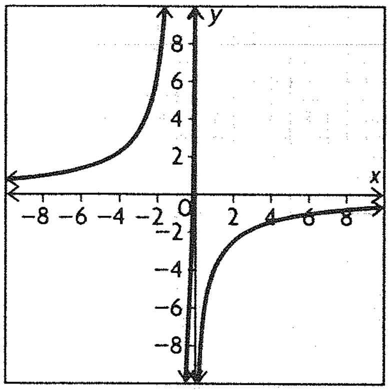
$$\left(\frac{x}{x}\right)\frac{-6}{x+1} - \frac{1}{x}\left(\frac{x+1}{x+1}\right) > 0$$

$$\frac{-6x}{x(x+1)} + \frac{-x-1}{x(x+1)} > 0$$

$$\frac{-7x-1}{x(x+1)} > 0$$

	$x < -1$	$-1 < x < -\frac{1}{7}$	$-\frac{1}{7} < x < 0$	$x > 0$
$-7x - 1$	+	+	−	−
x	−	−	−	+
$x + 1$	−	+	+	+
$\frac{-7x-1}{x(x+1)}$	+	−	+	−

The inequality is true on $x < -1$ and $-\frac{1}{7} < x < 0$.



f)

$$\frac{-5}{x-4} < \frac{3}{x+1}$$

$$\frac{-5}{x-4} - \frac{3}{x+1} < 0$$

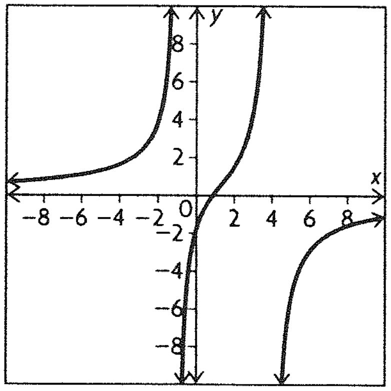
$$\left(\frac{x+1}{x+1}\right)\frac{-5}{x-4} - \frac{3}{x+1}\left(\frac{x-4}{x-4}\right) < 0$$

$$\frac{-5x-5}{(x+1)(x-4)} + \frac{-3x+12}{(x+1)(x-4)} < 0$$

$$\frac{-8x+7}{(x+1)(x-4)} < 0$$

	$x < -1$	$-1 < x < \frac{7}{8}$	$\frac{7}{8} < x < 4$	$x > 4$
$-8x + 7$	+	+	−	−
$x - 4$	−	−	−	+
$x + 1$	−	+	+	+
$\frac{-8x+7}{(x+1)(x-4)}$	+	−	+	−

The inequality is true on $-1 < x < \frac{7}{8}$ and $x < 4$.



5. a) $\frac{t^2 - t - 12}{t - 1} < 0$

$$\frac{t^2 - t - 12}{t-1} = \frac{(t-4)(t+3)}{t-1}$$

	$t < -3$	$-3 < t < 1$	$1 < t < 4$	$t > 4$
$t - 4$	−	−	−	+
$t + 3$	−	+	+	+
$t - 1$	−	−	+	+
$\frac{(t-4)(t+3)}{t-1}$	−	+	−	+

The inequality is true on $t < -3$ or $1 < t < 4$.

b) $\frac{t^2 + t - 6}{t - 4} \geq 0$

$$\frac{(t+3)(t-2)}{t-4} \geq 0$$

	$t < -3$	$-3 < t < 2$	$2 < t < 4$	$t > 4$
$t + 3$	−	+	+	+
$t - 2$	−	−	+	+
$t - 4$	−	−	−	+
$\frac{(t+3)(t-2)}{t-4}$	−	+	−	+

The inequality is true on $-3 \leq t \leq 2$ and $t > 4$.

c) $\frac{6t^2 - 5t + 1}{2t + 1} > 0$

$$\frac{(3t-1)(2t-1)}{2t-1} > 0$$

The function is true on $-\frac{1}{2} < x < \frac{1}{3}$ and $x > \frac{1}{2}$

	$t < -\frac{1}{2}$	$-\frac{1}{2} < t < \frac{1}{3}$	$\frac{1}{3} < t < \frac{1}{2}$	$t > \frac{1}{2}$
$3t - 1$	−	−	+	+
$2t - 1$	−	−	−	+
$2t + 1$	−	+	+	+
$\frac{(3t-1)(2t-1)}{2t+1}$	−	+	−	+

d)

$$t - 1 < \frac{30}{5t}$$

$$\left(\frac{5t}{5t}\right)t - \left(\frac{5t}{5t}\right)1 - \frac{30}{5t} < 0$$

$$\frac{5t^2}{5t} - \frac{5t}{5t} - \frac{30}{5t} < 0$$

$$\frac{5t^2 - 5t - 30}{5t} < 0$$

$$\frac{5(t^2 - t - 6)}{5t} < 0$$

$$\frac{(t-3)(t+2)}{t} < 0$$

	$t < -2$	$-2 < t < 0$	$0 < t < 3$	$t > 3$
$t - 3$	−	−	−	+
$t + 2$	−	+	+	+
t	−	−	+	+
$\frac{(t-3)(t+2)}{t}$	−	+	−	+

The inequality is true for $t < -2$ and $0 < t < 3$.

e)
$$\frac{2t - 10}{t} > t + 5$$
$$\frac{2t - 10}{t} - t - 5 > 0$$
$$\frac{2t - 10}{t} - t\left(\frac{t}{t}\right) - 5\left(\frac{t}{t}\right) > 0$$
$$\frac{2t - 10 - t^2 - 5t}{t} > 0$$
$$\frac{-t^2 - 7t - 10}{t} > 0$$
$$\frac{t^2 + 7t + 10}{t} < 0$$
$$\frac{(t + 5)(t + 2)}{t} < 0$$

	$t < -5$	$-5 < t < -2$	$-2 < t < 0$	$t > 0$
$t + 5$	−	+	+	+
$t + 2$	−	−	+	+
t	−	−	−	+
$\frac{(t+5)(t+2)}{t}$	−	+	−	+

The inequality is true on $t < -5$ and $-2 < t < 0$.

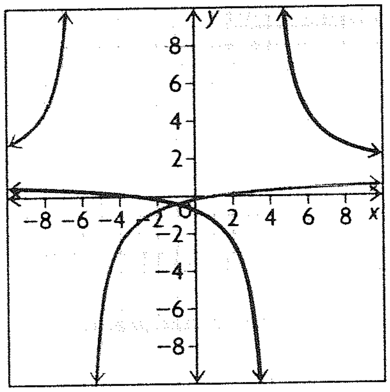
f)
$$\frac{-t}{4t - 1} \geq \frac{2}{t - 9}$$
$$\frac{-t}{4t - 1} - \frac{2}{t - 9} \geq 0$$
$$\left(\frac{t - 9}{t - 9}\right)\frac{-t}{4t - 1} - \frac{2}{t - 9}\left(\frac{4t - 1}{4t - 1}\right) \geq 0$$
$$\frac{-t^2 + 9t}{(4t - 1)(t - 9)} + \frac{-8t + 2}{(4t - 1)(t - 9)} \geq 0$$
$$\frac{-t^2 + t + 2}{(4t - 1)(t - 9)} \geq 0$$
$$\frac{(-t + 2)(t + 1)}{(4t - 1)t - 9)} \geq 0$$

	$t < -1$	$-1 < t < 0.25$	$0.25 < t < 2$	$2 < t < 9$	$t > 9$
$t + 1$	−	+	+	+	+
$-t + 2$	+	+	+	−	−
$4t - 1$	−	−	+	+	+
$t - 9$	−	−	−	−	+
$\frac{(-t+2)(t+1)}{(4t-1)(t-9)}$	−	+	−	+	−

The inequality is true on $-1 \leq t < 0.25$ and $2 \leq t < 9$.

6. Graph each expression to determine when the inequality is true.

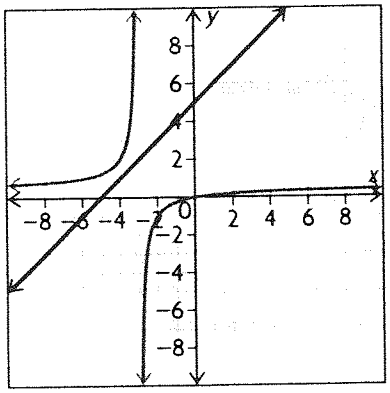
a) $\frac{x + 3}{x - 4} \geq \frac{x - 1}{x + 6}$



The two graphs intersect at $(-1, -0.4)$. The asymptotes are at $x = 4$ and $x = -6$. The graph of $y = \frac{x + 3}{x - 4}$ is above $y = \frac{x - 1}{x + 6}$ on $x < -6$ and on $-1 < x < 4$.

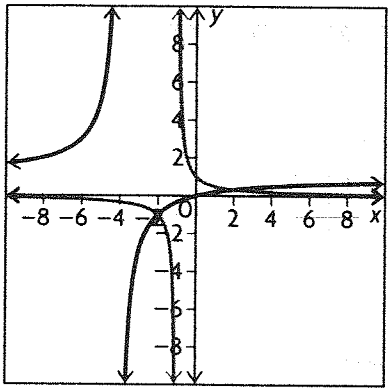
b) $x + 5 < \frac{x}{2x + 6}$

Graph each expression to determine when the inequality is true.



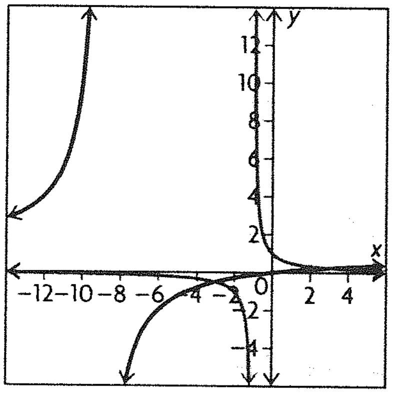
Notice that the graph of $y = x + 5$ is above the graph of $y = \frac{x}{2x + 6}$ after the vertical asymptote. The vertical asymptote occurs at $x = -3$. The solution is $x > 3$.

c) $\frac{x}{x + 4} \leq \frac{1}{x + 1}$
Graph each expression to determine when the inequality is true.



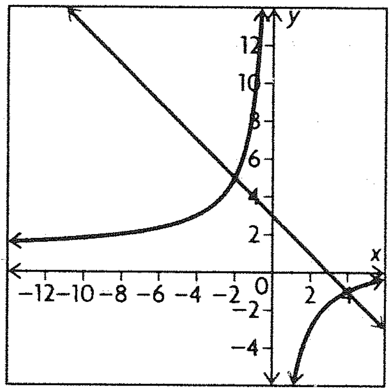
The graphs intersect at $(-2, -1)$ and $\left(2, \frac{1}{3}\right)$. The graph of $y = \frac{x}{x + 4}$ is below the other graph on $(-4, 2)$ and $(-1, 2)$.

d) $\frac{x}{x + 9} \geq \frac{1}{x + 1}$
Graph each expression to determine when the inequality is true.



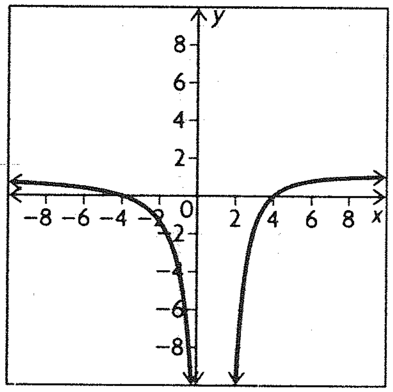
The graphs intersect at $(-3, -0.5)$ and $(3, 0.25)$. Because the inequality is $\geq$, the intervals that make the inequality true will include the points of intersection. The graph of $y = \frac{x}{x + 9}$ is above or intersecting with the other graph on $(-\infty, -9)$, $[-3, -1)$, and $[3, \infty)$.

e) $\frac{x - 8}{x} > 3 - x$
Graph each expression to determine when the inequality is true.



The two graphs intersect at $(-2, 5)$ and $(4, -1)$. The graph of $y = \frac{x - 8}{x}$ is above the other graph on $(-2, 0)$ and $(4, \infty)$.

f) $\frac{x^2 - 16}{(x - 1)^2} \geq 0$
Graph the expression and determine when the graph is above the x-axis.



The graph intersects the x-axis at $(-4, 0)$ and $(4, 0)$. The graph of $y = \frac{x^2 - 16}{(x - 1)^2}$ is above the x-axis on $(-\infty, -4)$ and $(4, \infty)$.

7. a)

$$\frac{3x-8}{2x-1} > \frac{x-4}{x+1}$$

$$\frac{3x-8}{2x-1} - \frac{x-4}{x+1} > 0$$

$$\left(\frac{x+1}{x+1}\right)\frac{3x-8}{2x-1} - \frac{x-4}{x+1}\left(\frac{2x-1}{2x-1}\right) > 0$$

$$\frac{3x^2-8x+3x-8}{(x+1)(2x-1)} \overset{-}{\cancel{+}} \frac{2x^2-x-8x+4}{(x+1)(2x-1)} > 0$$

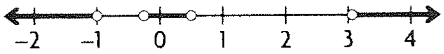
$$\frac{\cancel{3}5x^2-14x-4}{(x+1)(2x-1)} > 0$$

$$\frac{(x-3.065)(x+0.2614)}{(x+1)(2x-1)} > 0$$

	$x < -1$	$-1 < x < -0.2614$	$-0.2614 < x < 0.5$	$0.5 < x < 3.065$	$x > 3.065$
$x - 3.065$	−	−	−	−	+
$x + 0.2614$	−	−	+	+	+
$x + 1$	−	+	+	+	+
$2x - 1$	−	−	−	+	+
$\frac{(x-3.065)(x+0.2614)}{(x+1)(2x-1)}$	+	−	+	−	+

The inequality is true on
$x < -1$, $-0.2614 < x < 0.5$, and $x > 3.065$.

b)



Interval notation:
$(-\infty, -1)$, $(-0.2614, 0.5)$, $(3.065, \infty)$
Set notation:
$\{x \in \mathbf{R} \mid x < -1, -0.2614 < x < 0.5, \text{ or } x > 3.065\}$

8. a)

$$\frac{-6t}{t-2} < \frac{-30}{t-2}$$

$$\frac{-6t}{t-2} - \frac{-30}{t-2} < 0$$

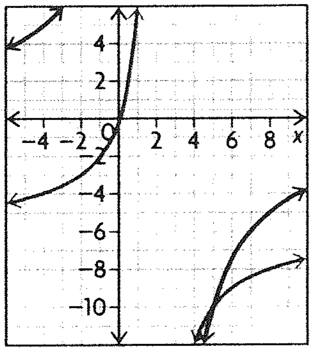
$$\frac{-6t+30}{t-2} < 0$$

$$\frac{-6(t-5)}{t-2} < 0$$

	$t < 2$	$2 < t < 5$	$t > 5$
$t - 5$	−	−	+
-6	−	−	−
$t - 2$	−	+	+
$\frac{-6(t+5)}{t-2}$	−	+	−

The inequality is true on $t < 2$ and $t < 5$.

b)



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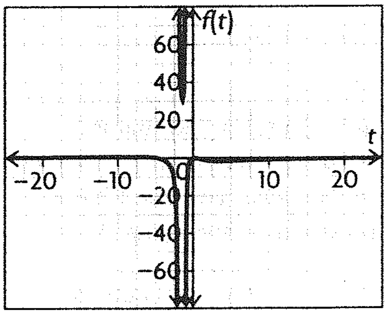
c) It would be difficult to find a situation that could be represented by these rational expressions because very few positive values of x yield a positive value of y.

9. The equation that gives the bacteria count over time for the tap water is $f(t) = \frac{5t}{t^2+3t+2}$. The equation that gives the bacteria count for the pond water over time is $g(t) = \frac{15t}{t^2+9}$. To see if the bacteria count in the tap water will ever exceed that of the pond water, set up the inequality $\frac{5t}{t^2+3t+2} > \frac{15t}{t^2+9}$. Solve this inequality graphically. Graph the expression $y = \frac{5t}{t^2+3t+2} - \frac{15t}{t^2+9}$ and determine when it is greater than 0 to find the solution to the inequality.



Notice that the only values that make the expression greater than 0 are negative. Because the values of t have to be positive, the bacteria count in the tap water will never be greater than that of the pond water.

10. a)

$$0.5x - 2 < \frac{5}{2x}$$

$$0.5x - 2 - \frac{5}{2x} < 0$$

$$\left(\frac{2x}{2x}\right)0.5x - \left(\frac{2x}{2x}\right)2 - \frac{5}{2x} < 0$$

$$\frac{x^2 - 4x - 5}{2x} < 0$$

$$\frac{(x - 5)(x + 1)}{2x} < 0$$

b)

	$x < -1$	$-1 < x < 0$	$0 < x < 5$	$x > 5$
$x - 5$	−	−	−	+
$x + 1$	−	+	+	+
$2x$	−	−	+	+
$\frac{(x - 5)(x + 1)}{2x}$	−	+	−	+

The inequality is true for $x < -1$ and $0 < x < 5$.

11. The profit would be the revenue minus the cost, $R(x) - C(x) = P(x)$. This is $P(x) = -x^2 + 10x - (4x + 5)$. Simplify.

$$\begin{aligned} P(x) &= -x^2 + 10x - (4x + 5) \\ &= -x^2 + 6x - 5 \\ &= x^2 - 6x + 5 \\ &= (x - 1)(x - 5) \end{aligned}$$

Divide this by x to get the average profit.

$$AP(x) = \frac{(x - 1)(x - 5)}{x}$$

Substitute the factors of this equation into a table to determine when $AP(x) > \frac{(x - 1)(x - 5)}{x}$.

	$x < 0$	$0 < x < 1$	$1 < x < 5$	$x > 5$
$(x - 5)$	−	−	−	+
$(x + 1)$	−	−	+	+
x	−	+	+	+
$\frac{(x - 1)(x - 5)}{x}$	−	+	−	+

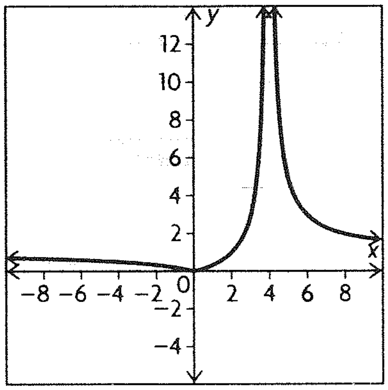
From the table, it can be determined that the average profit per snowboard is positive between $0 < x < 1$ and $x > 5$. Because you can't have a partial number of snowboards, the inequality is true on $x > 5$.

12. a) The first inequality can be manipulated algebraically to produce the second inequality.

b) You could graph the equation $y = \frac{x + 1}{x - 1} - \frac{x + 3}{x + 2}$ and determine when it is negative.

c) The values that make the factors of the second inequality zero are -5, -2, and 1. Determine the sign of each factor in the intervals corresponding to the zeros. Determine when the entire expression is negative by examining the signs of the factors.

13. You can graph the inequality to help you determine when the inequality is true.

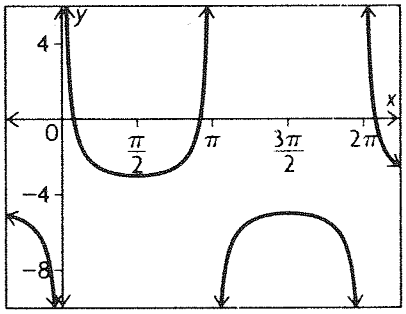


Notice that the function is greater than 1 on [2, 4) and (4, ∞).

14. $\frac{1}{\sin(x)} < 4, 0° \leq x \leq 360°$

$$\frac{1}{\sin(x)} - 4 < 0$$

You can graph this inequality and then determine when the graph is negative.



The graph is negative on $14.48 < x < 165.52$ and $180 < x < 360$.

15. $\frac{\cos(x)}{x} > 0.5 \quad 0° < x < 90°$

Examine the inequality. You want the numerator to be greater than half of the denominator.

$\cos(x) > 0.5x$

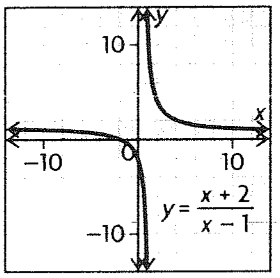
$x = 0$ would be the asymptote in this case, but values greater than zero could satisfy the inequality.

$\cos(1) > 0.5$

$0.999 > 0.5$
$\cos(x)$ becomes less than $0.5x$ after about 2.
$\cos(2) < 1$
$0.99 < 1$
So the inequality is true on $0 < x < 2$.

5.6 Rates of Change in Rational Functions, pp. 303–305

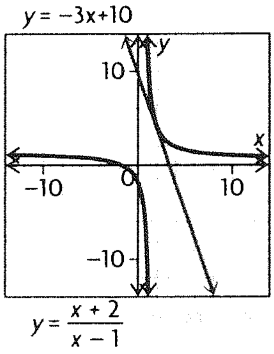
1. a)



Use the graph to determine $f(2)$ and $f(7)$.
$f(2) = 4$
$f(7) = 1.5$

$$y = \frac{1.5 - 4}{7 - 2}$$
$$= -0.5$$

b)



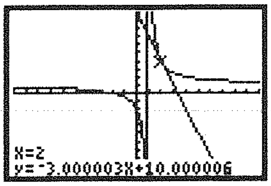
The slope of the tangent line is -3.

2. Use the difference quotient to help you estimate the average rate of change at $x = 2$. Use $h = 0.01$

$$\frac{f(a + h) - f(a)}{h}$$
$$= \frac{f(2) - f(2.01)}{0.01}$$
$$= \frac{3.97 - 4}{0.01}$$
$$= -3$$

The answers that I received match.

3. Use graphing technology to graph the function. Have the calculator draw a line tangent to point $x = 2$. Find the slope of that line.
The slope is -3, so the instantaneous rate of change at $x = 2$ is -3.


X=2
Y=-3.000003X+10.000006

4. Use the difference quotient to determine the instantaneous rate of change of $f(x) = \frac{x}{x - 4}$ at $(2, -1)$. The difference quotient is $f(x) = \frac{f(a + h) - f(a)}{h}$. You can use $h = 0.01$.

$$f(2.01) = \frac{2.01}{2.01 - 4}$$
$$f(2.01) = -1.01$$

Difference Quotient:

$$= \frac{-1.01 - (-1)}{0.01}$$
$$= -1$$

5. Use the difference quotient to determine the instantaneous rate of change for each function.
The difference quotient is $f(x) = \frac{f(a + h) - f(a)}{h}$.
Use 0.01 for h.

a) $y = \frac{1}{25 - x}, x = 13$

$$f(13) = \frac{1}{25 - 13}$$
$$= \frac{1}{12}$$
$$f(13.01) = \frac{1}{25 - 13.01}$$
$$= \frac{1}{11.999}$$

Difference Quotient:

$$= \frac{\frac{1}{11.99} - \frac{1}{12}}{0.001}$$
$$= 0.01$$

b) $y = \frac{17x + 3}{x^2 + 6}, x = -5$

$$f(5) = \frac{17(-5) + 3}{(-5)^2 + 6}$$
$$= -2.645$$
$$f(-4.99) = \frac{17(-4.99) + 3}{(-4.99)^2 + 6}$$
$$= \frac{-81.83}{30.9001}$$
$$= -2.648$$

Difference Quotient:

$$f(x) = \frac{f(a + h) - f(a)}{h}$$
$$= \frac{-2.648 - (-2.645)}{0.01}$$
$$= -\frac{0.003}{0.01}$$
$$= -0.3$$

c) $y = \frac{x + 3}{x - 2}, x = 4$

$$f(4) = \frac{4 + 3}{4 - 2}$$
$$= 3.5$$
$$f(4.01) = \frac{4.01 + 3}{4.01 - 2}$$
$$= 3.487$$

Difference Quotient:

$$f(x) = \frac{f(a + h) - f(a)}{h}$$
$$= \frac{3.487 - 3.5}{0.01}$$
$$= -\frac{0.013}{0.01}$$
$$= -1.3$$

d) $\frac{-3x^2 + 5x + 6}{x + 6}, x = -3$

$$f(-3) = \frac{-3(-3)^2 + 5(-3) + 6}{(-3) + 6}$$
$$= 6$$
$$f(-3.01) = \frac{-3(-3.01)^2 + 5(-3.01) + 6}{(-3.01) + 6}$$

Difference Quotient:

$$f(x) = \frac{f(a + h) - f(a)}{h}$$
$$= \frac{6.06 - 6}{0.01}$$
$$= \frac{0.06}{0.01}$$
$$= 6$$

6. Use the difference quotient to determine the instantaneous rate of change for each function. The difference quotient is $f(x) = \frac{f(a + h) - f(a)}{h}$. Use 0.01 for h. The point where there is no tangent line would be any vertical asymptotes.

a) $f(x) = \frac{-5x}{2x + 3}, x = 2$

$$f(2) = \frac{-5(2)}{2(2) + 3}$$
$$= \frac{-10}{7}$$
$$= -1.429$$
$$f(2.01) = \frac{-5(2.01)}{2(2.01) + 3}$$
$$= -1.432$$

Difference Quotient:

$$f(x) = \frac{f(a + h) - f(a)}{h}$$
$$= \frac{1.432 - (-1.429)}{0.01}$$
$$= 286.1$$

The vertical asymptote would occur at $x = -1.5$.

b) $f(x) = \frac{x - 6}{x + 5}, x = -7$

$$f(-7) = \frac{(-7) - 6}{(-7) + 5}$$
$$= \frac{-13}{-2}$$
$$= 6.5$$
$$f(-7.01) = \frac{(-7.01) - 6}{(-7.01) + 5}$$
$$= \frac{-13.01}{-2.01}$$
$$= 6.4726$$

Difference Quotient:

$$f(x) = \frac{f(a + h) - f(a)}{h}$$
$$= \frac{6.4726 - 6.5}{0.01}$$
$$= -2.74$$

The vertical asymptote would occur at $x = -5$.

c) $f(x) = \frac{2x^2 - 6x}{3x + 5}, x = -2$

$$f(-2) = \frac{2(-2)^2 - 6(-2)}{3(-2) + 5}$$
$$= -20$$
$$f(-2.01) = \frac{2(-2.01)^2 - 6(-2.01)}{3(-2.01) + 5}$$
$$= -19.5535$$

Difference Quotient:

$$f(x) = \frac{f(a + h) - f(a)}{h}$$
$$= \frac{-19.5535 - (-20)}{0.01}$$
$$= 44.65$$

The vertical asymptote would occur at $x = -\frac{5}{3}$.

d) $f(x) = \frac{5}{x - 6}, x = 4$

$$f(4) = \frac{5}{(4) - 6}$$
$$= -2.5$$
$$f(4.01) = \frac{5}{(4.01) - 6}$$
$$= -2.5126$$

Difference Quotient:

$$f(x) = \frac{f(a + h) - f(a)}{h}$$
$$= \frac{-2.5126 - (-2.5)}{0.01}$$
$$= -1.26$$

The vertical asymptote occurs at $x = 6$.

7. The function that models the concentration of the pollutant in the water is $c(t) = \frac{27t}{10\,000 + 3t}$, where the units of t is minutes.

a) There are sixty minutes in 1 hour so find $c(60)$.

$$c(60) = \frac{27(60)}{10\,000 + 3(60)}$$
$$= \frac{1620}{10\,180}$$
$$= 0.1591$$
$$c(60.01) = \frac{27(60.01)}{10\,000 + 3(60.01)}$$
$$= \frac{1620.27}{10\,180.03}$$
$$= 0.1592$$

Difference Quotient:

$$f(x) = \frac{f(a + h) - f(a)}{h}$$
$$= \frac{0.1592 - 0.1591}{0.01}$$
$$= 0.01$$

b) There are 10 080 minutes in one week. So determine $c(10\,080)$.

$$c(10\,080) = \frac{27(10\,080)}{10\,000 + 3(10\,080)}$$
$$= \frac{272\,160}{40\,240}$$
$$= 6.76$$
$$c(10\,080.01) = \frac{27(10\,080.01)}{10\,000 + 3(10\,080.01)}$$
$$= \frac{272\,160.27}{40\,240.03}$$
$$= 6.7634$$

Difference Quotient:

$$f(x) = \frac{f(a + h) - f(a)}{h}$$
$$= \frac{6.7634 - 6.76}{0.01}$$
$$= 0.34$$

8. a) The demand function is $p(x) = \frac{15}{2x^2 + 11x + 5}$. This function tells you the price of the cakes for every 1000 cakes. The revenue function would be the number of cakes sold, x, times the price of the cakes for that number of cakes sold $\frac{15}{2x^2 + 11x + 5}$.

So the revenue function is $p(x) = \frac{15x}{2x^2 + 11x + 5}$

b) Use the difference quotient to determine the marginal revenue for $x = 0.75$ and $x = 2.00$.

$$f(x) = \frac{15x}{2x^2 + 11x + 5}$$
$$f(0.75) = \frac{15(0.75)}{2(0.75)^2 + 11(0.75) + 5}$$
$$= \frac{11.25}{14.375}$$
$$= 0.7826$$
$$f(0.751) = \frac{15(0.751)}{2(0.751)^2 + 11(0.751) + 5}$$
$$= \frac{11.265}{14.389}$$
$$= 0.7829$$

Difference Quotient:

$$f(x) = \frac{f(a + h) - f(a)}{h}$$
$$= \frac{0.7829 - 0.7826}{0.001}$$
$$= 0.3$$

The marginal revenue at $x = 0.75$ is 0.3.

Now find the marginal revenue for $x = 2.00$.

$$f(2.00) = \frac{15(2.00)}{2(2.00)^2 + 11(2.00) + 5}$$
$$= \frac{30}{35}$$
$$= 0.8571$$
$$f(2.01) = \frac{15(2.01)}{2(2.01)^2 + 11(2.01) + 5}$$
$$= \frac{30.15}{35.1902}$$
$$= 0.8568$$

Difference Quotient:

$$f(x) = \frac{f(a+h) - f(a)}{h}$$
$$= \frac{0.8568 - 0.8571}{0.01}$$
$$= -0.03$$

The marginal revenue at $x = 2.00$ would be -0.03.

9. a) Since x is measured in thousands, find $C(3)$.

The function is $C(x) = \frac{x^2 - 4x + 20}{x}$.

$$C(x) = \frac{x^2 - 4x + 20}{x}$$
$$C(3) = \frac{(3)^2 - 4(3) + 20}{(3)}$$
$$= 5.67$$

The average cost per T-shirt is \$5.67.

b) Determine $C(3.01)$ and use this value in the difference quotient to help you estimate the average rate of change of the average price of a T-shirt when the factory is producing 3000 of them.

$$C(3.01) = \frac{(3.01)^2 - 4(3.01) + 20}{(3.01)}$$
$$= 5.65$$

Difference Quotient:

$$f(x) = \frac{f(a+h) - f(a)}{h}$$
$$= \frac{5.65 - 5.67}{0.01}$$
$$= -2$$

10. The function is $N(t) = \frac{100t^3}{100 + t^3}$.

a) $N(6) = \frac{100(6)^3}{100 + (6)^3}$

$$= \frac{21\,600}{316}$$
$$= 68.35$$
$$N(6.01) = \frac{100(6.01)^3}{100 + (6.01)^3}$$
$$= \frac{21\,708.18}{317.08}$$
$$= 68.46$$

Difference Quotient:

$$f(x) = \frac{f(a+h) - f(a)}{h}$$
$$= \frac{68.46 - 68.35}{0.01}$$
$$= -11$$

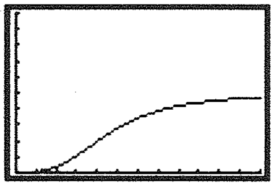
b) $N(12) = \frac{100(12)^3}{100 + (12)^3}$

$$= \frac{172\,800}{1828}$$
$$= 94.53$$
$$N(12.01) = \frac{100(12.01)^3}{100 + (12.01)^3}$$
$$= \frac{173\,232.36}{1832.32}$$
$$= 94.54$$

Difference Quotient:

$$f(x) = \frac{f(a+h) - f(a)}{h}$$
$$= \frac{94.54 - 94.53}{0.01}$$
$$= 1$$

c)



The number of houses that were built increases slowly at first, but rises rapidly between the third and sixth months. During the last six months, the rate at which the houses were built decreases.

11. Examine the interval $14 \le x \le 15$. Find the rate of change over the interval.

$$f(15) = \frac{15 - 2}{15 - 5} = \frac{13}{10} = 1.3$$
$$f(14) = \frac{14 - 2}{14 - 5} = \frac{12}{9} = 1.33$$
$$\frac{1.3 - 1.33}{15 - 14} = -0.03$$

The slope over the interval $14 \le x \le 15$ is $m = -0.03$, which is approximately 0. Now find the instantaneous rate of change at 14.5.

$$f(14.5) = \frac{14.5 - 2}{14.5 - 5} = \frac{12.5}{9.5} = 1.316$$
$$f(14.51) = \frac{14.51 - 2}{14.51 - 5} = \frac{12.51}{9.51} = 1.315$$
$$\frac{1.315 - 1.316}{14.51 - 14.5} = -0.1$$

The instantaneous rate of change at the point 14.5 is -0.01, which is approximately 0. The rate of change over the interval and at the specific point is about 0.

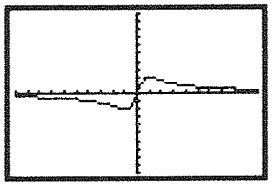
12. a) I would find $s(0)$ and $s(6)$ and would then solve $\frac{s(6) - s(0)}{6 - 0}$.

b) The average rate of change over this interval gives the object's speed.

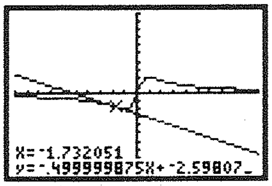
c) To find the instantaneous rate of change at a specific point, you could find the slope of the line that is tangent to the function $s(t)$ at the specific point. You could also find the average rate of change on either side of the point for smaller and smaller intervals until it stabilizes to a constant. It is generally easier to find the instantaneous rate using a graph, but the second method is more accurate.

d) The instantaneous rate of change for a specific time, t, is the acceleration of the object at this time.

13. Use your graphing calculator to graph the equation $f(x) = \frac{4x}{x^2 + 1}$.

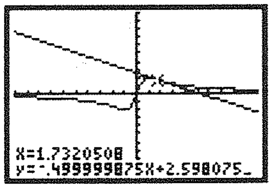


Find $x = -\sqrt{3}$ on the graph and use the calculator to find the line tangent to the graph at this point.



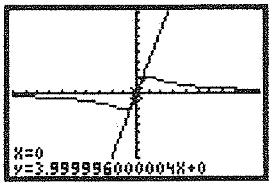
```
X=-1.732051
y=-.499999875X+-2.59807_
```

The equation of the line tangent to the function at $(-\sqrt{3}, -\sqrt{3})$ is $y = -0.5x - 2.598$.



```
X=1.7320508
y=-.499999875X+2.598075_
```

The equation of the line tangent to the function at $(\sqrt{3}, \sqrt{3})$ is $y = -0.5x + 2.598$.



```
X=0
y=3.99999600000004X+0
```

The equation of the line tangent to the function at $(0, 0)$ is $y = 4x$.

14. Examine the graph of the function. Use your calculator to zoom into points around the origin. The graph appears to be a straight line at this point, and so the instantaneous rates of change at $(0, 0)$ will likely be pretty close to the instantaneous rate of change at $(0, 0)$, which is 4.

Because the graph is almost a straight line at $(0, 0)$ the rate of change is neither increasing nor decreasing; it will remain constant. Therefore the rate of change at this rate of change will be 0.

Chapter Review, pp. 308–309

1. a) The function is $f(x) = 3x + 2$. The function will be a straight line and so the domain and range will be all real numbers—$\{x \in \mathbf{R}\}$ and $\{y \in \mathbf{R}\}$.

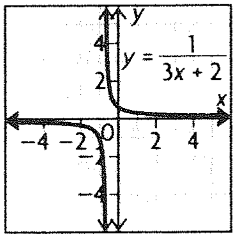
x-intercept:

$$\begin{aligned} 0 &= 3x + 2 \\ -2 &= 3x \\ \frac{-2}{3} &= \frac{3x}{3} \\ -\frac{2}{3} &= x \end{aligned}$$

y-intercept:

$$\begin{aligned} y &= 3(0) + 2 \\ &= 2 \end{aligned}$$

The slope of the equation is positive and so the function will always be increasing. Additionally, this means that the function will be negative on $\left(-\infty, -\frac{2}{3}\right)$ and positive on $\left(-\frac{2}{3}, \infty\right)$. Use this information to help you graph the function.



b) The function is $f(x) = 2x^2 + 7x - 4$. The function is a quadratic and so the domain will be $\{x \in \mathbf{R}\}$. The coefficient of the first term is positive and so the graph will be pointed up. The function factors to $f(x) = (2x - 1)(x + 4)$. Therefore, the x-intercepts would be $0 = 2x - 1$ and $0 = x + 4$, or $x = 0.5$ and -4.

Use the x-intercepts and direction that the graph is facing to determine the positive and negative intervals. The graph will be positive on $(-\infty, -4)$ and $(0.5, \infty)$. The graph will be negative on $(-4, 0.5)$. You can also use the x-intercepts to determine the range. The range corresponds to the vertex of the function and the vertex can be found halfway in between the two x-intercepts.

$$\text{vertex } \frac{0.5 + (-4)}{2} = -1.75$$

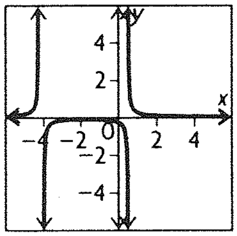
This is the x-value of the vertex. Now you can substitute -1.75 into the function to find the graphs minimum and its subsequent range.

$$\begin{aligned} f(-1.75) &= 2(-1.75)^2 + 7(-1.75) - 4 \\ &= -10.125 \end{aligned}$$

The range of the function would be $\{y \in \mathbf{R} | y > -10.125\}$.

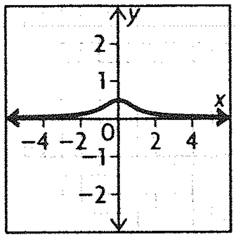
Because you know the vertex and the direction the graph is pointed, you can also find the increasing/decreasing intervals. The graph will be decreasing on $(-\infty, -10.125)$ and increasing on $(-10.125, \infty)$. Use this information to help you graph the function.

$y = \dfrac{1}{2x^2 + 7x - 4}$



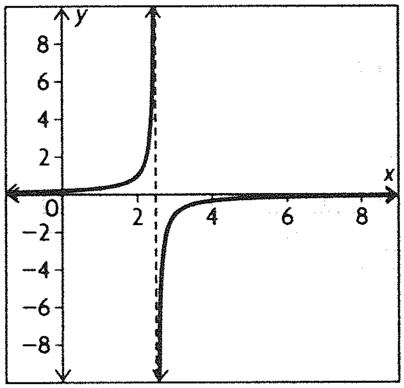
c) The function is $f(x) = 2x^2 + 2$. The function is a quadratic function. Therefore, the domain will be $\{x \in \mathbf{R}\}$. The graph will be facing upward because the coefficient of the first term is positive. Because there are no real solutions to $0 = 2x^2 + 2$, the graph will have no x-intercepts. The y-intercept is $y = 2(0)^2 + 2 = 2$. This means that the domain will be $\{y \in \mathbf{R} | y > 2\}$. The function will be decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. Because the function has no x-intercepts and because the graph is facing up, the function will never be negative. Use this information to help you graph the reciprocal of the function.

$y = \dfrac{1}{2x^2 + 2}$



2. Examine each graph. Glean any information from these graphs that might help you to graph their function's reciprocals.

a) The function is linear. The domain and range are all real numbers. It is always decreasing. The x-intercept is $x = 2.5$. The y-intercept is $y = 5$. The function is positive on $(-\infty, 2.5)$ and negative on $(2.5, \infty)$. The graph of the reciprocal function would be:

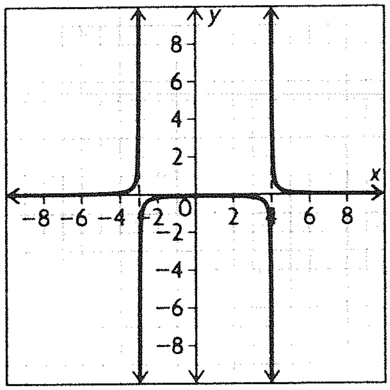


b) The function is a quadratic equation, which means that the domain of the function is $\{x \in \mathbf{R}\}$. The x-intercepts of the function are $x = -3$ and $x = 4$. The function is $f(x) = (x + 3)(x - 4) = x^2 - x - 12$. The y-intercept of the function is $(0, -12)$. The vertex can be found by finding the half-way point between the x-intercepts.

$$\frac{-3 + 4}{2} = 0.5$$

$$\begin{aligned} f(0.5) &= ((0.5) + 3)((0.5) - 4) \\ &= (3.5)(-3.5) \\ &= -12.25 \end{aligned}$$

The vertex is $(0.5, -12.25)$. This means that the range of the function is $\{y \in \mathbf{R} | y > -12.25\}$. The function is decreasing on $(-\infty, -12.25)$ and increasing on $(-12.25, \infty)$. Use this information to graph the reciprocal of the function.



3. a) The function is $y = \frac{1}{x + 17}$. To find the vertical asymptotes, find the zeros of the expression in the denominator.

$0 = x + 17$
$0 - 17 = x + 17 - 17$
$-17 = x$

Because the numerator of the rational function is a constant the horizontal asymptote would be $y = 0$.

b) The function is $y = \frac{2x}{5x + 3}$. To find the vertical asymptotes, find the zeros of the expression in the denominator.

$0 = 5x + 3$
$0 - 3 = 5x + 3 - 3$
$-3 = 5x$
$\frac{-3}{5} = \frac{5x}{5}$
$-\frac{3}{5} = x$

Divide the leading coefficients of the numerator and denominator to find the equation of the horizontal asymptote.

$y = \frac{2x}{5x} = \frac{2}{5}$

c) The function is $y = \frac{3x + 33}{-4x^2 - 42x + 22}$. To find the vertical asymptotes, find the zeros of the expression in the denominator.

$0 = -4x^2 - 42x + 22$

You can use the quadratic formula to help you factor the equation.

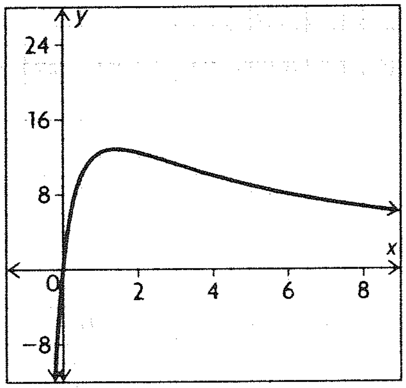
$0 = (x + 11)(x - 0.5)$
$x = -11$ and $x = 0.5$

Notice that $(x + 11)$ is a factor in both the numerator and the denominator. This means that there will be a hole at $x = -11$. Because the degree of the expression in the denominator is 2 and the degree of the expression in the numerator is 1, the horizontal asymptote will be $y = 0$.

d) The function is $y = \frac{3x^2 - 2}{x - 1}$. To find the vertical asymptotes, find the zeros of the expression in the denominator.

$0 = x - 1$, so there is a vertical asymptote at $x = 1$. The numerator does not factor, so no common factors from the numerator and denominator will cancel. Because the degree of the expression in the numerator is 2 and the degree of the expression in the denominator is 1, there will be an oblique asymptote. Divide the numerator by the denominator; the non-remainder part is the equation of the oblique asymptote: $y = 3x + 3$.

4. The function that models the population of the locusts is $f(x) = \frac{75x}{x^2 + 3x + 2}$. You can graph the function using a graphing calculator and then use the graph to describe the locust population over time.



The locust population increased during the first 1.75 years, to reach a maximum of 1 248 000. The population gradually decreased until the end of the 50 years, when the population was 128 000.

5. a) $f(x) = \frac{2}{x + 5}$

x-intercept:

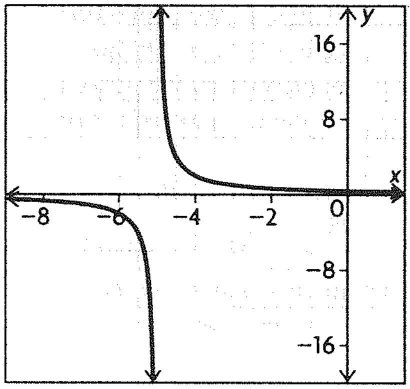
$0 = \frac{2}{x + 5}$
$(x + 5)0 = \frac{2}{x + 5}(x + 5)$
$0 = 2$

The function does not intercept the x-axis. This means that the horizontal asymptote of the function will be $y = 0$.

y-intercept:

$$f(0) = \frac{2}{0 + 5}$$
$$= \frac{2}{5}$$

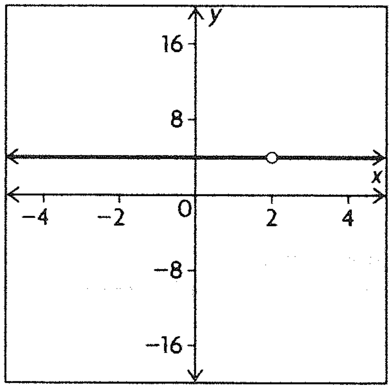
The function has a vertical asymptote at $x = -5$. This means that the domain of the function will be $\{x \in \mathbf{R} | x \neq -5\}$. The function will be negative for $x < -5$ and positive for $x > -5$.



This function is never increasing and is decreasing on $(-\infty, -5)$ and $(-5, \infty)$.
$D = \{x \in \mathbf{R} | x \neq -5\}$;
negative for $x < -5$;
positive for $x > -5$

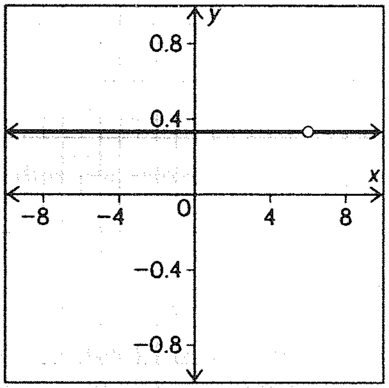
b) $f(x) = \frac{4x - 8}{x - 2}$

Notice that the function factors to $f(x) = \frac{4(x - 2)}{(x - 2)}$.
This means that there will be a hole at $x = 2$ and that the graph of the function will be the horizontal line $y = 4$. The function is positive everywhere except $x = 2$. There is no x-intercept, and the y-intercept is 4. Use this information to help you graph the function.



The function is not increasing or decreasing.

c) The function is $f(x) = \frac{x - 6}{3x - 18}$. Notice that the function factors to $f(x) = \frac{x - 6}{3(x - 6)} = \frac{1}{3}$.
This means that the graph will have a hole at $x = 6$ and will be a horizontal line at $y = \frac{1}{3}$. The function is positive everywhere except $x = 6$. There is no x-intercept, and the y-intercept is $\frac{1}{3}$.



The function is not increasing or decreasing.

d) The function is $f(x) = \frac{4x}{2x + 1}$. To find the function's vertical asymptote, find the zero(s) of the expression in the denominator.

$$0 = 2x + 1$$
$$-1 = 2x$$
$$-0.5 = x$$

The function will have a vertical asymptote at $x = -0.5$. This means that the domain will be $\{x \in \mathbf{R} | x \neq -0.5\}$.
x-intercept:

$$0 = \frac{4x}{2x + 1}$$
$$(2x + 1)0 = \frac{4x}{2x + 1}(2x + 1)$$
$$0 = 4x$$
$$0 = x$$

The graph will intersect the x-axis at 0.
y-intercept:

$$\frac{4(0)}{2(0) + 1} = 0$$

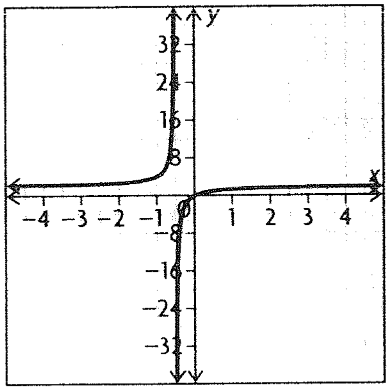
The y-intercept will also be 0.
Divide the first terms of the expressions in the numerator and the denominator to find the equation of the horizontal asymptote.

$$y = \frac{4x}{2x} = 2$$

Use the following table to determine when the function is positive and negative.

	$x < -0.5$	$-0.5 < x < 0$	$x > 0$
$4x$	−	−	+
$2x + 1$	−	+	+
$\frac{4x}{2x+1}$	+	−	+

The function is positive on $x < -0.5$ and $x > 0$.
The function is negative on $-0.5 < x < 0$.
Use this information to help you graph the function.



The function is never decreasing and is increasing on $(-\infty, -0.5)$ and $(-0.5, \infty)$.

6. Answers may vary. For example, consider the function $f(x) = \frac{1}{x - 6}$. You know that the vertical asymptote would be $x = 6$. If you were to find the value of the function very close to $x = 6$ (say $f(5.99)$ or $f(6.01)$) you would be able to determine the behaviour of the function on either side of the asymptote.

$$f(5.99) = \frac{1}{(5.99) - 6} = -100$$

$$f(6.01) = \frac{1}{(6.01) - 6} = 100$$

To the left of the vertical asymptote the function moves towards $-\infty$. To the right of the asymptote the function moves towards ∞.

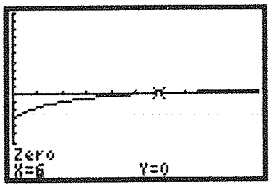
7. Make sure to use your graphing calculator to verify each solution.

a) $$\frac{x-6}{x+2} = 0$$

$$(x+2)\frac{x-6}{x+2} = 0(x+2)$$

$$x - 6 = 0$$

$$x = 6$$


Zero
X=6 Y=0

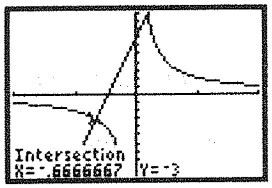
b) $$15x + 7 = \frac{2}{x}$$

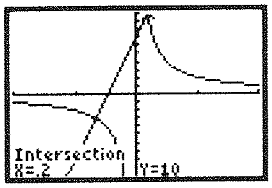
$$x\left(15x + 7 = \frac{2}{x}\right)$$

$$15x^2 + 7x = 2$$

$$15x^2 + 7x - 2 = 0$$

You can use the quadratic formula to help you solve this quadratic equation. The roots of the function are $x = 0.2$ and $x = -\frac{2}{3}$.


Intersection
X=-.6666667 Y=-3


Intersection
X=.2 Y=10

c) $$\frac{2x}{x-12} = \frac{-2}{x+3}$$

$$2x(x + 3) = -2(x - 12)$$

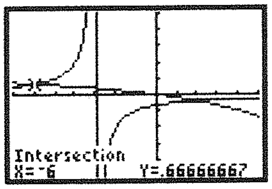
$$2x^2 + 6x = -2x + 24$$

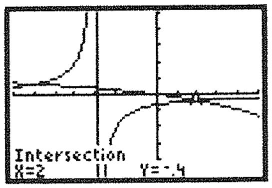
$$2x^2 + 8x - 24 = 0$$

$$2(x^2 + 4x - 12) = 0$$

$$2(x + 6)(x - 2) = 0$$

$$x = -6 \text{ or } x = 2$$


Intersection
X=-6 Y=.6666667

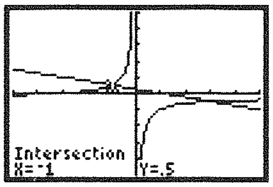

Intersection
X=2 Y=-.4

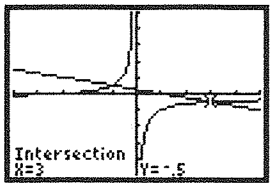
d) $$\frac{x+3}{-4x} = \frac{x-1}{-4}$$

$$-4(x + 3) = -4x(x - 1)$$

$$-4x - 12 = -4x^2 + 4x$$

$$0 = -4x^2 + 8x + 12$$
$$0 = -4(x^2 - 2x - 3)$$
$$0 = -4(x + 1)(x - 3)$$
$$x = -1 \text{ and } x = 3$$





8. Janet and Nick's rate would be $\frac{1}{m}$ and Rodriguez's rate would be $\frac{1}{m - 5}$. Working together their rate would be $\frac{2}{m} + \frac{1}{m - 5}$, which is equal to $\frac{1}{32.3}$.

$$\frac{2}{m} + \frac{1}{m - 5} = \frac{1}{3.23}$$
$$3.23m(m - 5)\left(\frac{2}{m} + \frac{1}{m - 5} = \frac{1}{3.23}\right)$$
$$3.23(m - 5)(2) + 3.23m = m(m - 5)$$
$$6.46m - 32.3 + 3.23m = m^2 - 5m$$
$$0 = m^2 - 14.69m + 32.3$$

Use the quadratic formula to determine the roots of this quadratic equation.

$m = 12$ and 2.69

The possible answers are 12 minutes and 2.69 minutes. But because Janet's time has to be greater than 5 minutes, the answer must be 12 minutes. It takes Janet about 12 minutes to wash the car.

9. The function that represents the concentration of a toxic chemical is $c(x) = \frac{50x}{x^2 + 3x + 6}$.

To determine when the concentration is 6.16 g/L solve the equation $6.16 = \frac{50x}{x^2 + 3x + 6}$.

$$6.16 = \frac{50x}{x^2 + 3x + 6}$$
$$6.16(x^2 + 3x + 6) = \frac{50x}{x^2 + 3x + 6}(x^2 + 3x + 6)$$
$$6.16x^2 + 18.48x + 36.96 = 50x$$
$$6.16x^2 - 31.52x + 36.96 = 0$$

Use the quadratic formula to solve this equation.

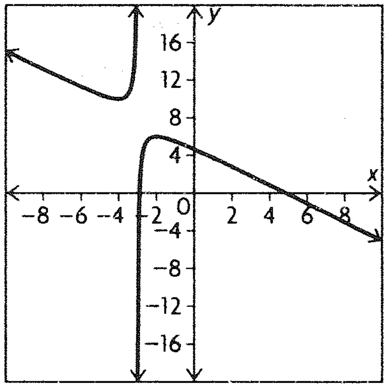
$x = 3.297$ and 1.82

The concentration of the chemical will be 6.16 g/L at 1.82 and 3.297 days.

10. a)
$$-x + 5 < \frac{1}{x + 3}$$
$$-x + 5 - \frac{1}{x + 3} < 0$$
$$\left(\frac{x + 3}{x + 3}\right)(-x) + \left(\frac{x + 3}{x + 3}\right)5 - \frac{1}{x + 3} < 0$$
$$\frac{-x^2 - 3x}{x + 3} + \frac{5x + 15}{x + 3} - \frac{1}{x + 3} < 0$$
$$\frac{-x^2 + 2x + 14}{x + 3} < 0$$
$$\frac{(x + 2.873)(x - 4.873)}{x + 3} < 0$$

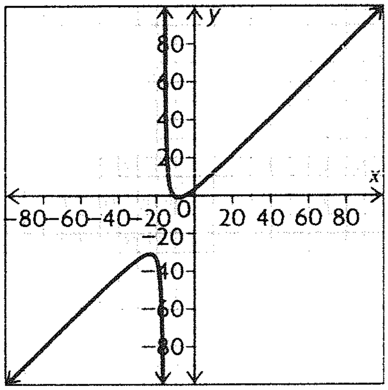
	$x < -3$	$-3 < x < -2.873$	$-2.873 < x < 4.873$	$x > 4.873$
$x + 2.873$	−	−	+	+
$x - 2.873$	−	−	−	+
$x + 3$	−	+	+	+
$\frac{(x + 2.873)(x - 4.873)}{x + 3}$	−	+	−	+

The inequality is true on $x < -3$ and $-2.873 < x < 4.873$.



b)
$$\frac{55}{x + 16} > -x$$
$$\frac{55}{x + 16} + x > 0$$
$$\frac{55}{x + 16} + \left(\frac{x + 16}{x + 16}\right)x > 0$$
$$\frac{x^2 + 16x + 55}{x + 16} > 0$$
$$\frac{(x + 5)(x + 11)}{x + 16} > 0$$

	$x < -16$	$-16 < x < -11$	$-11 < x < -5$	$x > -5$
$x + 5$	−	−	−	+
$x + 11$	−	−	+	+
$x + 16$	−	+	+	+
$\frac{(x + 5)(x + 11)}{x + 16}$	−	+	−	+



c)

$$\frac{2}{3x + 4} > \frac{x}{x + 1}$$

$$\frac{2x}{3x + 4} - \frac{x}{x + 1} > 0$$

$$\left(\frac{x + 1}{x + 1}\right)\frac{2x}{3x + 4} - \frac{x}{x + 1}\frac{3x + 4}{3x + 4} > 0$$

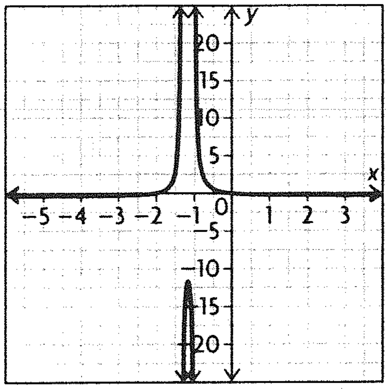
$$\frac{2x^2 + 2x}{(x + 1)(3x + 4)} - \frac{3x^2 + 4x}{(x + 1)(3x + 4)} > 0$$

$$\frac{-x^2 - 2x}{(x + 1)(3x + 4)} > 0$$

$$\frac{-x(x + 2)}{(x + 1)(3x + 4)} > 0$$

	$x < -2$	$-2 < x < -1.33$	$-1.33 < x < -1$	$-1 < x < 0$	$x > 0$
$-x$	+	+	+	+	−
$x + 2$	−	+	+	+	+
$x + 1$	−	−	−	+	+
$3x + 4$	−	−	+	+	+
$\frac{-x(x + 2)}{(x + 1)(3x + 4)} > 0$	−	+	−	+	−

The inequality is true on $-2 < x < -1.33$ and $-1 < x < 0$.



d)

$$\frac{x}{6x - 9} \leq \frac{1}{x}$$

$$\frac{x}{6x - 9} - \frac{1}{x} \leq 0$$

$$\left(\frac{x}{x}\right)\frac{x}{6x - 9} - \frac{1}{x}\left(\frac{6x - 9}{6x - 9}\right) \leq 0$$

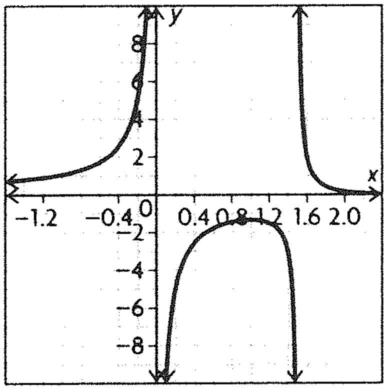
$$\frac{x^2}{(x)(6x - 9)} + \frac{-6x + 9}{(x)(6x - 9)} \leq 0$$

$$\frac{x^2 - 6x + 9}{(x)(6x - 9)} \leq 0$$

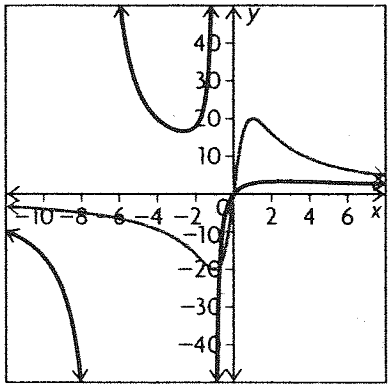
$$\frac{(x - 3)(x - 3)}{(x)(6x - 9)} \leq 0$$

	$x \leq 0$	$0 \leq x \leq 1.5$	$1.5 \leq x \leq 3$	$x \geq 3$
$x - 3$	−	−	−	+
$x - 3$	−	−	−	+
x	−	+	+	+
$6x - 9$	−	−	+	+
$\frac{(x - 3)(x - 3)}{(x)(6x - 9)} \leq 0$	+	−	+	+

The inequality is true on $0 < x < 1.5$.



11. The function that models the biologist's prediction of the population of the tadpoles in the pond is $f(t) = \frac{40t}{t^2 + 1}$. The actual population is modelled by $g(t) = \frac{45t}{t^2 + 8t + 7}$. Graph $\frac{45t}{t^2 + 8t + 7}$ and $\frac{40t}{t^2 + 1}$ to determine where $g(t) > f(t)$.



$g(t)$ appears to be greater than $f(t)$ on $-6.7 < x < -1.01$ and $-0.73 < x < 0$.

12. The slope of the line tangent to the graph for the given point is equal to the instantaneous rate of change at that point. Use the difference quotient to determine the instantaneous rate of change. The point where a vertical asymptote occurs is the point where no tangent line could be drawn.

a) $\frac{x + 3}{x - 3}, x = 4$

$$f(4) = \frac{4 + 3}{4 - 3} = 7$$

$$f(4.01) = \frac{4.01 + 3}{4.01 - 3} = 6.94$$

Difference Quotient:

$$f(x) = \frac{f(a + h) - f(a)}{h} = \frac{6.94 - 7}{0.01} = \frac{-0.06}{0.01} = -6$$

The vertical asymptote occurs at $x = 3$.

b) $f(x) = \frac{2x - 1}{x^2 + 3x + 2}, x = 1$

$$f(1) = \frac{2(1) - 1}{(1)^2 + 3(1) + 2} = 0.167$$

$$f(1.01) = \frac{2(1.01) - 1}{(1.01)^2 + 3(1.01) + 2} = 0.169$$

Difference Quotient:

$$f(x) = \frac{f(a + h) - f(a)}{h} = \frac{0.169 - 0.167}{0.01} = \frac{0.002}{0.01} = 0.2$$

The vertical asymptotes occur at $x = -2$ and $x = -1$.

13. a) To find the average rate of change during the first two hours of the drug's ingestion, find $c(2)$ and $c(0)$.

$$c(2) = \frac{5(2)}{(2)^2 + 7} = \frac{10}{11} = 0.91$$

$$c(0) = \frac{5(0)}{(0)^2 + 7} = 0$$

Average Rate of Change:

$$\frac{0.91 - 0}{2 - 0} = 0.455 \text{ mg/L/h}$$

b) Use the difference quotient to help you determine the drug's instantaneous rate of change at $t = 3$.

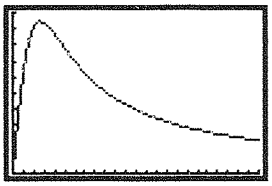
$$c(3) = \frac{5(3)}{(3)^2 + 7} = 0.9375$$

$$c(3.01) = \frac{5(3.01)}{(3.01)^2 + 7} = \frac{15.05}{16.0601} = 0.9371$$

Difference Quotient:

$$f(x) = \frac{f(a + h) - f(a)}{h} = \frac{0.9371 - 0.9375}{0.01} = \frac{-0.004}{0.01} = -0.04 \text{ mg/L/h}$$

c) Graph the function $c(t) = \frac{5t}{t^2 + 7}$.



The concentration of the drug in the blood stream appears to be increasing most rapidly during the first hour and a half; the graph is steep and increasing during this time.

14. The slope of the tangent line is going to be $\frac{4-10}{8-5} = -2$. This means that the equation of the line would be $y = -2x + 20$. You are looking for the point of intersection between the two functions, and so you need to solve $-2x + 20 = \frac{2x}{x-4}$.

$$\begin{aligned} -2x + 20 &= \frac{2x}{x-4} \\ (-2x + 20)(x-4) &= 2x \\ -2x^2 + 8x + 20x - 80 &= 2x \\ -2x^2 + 26x - 80 &= 0 \\ x^2 - 13x + 40 &= 0 \\ (x-5)(x-8) &= 0 \end{aligned}$$

The two points of intersection would be $x = 5$ and $x = 8$. The point with the line parallel to the secant line would lie half-way in between these two points, $x = 6.5$.

15. a) As the x-coordinate approaches the vertical asymptote of a rational function, the line tangent to the graph will get closer and closer to being a vertical line. This means that the slope of the line tangent to the graph will get larger and larger, approaching positive or negative infinity depending on the function, as x gets closer to the vertical asymptote.

b) As the x-coordinate grows larger and larger in either direction, the line tangent to the graph will get closer and closer to being a horizontal line. This means that the slope of the line tangent to the graph will always approach zero as x gets larger and larger.

Chapter Self-Test, p. 310

1. a) The graph indicates that there is a vertical asymptote at $x = 0.5$ and a horizontal asymptote at $y = 0$. This matches equation B.

b) The graph indicates that there is a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 5$. This matches equation A.

2. a) If $f(n)$ is very large, then that would make $\frac{1}{f(n)}$ a very small fraction.

b) If $f(n)$ is very small (less than 1), then that would make $\frac{1}{f(n)}$ very large.

c) If $f(n) = 0$, then that would make $\frac{1}{f(n)}$ undefined at that point because you cannot divide by 0.

d) If $f(n)$ is positive, then that would make $\frac{1}{f(n)}$ also positive because you are dividing two positive numbers.

3. The horizontal asymptote of the function can be found by finding the zeros of the expression in the denominator.

$0 = x - 2$

$2 = x$

The horizontal asymptotes of the function can be found by dividing the first two terms of the expressions in the numerator and denominator.

x-intercept:

$$\begin{aligned} 0 &= \frac{2x + 6}{x - 2} \\ 0 &= 2x + 6 \\ -6 &= 2x \\ -3 &= x \end{aligned}$$

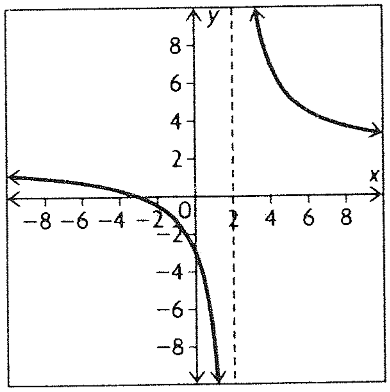
y-intercept:

$$\begin{aligned} y &= \frac{2(0) + 6}{0 - 2} \\ &= -3 \end{aligned}$$

Use a table to determine when the graph is positive and negative.

	$x < -3$	$-3 < x < 2$	$x > 2$
$2x + 6$	$-$	$+$	$+$
$x - 2$	$-$	$-$	$+$
$\frac{2x+6}{x-2}$	$+$	$-$	$+$

Because the expression in the denominator is always increasing, this function will always be decreasing. Use all of this information to sketch the graph.



4. The average cost for a kilogram of steel before it has been processed would be $\frac{2249.52}{x}$.

The company has made \$2 profit on each pound of steel. So the price of steel after it has been processed would be $\frac{2249.52}{x} + 2$. The mass of the steel has lost 25 kilograms. The value of the steel would be the amount left multiplied by the current price.

$$\left(\frac{2249.52}{x} + 2\right)(x - 25) = 10\,838.52$$
$$(2249.52 + 2x)(x - 25) = 10\,838.52x$$
$$2249.52x - 56\,238 + 2x^2 - 50x = 10\,838.52x$$
$$2x^2 - 8639x - 56\,238 = 0$$
$$x = 4326$$

The original weight was 4326 kg. The original cost would be \$0.52/kg.

5. a) Algebraic

$$\frac{-x}{x - 1} = \frac{-3}{x + 7}$$
$$-x(x + 7) = -3(x - 1)$$
$$-x^2 - 7x = -3x + 3$$
$$-x^2 - 4x - 3 = 0$$
$$x^2 + 4x + 3 = 0$$
$$(x + 1)(x + 3) = 0$$
$$x = -1 \text{ and } x = -3$$

b)

$$\frac{2}{x + 5} > \frac{3x}{x + 10}$$
$$\frac{2}{x + 5} - \frac{3x}{x + 10} > 0$$
$$\left(\frac{x + 10}{x + 10}\right)\frac{2}{x + 5} - \frac{3x}{x + 10}\left(\frac{x + 5}{x + 5}\right) > 0$$
$$\frac{2x + 20}{(x + 10)(x + 5)} - \frac{3x^2 + 15x}{(x + 10)(x + 5)} > 0$$
$$\frac{-3x^2 - 13x + 20}{(x + 10)(x + 5)} > 0$$
$$\frac{(x + 5.5)(x - 1.2)}{(x + 10)(x + 5)} > 0$$

	$x < -10$	$-10 < x < -5.5$	$-5.5 < x < -5$	$-5 < x < 1.2$	$x > 1.2$
$(x + 5.5)$	−	−	+	+	+
$(x - 1.2)$	−	−	−	−	+
$(x + 10)$	−	+	+	+	+
$(x + 5)$	−	−	−	+	+
$\frac{(x + 5.5)(x - 1.2)}{(x + 10)(x + 5)} > 0$	+	−	+	−	+

The inequality is true on $x < -10$, $-5.5 < x < -5$, and $x > 1.2$.

6. a) To find the vertical asymptotes of the function, find the zeros of the expression in the denominator. To find the equation of the horizontal asymptotes, divide the first two terms of the expressions in the numerator and denominator.

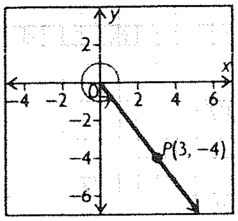
b) This type of function will have a hole when both the numerator and the denominator share the same factor $(x + a)$.

CHAPTER 6
Trigonometric Functions

Getting Started, p. 314

1. a) 28°
b) 360° − 28° = 332°

2. a)



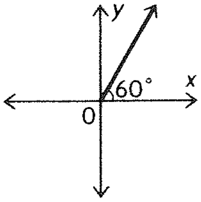
Side opposite: −4
Side adjacent: 3
Hypotenuse: $h^2 = 3^2 + 4^2$
$h^2 = 25$
$h = 5$

$\sin\theta = -\frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = -\frac{4}{3}$,

$\csc\theta = -\frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = -\frac{3}{4}$,

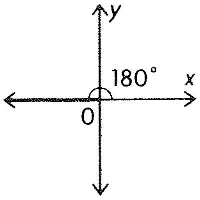
b) $\theta = \sin^{-1}\left(-\frac{4}{5}\right)$
$\theta = 307°$
The principal angle is 360° − 53° = 307°

3. a)



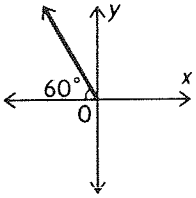
$\sin 60° = \frac{\sqrt{3}}{2}$

b)



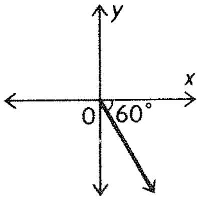
$\tan 180° = \frac{0}{1} = 0$

c)



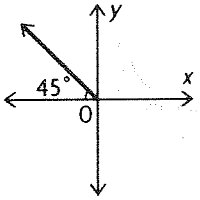
$\sin 120° = \frac{\sqrt{3}}{2}$

d)



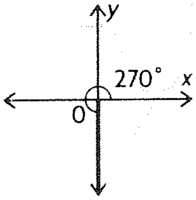
$\cos 300° = \frac{1}{2}$

e)



$\sec 135° = -\frac{\sqrt{2}}{1} = -\sqrt{2}$

f)

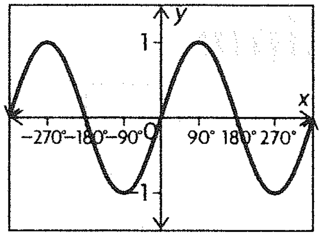


$\csc 270° = \frac{1}{-1} = -1$

4. a) Since cosine is positive in the first and fourth quadrants, $\theta = 60°, 300°$
b) Since tangent is positive in the first and third quadrants, $\theta = 30°, 210°$
c) Since tangent is positive in the first and third quadrants, $\theta = 45°, 225°$
d) Cosine equals −1 at $\theta = 180°$

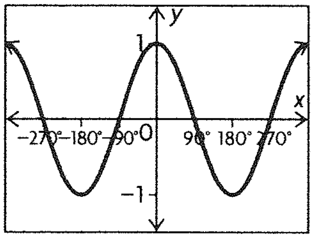
e) Cotangent equals -1 at $\theta = 135°, 315°$
f) Sine equals 1 at $\theta = 90°$

5. a)



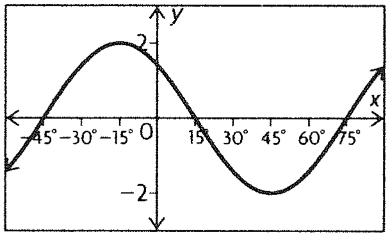
period $= 360°$; amplitude $= 1$; $y = 0$;
$R = \{y \in \mathbf{R} \mid -1 \le y \le 1\}$

b)

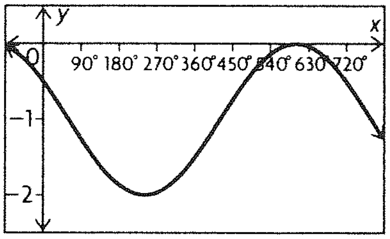


period $= 360°$; amplitude $= 1$; $y = 0$;
$R = \{y \in \mathbf{R} \mid -1 \le y \le 1\}$

6. a) period $= \frac{2(180)}{3} = 120$;
$y = 0$; 45° to the left; amplitude $= 2$



b) period $= \frac{2(180)}{\frac{1}{2}} = 360(2) = 720°$;
$y = -1$; 60° to the right; amplitude $= 1$



7. a is the amplitude, which determines how far above and below the axis of the curve of the function rises and falls; k defines the period of the function, which is how often the function repeats itself; d is the horizontal shift, which shifts the function to the right or the left; and c is the vertical shift of the function.

6.1 Radian Measure, pp. 320–322

1. a) π radians;

$$\cancel{\pi \text{ radians}} \times \left(\frac{180°}{\cancel{\pi \text{ radians}}}\right) = 180°$$

b) $\frac{\pi}{2}$ radians;

$$\frac{\cancel{\pi}}{2} \cancel{\text{radians}} \times \left(\frac{180°}{\cancel{\pi \text{ radians}}}\right) = 90°$$

c) $-\pi$ radians;

$$-\cancel{\pi \text{ radians}} \times \left(\frac{180°}{\cancel{\pi \text{ radians}}}\right) = -180° = 180°$$

d) $-\frac{3\pi}{2}$ radians $= \frac{\pi}{2}$ radians;

$$-\frac{3\cancel{\pi}}{2} \cancel{\text{radians}} \times \left(\frac{180°}{\cancel{\pi \text{ radians}}}\right) = -270°$$

e) -2π radians;

$$-2\cancel{\pi \text{ radians}} \times \left(\frac{180°}{\cancel{\pi \text{ radians}}}\right) = -360°$$

f) $\frac{3\pi}{2}$ radians;

$$\frac{3\cancel{\pi}}{2} \cancel{\text{radians}} \times \left(\frac{180°}{\cancel{\pi \text{ radians}}}\right) = 270°$$

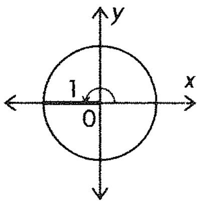
g) $-\frac{4\pi}{3}$ radians;

$$-\frac{4\cancel{\pi}}{3} \cancel{\text{radians}} \times \left(\frac{180°}{\cancel{\pi \text{ radians}}}\right) = -240°$$

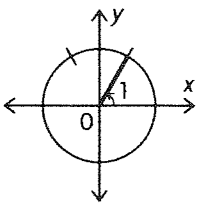
h) $\frac{2\pi}{3}$ radians;

$$\frac{2\cancel{\pi}}{3} \cancel{\text{radians}} \times \left(\frac{180°}{\cancel{\pi \text{ radians}}}\right) = 120°$$

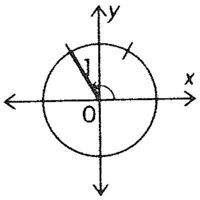
2. a)



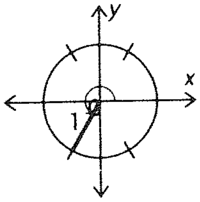
b)



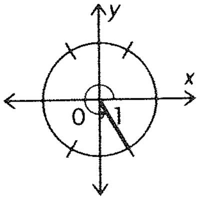
c)



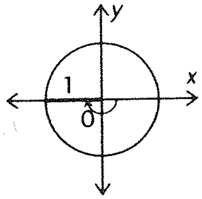
d)



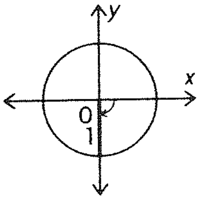
e)



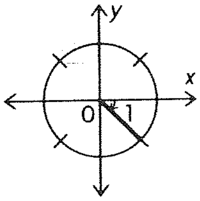
f)



g)



h)



3. a) $75° = 75° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{5\pi}{12} \text{ radians}$

b) $200° = 200° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{10\pi}{9} \text{ radians}$

c) $400° = 400° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{20\pi}{9} \text{ radians}$

d) $320° = 320° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{16\pi}{9} \text{ radians}$

4. a) $\frac{5\pi}{3} = \frac{5\pi}{3} \times \left(\frac{180°}{\pi \text{ radians}}\right) = 300°$

b) $0.3\pi = 0.3\pi \times \left(\frac{180°}{\pi \text{ radians}}\right) = 54°$

c) $3 = 3 \times \left(\frac{180°}{\pi \text{ radians}}\right) = 171.89°$

d) $\frac{11\pi}{4} = \frac{11\pi}{4} \times \left(\frac{180°}{\pi \text{ radians}}\right) = 495°$

5. a) $5 = \frac{x°}{360°}(2\pi)(2.5)$

$1800 = (x)(2\pi)(2.5)$

$114.6° = x$

$114.6° = 114.6° \times \left(\frac{\pi \text{ radians}}{180°}\right) = 2 \text{ radians}$

b) $200° = 200° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{10\pi}{9} \text{ radians}$

$$x = \frac{\frac{10\pi}{9}}{2\pi}(2\pi)(2.5)$$

$$x = \frac{5}{9}(2\pi)(2.5)$$

$$x = \frac{25\pi}{9} \text{ cm}$$

6. a) $3.5 = 3.5 \times \left(\frac{180°}{\pi \text{ radians}}\right) = 200.5°$

$$x = \frac{200.5°}{360°}(2\pi)(8)$$

$$x = 28 \text{ cm}$$

b) $300° = 300° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{5\pi}{3} \text{ radians}$

$$x = \frac{\frac{5\pi}{3}}{2\pi}(2\pi)(8)$$

$$x = \frac{5}{6}(2\pi)(8)$$

$$x = \frac{40\pi}{3} \text{ cm}$$

7. a) $90° = 90° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{\pi}{2} \text{ radians}$

b) $270° = 270° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{3\pi}{2} \text{ radians}$

c) $-180° = -180° \times \left(\frac{\pi \text{ radians}}{180°}\right)$

$= -\pi = \pi \text{ radians}$

d) $45° = 45° \times \left(\frac{\pi\text{ radians}}{180°}\right) = \frac{\pi}{4}\text{radians}$

e) $-135° = -135° \times \left(\frac{\pi\text{ radians}}{180°}\right)$

$$= -\frac{3\pi}{4} = \frac{5\pi}{4}\text{radians}$$

f) $60° = 60° \times \left(\frac{\pi\text{ radians}}{180°}\right) = \frac{\pi}{3}\text{radians}$

g) $240° = 240° \times \left(\frac{\pi\text{ radians}}{180°}\right) = \frac{4\pi}{3}\text{radians}$

h) $-120° = -120° \times \left(\frac{\pi\text{ radians}}{180°}\right)$

$$= -\frac{2\pi}{3} = \frac{4\pi}{3}\text{radians}$$

8. a) $\frac{2\pi}{3} = \frac{2\pi}{3} \times \left(\frac{180°}{\pi\text{ radians}}\right) = 120°$

b) $-\frac{5\pi}{3} = -\frac{5\pi}{3} \times \left(\frac{180°}{\pi\text{ radians}}\right) = -300° = 60°$

c) $\frac{\pi}{4} = \frac{\pi}{4} \times \left(\frac{180°}{\pi\text{ radians}}\right) = 45°$

d) $-\frac{3\pi}{4} = -\frac{3\pi}{4} \times \left(\frac{180°}{\pi\text{ radians}}\right) = -135° = 225°$

e) $\frac{7\pi}{6} = \frac{7\pi}{6} \times \left(\frac{180°}{\pi\text{ radians}}\right) = 210°$

f) $-\frac{3\pi}{2} = -\frac{3\pi}{2} \times \left(\frac{180°}{\pi\text{ radians}}\right) = -270° = 90°$

g) $\frac{11\pi}{6} = \frac{11\pi}{6} \times \left(\frac{180°}{\pi\text{ radians}}\right) = 330°$

h) $-\frac{9\pi}{2} = -\frac{9\pi}{2} \times \left(\frac{180°}{\pi\text{ radians}}\right) = -810°$

$$= -90° = 270°$$

9. a) $x = \frac{\frac{19\pi}{20}}{2\pi}(2\pi)(65)$

$$x = \frac{19}{40}(2\pi)(65)$$

$$x = \frac{2470\pi}{40}$$

$$x = \frac{247\pi}{4}\text{ m}$$

b) $1.25 = 1.25 \times \left(\frac{180°}{\pi\text{ radians}}\right) = 143.2°$

$$x = \frac{143.2°}{360°}(2\pi)(65)$$

$$x = 162.5\text{ m}$$

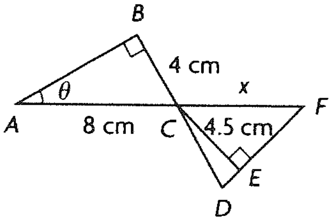
c) $150° = 150° \times \left(\frac{\pi\text{ radians}}{180°}\right) = \frac{5\pi}{6}\text{radians}$

$$x = \frac{\frac{5\pi}{6}}{2\pi}(2\pi)(65)$$

$$x = \frac{5}{12}(2\pi)(65)$$

$$x = \frac{325\pi}{6}\text{ cm}$$

10.



$\sin\theta = \frac{4}{8}$ or $\frac{1}{2}$, so $\theta = \frac{\pi}{6}$.

$$\angle BCA = \frac{\pi}{2} - \frac{\pi}{6}$$

$$= \frac{\pi}{3}$$

Because they are vertical angles, $\angle BCA = \angle DCF$.

$\angle DCF = \angle DCE + \angle ECF$

$$\frac{\pi}{3} = \frac{\pi}{12} + \angle ECF$$

$$\frac{\pi}{3} - \frac{\pi}{12} = \angle ECF$$

$$\frac{\pi}{4} = \angle ECF$$

Since $\angle ECF = \frac{\pi}{4}$, $x = \sqrt{2}(CE)$

$$= 4.50\sqrt{2}\text{ cm}$$

11. a) It rotates 4 times per min. So it rotates once every 15 seconds.

$$\omega = \frac{2\pi\text{ radians}}{15\text{ s}} \doteq 0.418\,88\text{ radians/s}$$

b) Radius = 3 m

Revolutions, $n = (4\text{ rev/min})(5\text{ min}) = 20\text{ rev}$

distance travelled $= 20(2\pi)(3) \doteq 377.0$ m

12. a) $\omega = 1.2\pi\text{ rad/s}(60\text{ s/min})$

$$= 72\pi\text{ rad/min}$$

$$72\cancel{\pi}\ \cancel{\text{rad}}/\text{min}\left(\frac{180°}{\cancel{\pi}\ \cancel{\text{radians}}}\right) = 12\,960°$$

In one minute, the wheel rotates 12 960°. So,

Revolutions, $n = 12\,960° \div 360° = 36$

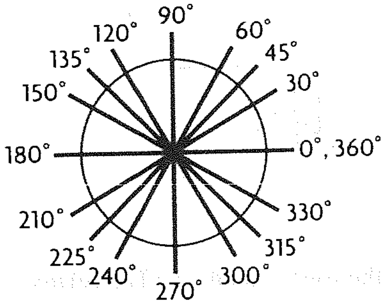
b) The wheel travels $(9.6\pi)(6) = 57.6\pi$ metres in a minute.

$$57.6\pi = 36(2\pi)(r)$$

$$0.8\text{ m} = r$$

13. a) The angular velocity of piece A is equal to piece B because they rotate at the same speed around the centre.
b) The velocity of piece A is greater than piece B because the radius of A is greater than the radius of B.
c) The percentage would stay the same.

14.



$$0° = 0° \times \left(\frac{\pi \text{ radians}}{180°}\right) = 0 \text{ radians;}$$

$$30° = 30° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{\pi}{6} \text{ radians;}$$

$$45° = 45° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{\pi}{4} \text{ radians;}$$

$$60° = 60° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{\pi}{3} \text{ radians;}$$

$$90° = 90° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{\pi}{2} \text{ radians;}$$

$$120° = 120° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{2\pi}{3} \text{ radians;}$$

$$135° = 135° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{3\pi}{4} \text{ radians;}$$

$$150° = 150° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{5\pi}{6} \text{ radians;}$$

$$180° = 180° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \pi \text{ radians;}$$

$$210° = 210° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{7\pi}{6} \text{ radians;}$$

$$225° = 225° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{5\pi}{4} \text{ radians;}$$

$$240° = 240° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{4\pi}{3} \text{ radians;}$$

$$270° = 270° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{3\pi}{2} \text{ radians;}$$

$$300° = 300° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{5\pi}{3} \text{ radians;}$$

$$315° = 315° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{7\pi}{4} \text{ radians;}$$

$$330° = 330° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{11\pi}{6} \text{ radians;}$$

$$360° = 360° \times \left(\frac{\pi \text{ radians}}{180°}\right) = 2\pi \text{ radians}$$

15. Circle A: $\frac{\frac{\pi}{6}}{2\pi}(2\pi)(15) \doteq 7.85$ cm

Circle B: $\frac{\frac{\pi}{7}}{2\pi}(2\pi)(17) \doteq 7.62$ cm

Circle C: $\frac{\frac{\pi}{5}}{2\pi}(2\pi)(14) \doteq 8.80$ cm

So, from smallest to largest, the order of the arcs would be Circle B, Circle A, and Circle C.

16. $C = 2\pi r = 2\pi(32) = 64\pi$ cm

$$64\pi \text{ cm} \times \left(\frac{1 \text{ m}}{100 \text{ cm}}\right) \times \left(\frac{1 \text{ km}}{1000 \text{ m}}\right) = 0.000\ 64\pi \text{ km}$$

$$\text{Revolutions} = \frac{675 \text{ km}}{0.000\ 64\pi \text{ km}} \doteq 1\ 054\ 687.5$$

$$6 \text{ hr } 45 \text{ min} = 765 \text{ min} \times \left(\frac{60 \text{ s}}{1 \text{ min}}\right) = 45\ 900 \text{ s}$$

$$\text{Rev/sec} = \frac{1\ 054\ 687.5 \text{ rev}}{45\ 900 \text{ s}} \doteq 23 \text{ rev/s}$$

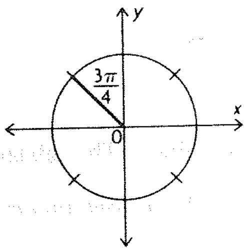
$$23 \text{ rev} \times 360° = 8280°/\text{s}$$

$$8280°/\text{s} = 8280°/\text{s} \times \left(\frac{\pi \text{ radians}}{180°}\right)$$

$$\doteq 144.5 \text{ radians/s}$$

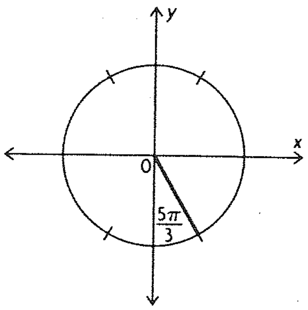
6.2 Radian Measure and Angles on the Cartesian Plane, pp. 330–332

1. a)



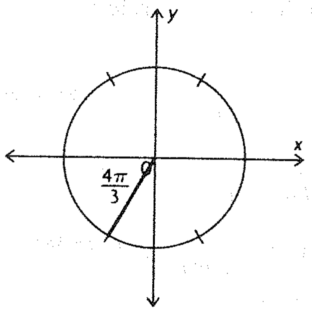
$\frac{3\pi}{4}$ is located in the second quadrant. The related angle is $\frac{\pi}{4}$, and sine is positive in the second quadrant.

b)



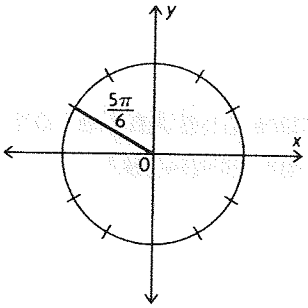
$\frac{5\pi}{3}$ is located in the fourth quadrant. The related angle is $\frac{\pi}{3}$, and cosine is positive in the fourth quadrant.

c)



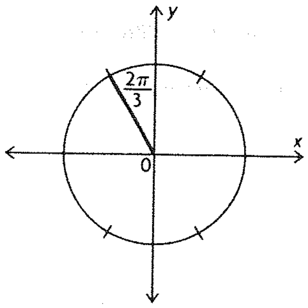
$\frac{4\pi}{3}$ is located in the third quadrant. The related angle is $\frac{\pi}{3}$, and tangent is positive in the third quadrant.

d)



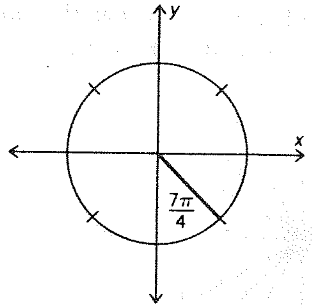
$\frac{5\pi}{6}$ is located in the second quadrant. The related angle is $\frac{\pi}{6}$, and secant is negative in the second quadrant.

e)



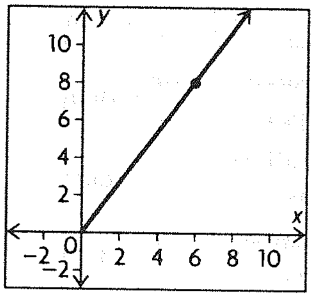
$\frac{2\pi}{3}$ is located in the second quadrant. The related angle is $\frac{\pi}{3}$, and cosine is negative in the second quadrant.

f)



$\frac{7\pi}{4}$ is located in the fourth quadrant. The related angle is $\frac{\pi}{4}$, and cotangent is negative in the fourth quadrant.

2. a) i)



ii) $r^2 = 6^2 + 8^2$

$r^2 = 36 + 64$

$r^2 = 100$

$r = 10$

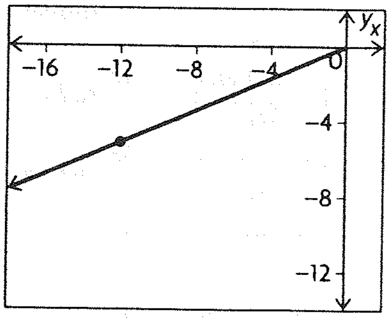
iii) $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$,

$\csc\theta = \frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = \frac{3}{4}$

iv) $\sin^{-1}\left(\frac{4}{5}\right) \doteq 0.93$

$\theta \doteq 0.93$

b) i)



ii) $r^2 = (-12)^2 + (-5)^2$

$r^2 = 144 + 25$

$r^2 = 169$

$r = 13$

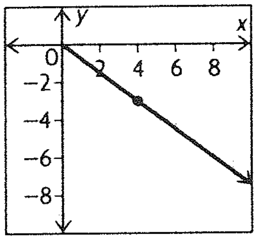
iii) $\sin\theta = -\frac{5}{13}$, $\cos\theta = -\frac{12}{13}$, $\tan\theta = \frac{5}{12}$,

$\csc\theta = -\frac{13}{5}$, $\sec\theta = -\frac{13}{12}$, $\cot\theta = \frac{12}{5}$

iv) $\sin^{-1}\left(-\frac{5}{13}\right) \doteq -0.395$

$\theta \doteq \pi + 0.395 \doteq 3.54$

c) i)



ii) $r^2 = 4^2 + (-3)^2$

$r^2 = 16 + 9$

$r^2 = 25$

$r = 5$

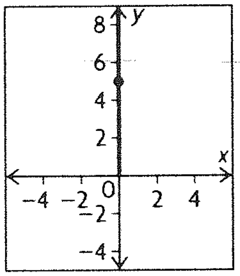
iii) $\sin\theta = -\frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = -\frac{3}{4}$,

$\csc\theta = -\frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = -\frac{4}{3}$

iv) $\sin^{-1}\left(-\frac{3}{5}\right) \doteq -0.64$

$\theta \doteq 2\pi - 0.64 \doteq 5.64$

d) i)



ii) $r^2 = 0^2 + 5^2$

$r^2 = 0 + 25$

$r^2 = 25$

$r = 5$

iii) $\sin\theta = \frac{5}{5} = 1$, $\cos\theta = \frac{0}{5} = 0$,

$\tan\theta = \frac{5}{0} = \text{undefined}$,

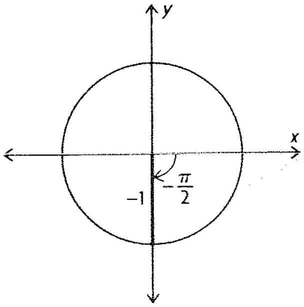
$\csc\theta = \frac{5}{5} = 1$, $\sec\theta = \frac{5}{0} = \text{undefined}$,

$\cot\theta = \frac{0}{5} = 0$

iv) $\sin^{-1}(1) \doteq 1.57$

$\theta \doteq \frac{\pi}{2}$

3. a)



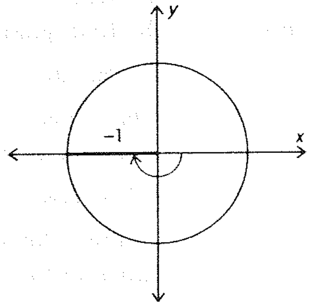
$x = 0, y = -1, r = 1$

$\sin\left(-\frac{\pi}{2}\right) = -1$, $\cos\left(-\frac{\pi}{2}\right) = 0$,

$\tan\left(-\frac{\pi}{2}\right) = \text{undefined}$, $\csc\left(-\frac{\pi}{2}\right) = -1$,

$\sec\left(-\frac{\pi}{2}\right) = \text{undefined}$, $\cot\left(-\frac{\pi}{2}\right) = 0$

b)



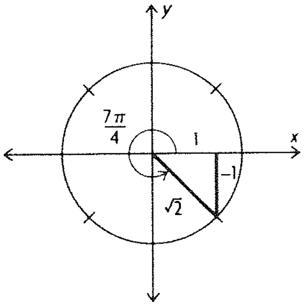
$x = -1, y = 0, r = 1$

$\sin(-\pi) = 0$, $\cos(-\pi) = -1$, $\tan(-\pi) = 0$,

$\csc(-\pi) = \text{undefined}$, $\sec(-\pi) = -1$,

$\cot(-\pi) = \text{undefined}$

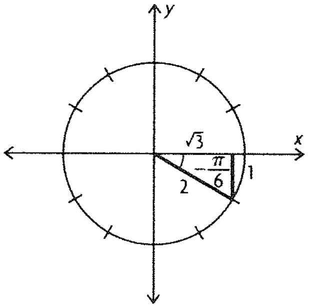
c)



$x = 1, y = -1, r = \sqrt{2}$

$$\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}, \cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2},$$
$$\tan\left(\frac{7\pi}{4}\right) = -1, \csc\left(\frac{7\pi}{4}\right) = -\sqrt{2},$$
$$\sec\left(\frac{7\pi}{4}\right) = \sqrt{2}, \cot\left(\frac{7\pi}{4}\right) = -1$$

d)



$$x = \sqrt{3}, y = -1, r = 2$$
$$\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}, \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},$$
$$\tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}, \csc\left(-\frac{\pi}{6}\right) = -2,$$
$$\sec\left(-\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3}, \cot\left(-\frac{\pi}{6}\right) = -\sqrt{3}$$

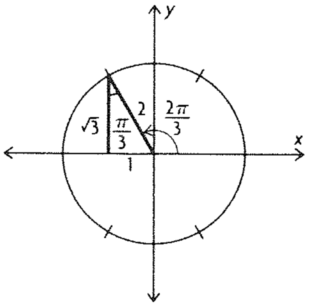
4. a) This is in the second quadrant where sine is positive. Sine is also positive in the first quadrant. So, an equivalent expression would be $\sin\frac{\pi}{6}$.

b) This is in the fourth quadrant where cosine is positive. Cosine is also positive in the first quadrant. So, an equivalent expression would be $\cos\frac{\pi}{3}$.

c) This is in the fourth quadrant where cotangent is negative. Cotangent is also negative in the second quadrant. So, an equivalent expression would be $\cot\frac{3\pi}{4}$.

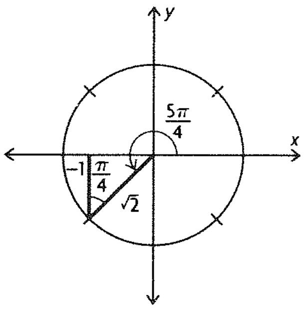
d) This is in the third quadrant where secant is negative. Secant is also negative in the second quadrant. So, an equivalent expression would be $\sec\frac{5\pi}{6}$.

5. a)



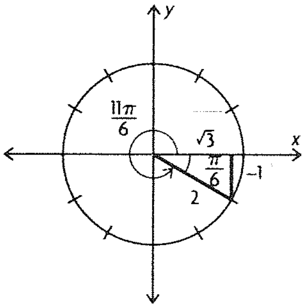
$$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

b)



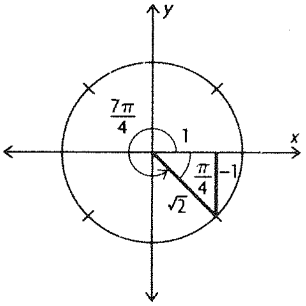
$$\cos\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

c)



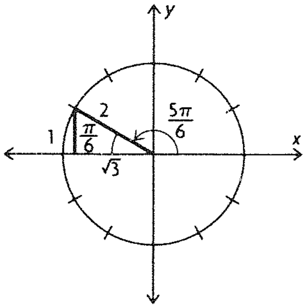
$$\tan\left(\frac{11\pi}{6}\right) = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

d)



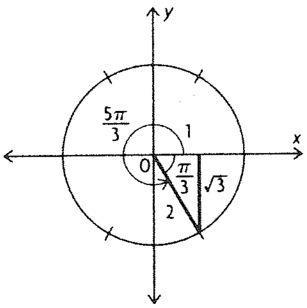
$$\sin\left(\frac{7\pi}{4}\right) = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

e)



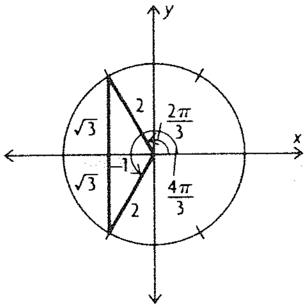
$$\csc\left(\frac{5\pi}{6}\right) = \frac{2}{1} = 2$$

f)



$$\sec\left(\frac{5\pi}{3}\right) = \frac{2}{1} = 2$$

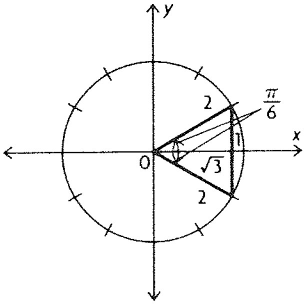
6. a)



$$\cos\theta = -\frac{1}{2}$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

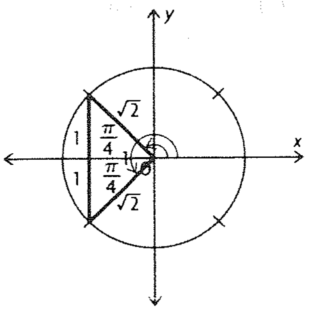
b)



$$\cos\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{6}, \frac{11\pi}{6}$$

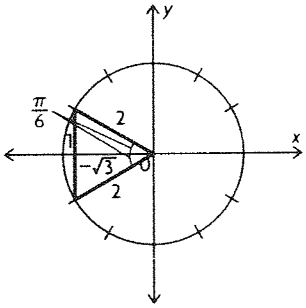
c) $-\dfrac{\sqrt{2}}{2} = -\dfrac{1}{\sqrt{2}}$



$$\cos\theta = -\frac{1}{\sqrt{2}}$$

$$\theta = \frac{3\pi}{4}, \frac{5\pi}{4}$$

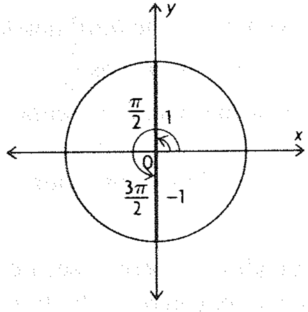
d)



$$\cos\theta = -\frac{\sqrt{3}}{2}$$

$$\theta = \frac{5\pi}{6}, \frac{7\pi}{6}$$

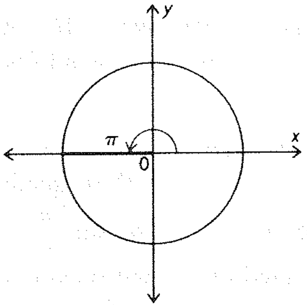
e)



$$\cos\theta = 0$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

f)



$\cos\theta = -1$

$\theta = \pi$

7. a) $(-7, 8)$ is in the second quadrant.

$$\tan^{-1}\left(\frac{8}{-7}\right) \doteq -0.852$$

$$\theta \doteq \pi - 0.852 \doteq 2.29$$

b) $(12, 2)$ is in the first quadrant.

$$\tan^{-1}\left(\frac{2}{12}\right) \doteq 0.17$$

$$\theta \doteq 0.17$$

c) (3, 11) is in the first quadrant.

$$\tan^{-1}\left(\frac{11}{3}\right) \doteq 1.30$$

$$\theta \doteq 1.30$$

d) $(-4, -2)$ is in the third quadrant.

$$\tan^{-1}\left(\frac{-2}{-4}\right) \doteq 0.464$$

$$\theta \doteq \pi + 0.464 \doteq 3.61$$

e) (9, 10) is in the first quadrant.

$$\tan^{-1}\left(\frac{10}{9}\right) \doteq 0.84$$

$$\theta \doteq 0.84$$

f) $(6, -1)$ is in the fourth quadrant.

$$\tan^{-1}\left(\frac{-1}{6}\right) \doteq -0.165$$

$$\theta \doteq 2\pi - 0.165 \doteq 6.12$$

8. a) This is in the second quadrant where cosine is negative. Cosine is also negative in the third quadrant. So, an equivalent expression would be $\cos\frac{5\pi}{4}$.

b) This is in the fourth quadrant where tangent is negative. Tangent is also negative in the second quadrant. So, an equivalent expression would be $\tan\frac{5\pi}{6}$.

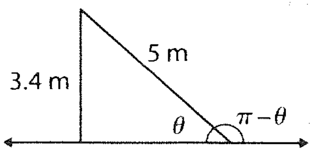
c) This is in the fourth quadrant where cosecant is negative. Cosecant is also negative in the third quadrant. So, an equivalent expression would be $\csc\frac{4\pi}{3}$.

d) This is in the second quadrant where cotangent is negative. Cotangent is also negative in the fourth quadrant. So, an equivalent expression would be $\cot\frac{5\pi}{3}$.

e) This is in the fourth quadrant where sine is negative. Sine is also negative in the third quadrant. So, an equivalent expression would be $\sin\frac{7\pi}{6}$.

f) This is in the fourth quadrant where secant is positive. Secant is also positive in the first quadrant. So, an equivalent expression would be $\sec\frac{\pi}{4}$.

9.

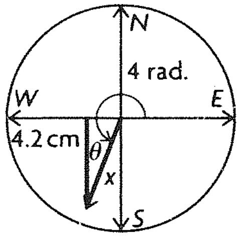


$$\sin\theta = \frac{3.4}{5}$$

$$\theta = \sin^{-1}\left(\frac{3.4}{5}\right) \doteq 0.748$$

$$\pi - 0.748 \doteq 2.39$$

10.

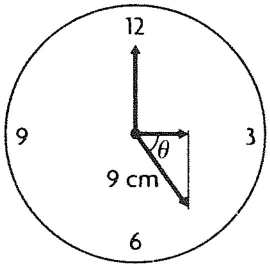


$$\theta = 4 - \pi \doteq 0.8584$$

$$\sin(0.8584) = \frac{4.2}{x}$$

$$x = \frac{4.2}{\sin(0.8584)} \doteq 5.55 \text{ cm}$$

11.



Use a proportion to find θ.

$$\frac{10\text{ sec}}{60\text{ sec}} = \frac{r\text{ rad}}{2\pi\text{ rad}}$$

$$r \doteq 1.05 \text{ rad}$$

$$\cos(1.05) = \frac{x}{9}$$

$$x = 9\cos(1.05)$$

$$x \doteq 4.5 \text{ cm}$$

12. Draw the angle and determine the measure of the reference angle. Use the CAST rule to determine the sign of each of the ratios in the quadrant in which the angle terminates. Use this sign and the value of the ratios of the reference angle to determine the values of the primary trigonometric ratios for the given angle.

13. a) It lies in the second or third quadrant because cosine is negative in these quadrants.

b) $x = -5, r = 13, y = ?$

$$13^2 - (-5)^2 = y^2$$

$$169 - 25 = y^2$$

$$144 = y^2$$

$$12 = y$$

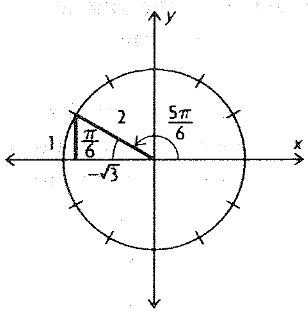
$$\sin\theta = \frac{12}{13} \text{ or } -\frac{12}{13},$$
$$\tan\theta = \frac{12}{5} \text{ or } -\frac{12}{5},$$
$$\sec\theta = -\frac{13}{5},$$
$$\csc\theta = \frac{13}{12} \text{ or } -\frac{13}{12},$$
$$\cot\theta = \frac{5}{12} \text{ or } -\frac{5}{12}$$

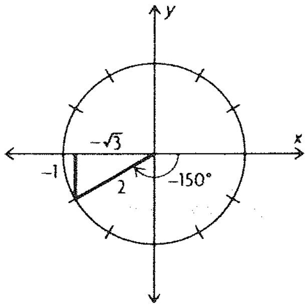
c) $\sin\theta = \dfrac{12}{13}$

$$\theta = \sin^{-1}\left(\frac{12}{13}\right)$$
$$\theta \doteq 1.176$$

In the second quadrant, $\pi - 1.176 \doteq 1.97$.
In the second quadrant, $\pi + 1.176 \doteq 4.32$.

14.





By examining the special triangles, we see
$$\cos\left(\frac{5\pi}{6}\right) = \cos(-150°) = -\frac{\sqrt{3}}{2}$$

15. $2\left(\sin^2\left(\dfrac{11\pi}{6}\right)\right) - 1$

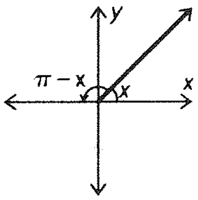
$$= 2\left(-\frac{1}{2}\right)^2 - 1$$
$$= 2\left(\frac{1}{4}\right) - 1$$
$$= -\frac{1}{2}$$

$$\left(\sin^2\frac{11\pi}{6}\right) - \left(\cos^2\frac{11\pi}{6}\right)$$
$$= \left(-\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$$
$$= \frac{1}{4} - \frac{3}{4}$$
$$= -\frac{1}{2}$$
$$2\left(\sin^2\left(\frac{11\pi}{6}\right)\right) - 1 = \left(\sin^2\frac{11\pi}{6}\right) - \left(\cos^2\frac{11\pi}{6}\right)$$

16. $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{8}{AB}$

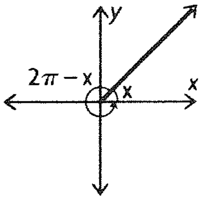
$$AB = \frac{8}{\sin\left(\frac{\pi}{6}\right)}$$
$$AB = 16$$
$$(AD)^2 = 8^2 + 8^2$$
$$(AD)^2 = 64 + 64$$
$$(AD)^2 = 128$$
$$(AD) = \sqrt{128} = 8\sqrt{2}$$
$$\sin D = \frac{8}{8\sqrt{2}} = \frac{\sqrt{2}}{2};$$
$$\cos D = \frac{8}{8\sqrt{2}} = \frac{\sqrt{2}}{2};$$
$$\tan D = \frac{8}{8} = 1$$

17. a)



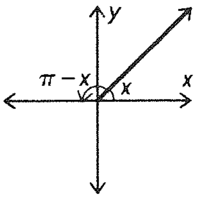
The first and second quadrants both have a positive y-value.

b)



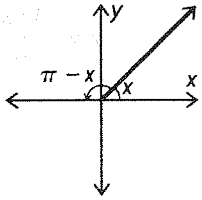
The first quadrant has a positive y-value, and the fourth quadrant has a negative y-value.

c)



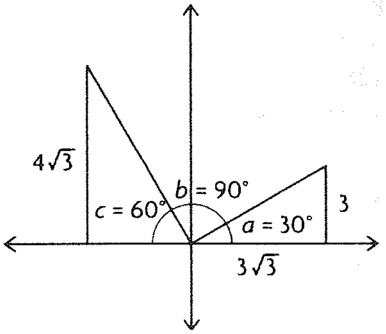
The first quadrant has a positive x-value, and the second quadrant has a negative x-value.

d)



The first quadrant has a positive x-value and a positive y-value, and the third quadrant has a negative x-value and a negative y-value.

18.



$$\tan a = \frac{3}{3\sqrt{3}}$$

$$a = \tan^{-1}\left(\frac{3}{3\sqrt{3}}\right)$$

$$a = 30°$$

$$\tan c = \frac{4\sqrt{3}}{4}$$

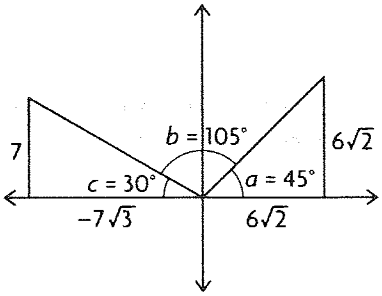
$$c = \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right)$$

$$c = 60°$$

$$b = 180° - 30° - 60° = 90°$$

$\sin 90° = 1$

19.



$$\tan a = \frac{6\sqrt{2}}{6\sqrt{2}}$$

$$a = \tan^{-1}\left(\frac{1}{1}\right)$$

$$a = 45°$$

$$\tan c = \frac{7}{7\sqrt{3}}$$

$$c = \tan^{-1}\left(\frac{7}{7\sqrt{3}}\right)$$

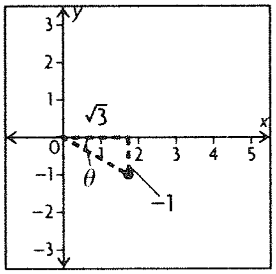
$$c = 30°$$

$$b = 180° - 30° - 45° = 105°$$

$\cos 150° \doteq -0.26$

20. The ranges of the cosecant and secant functions are both $\{y \in \mathbf{R} \mid -1 \geq y \text{ or } y \geq 1\}$. In other words, the values of these functions can never be between -1 and 1. For the values of these functions to be between -1 and 1, the values of the sine and cosine functions would have to be greater than 1 and less than -1, which is never the case.

21. The terminal arm is in the fourth quadrant. Cotangent is the ratio of adjacent side to opposite side. The given information leads to the figure shown below.



This is a special triangle, and the hypotenuse is 2.

$$\sin\theta\cot\theta - \cos^2\theta = -\frac{1}{2}(-\sqrt{3}) - \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{\sqrt{3}}{2} - \frac{3}{4}$$

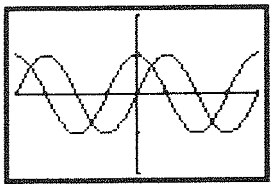
$$= \frac{2\sqrt{3} - 3}{4}$$

6.3 Exploring Graphs of the Primary Trigonometric Functions, p. 336

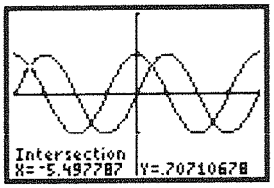
1. a) $y = \sin\theta$ and $y = \cos\theta$ have the same period, axis, amplitude, maximum value, minimum value, domain, and range. They have different y- and θ-intercepts.

b) $y = \sin\theta$ and $y = \tan\theta$ have no characteristics in common except for their y-intercept and zeros.

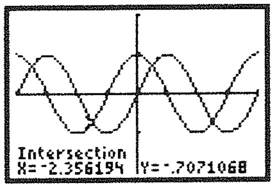
2. a)



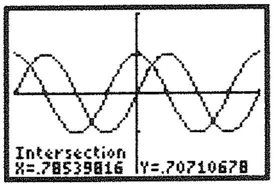
b)



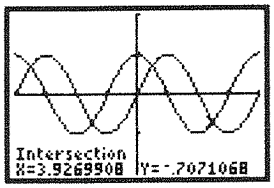
$\theta = -5.50$



$\theta = -2.36$



$\theta = 0.79$



$\theta = 3.93$

c) i) The graph of $y = \sin\theta$ intersects the θ-axis at $0, \pm\pi, \pm 2\pi, \ldots$

$t_n = n\pi, n \in \mathbf{I}$

ii) The maximum value occurs at $\frac{\pi}{2}$ and every 2π, since the period is 2π.

$t_n = \frac{\pi}{2} + 2n\pi, n \in \mathbf{I}$

iii) The minimum value occurs at $\frac{3\pi}{2}$ and every 2π, since the period is 2π.

$t_n = \frac{3\pi}{2} + 2n\pi, n \in \mathbf{I}$

3. a) The graph of $y = \cos\theta$ intersects the θ-axis at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

$t_n = \frac{\pi}{2} + n\pi, n \in \mathbf{I}$

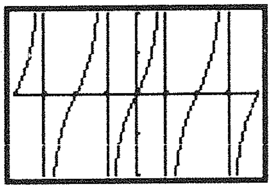
b) The maximum values occur at 0 and every 2π, since the period is 2π.

$t_n = 2n\pi, n \in \mathbf{I}$

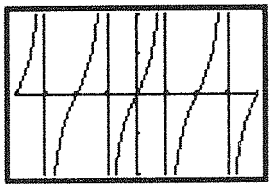
c) The minimum value occurs at π and every 2π, since the period is 2π.

$t_n = -\pi + 2n\pi, n \in \mathbf{I}$

4. Here is the graph of $y = \frac{\sin x}{\cos x}$:



Here is the graph of $y = \tan x$:



The two graphs appear to be identical.

5. a) The graph of $y = \tan\theta$ intersects the θ-axis at $0, \pm\pi, \pm 2\pi, \ldots$

$t_n = n\pi, n \in \mathbf{I}$

b) The graph of $y = \tan\theta$ has vertical asymptotes at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

$t_n = \frac{\pi}{2} + n\pi, n \in \mathbf{I}$

6.4 Transformations of Trigonometric Functions, pp. 343–346

1. a) period: $\frac{2\pi}{|k|} = \frac{2\pi}{|4|} = \frac{\pi}{2}$

amplitude: $|a| = |0.5| = 0.5$

horizontal translation: $d = 0$

equation of the axis: $y = 0$

b) period: $\frac{2\pi}{|k|} = \frac{2\pi}{|1|} = 2\pi$

amplitude: $|a| = |1| = 1$

horizontal translation: $d = \frac{\pi}{4}$

equation of the axis: $y = 3$

c) period: $\frac{2\pi}{|k|} = \frac{2\pi}{3}$

amplitude: $|a| = |2| = 2$

horizontal translation: $d = 0$

equation of the axis: $y = -1$

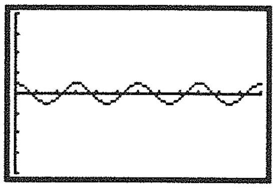
d) period: $\frac{2\pi}{|k|} = \frac{2\pi}{|-2|} = \pi$

amplitude: $|a| = |5| = 5$

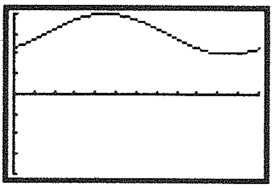
horizontal translation: $d = \frac{\pi}{6}$

equation of the axis: $y = -2$

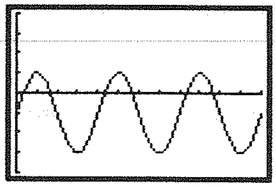
2. For $y = 0.5\cos(4x)$



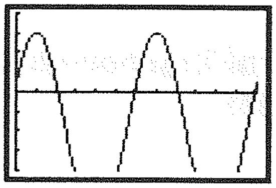
For $y = \sin\left(x - \frac{\pi}{4}\right) + 3$



For $y = 2\sin(3x) - 1$

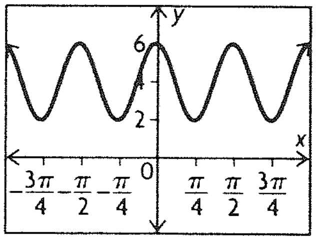


For $y = 5\cos\left(-2x + \frac{\pi}{3}\right) - 2$



Only the last one is cut off.

3.



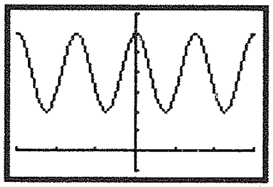
$y = -2\cos\left(4\left(x + \frac{\pi}{4}\right)\right) + 4$

period: $\frac{2\pi}{|k|} = \frac{2\pi}{|4|} = \frac{\pi}{2}$

amplitude: $|a| = |-2| = 2$

horizontal translation: $d = -\frac{\pi}{4}$ units to the left

equation of the axis: $y = 4$



4. $y = a\sin(k(x - d)) + c$

a) $a = 25$

period: $\frac{2\pi}{|k|} = \pi$

$k = 2$

$f(x) = 25\sin(2x) - 4$

b) $a = \frac{2}{5}$

period: $\frac{2\pi}{|k|} = 10$

$k = \frac{\pi}{5}$

$f(x) = \frac{2}{5}\sin\left(\frac{\pi}{5}x\right) + \frac{1}{15}$

c) $a = 80$

period: $\frac{2\pi}{|k|} = 6\pi$

$k = \frac{1}{3}$

$f(x) = 80\sin\left(\frac{1}{3}x\right) - \frac{9}{10}$

d) $a = 11$

period: $\frac{2\pi}{|k|} = \frac{1}{2}$

$k = 4\pi$

$f(x) = 11\sin(4\pi x)$

5. a) period $= 2\pi$, amplitude $= 18$, equation of the axis is $y = 0$;

$y = 18\sin x$

b) period $= 4\pi$, amplitude $= 6$, equation of the axis is $y = -2$;

$y = -6\sin(0.5x) - 2$

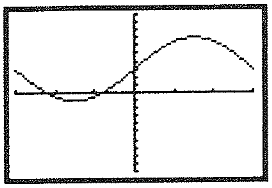
c) period $= 6\pi$, amplitude $= 2.5$, equation of the axis is $y = 6.5$;

$y = -2.5\cos\left(\frac{1}{3}x\right) + 6.5$

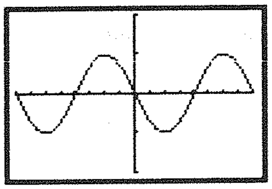
d) period $= 4\pi$, amplitude $= 2$, equation of the axis is $y = -1$;

$y = -2\cos\left(\frac{1}{2}x\right) - 1$

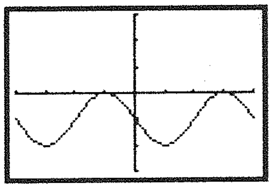
6. a) vertical stretch by a factor of 4, vertical translation 3 units up



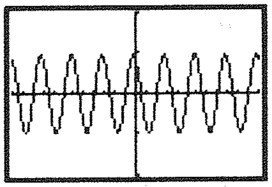
b) reflection in the x-axis, horizontal stretch by a factor of 4



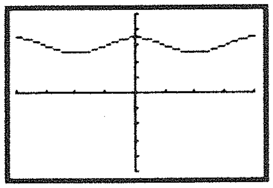
c) horizontal translation π to the right, vertical translation 1 unit down



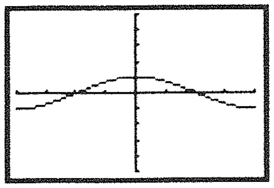
d) horizontal compression by a factor of $\frac{1}{4}$, horizontal translation $\frac{\pi}{6}$ to the left



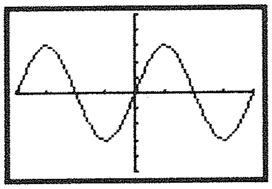
7. a) $f(x) = \frac{1}{2}\cos x + 3$



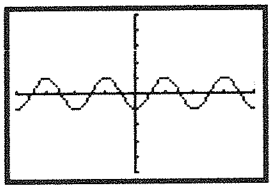
b) $f(x) = \cos\left(-\frac{1}{2}x\right)$



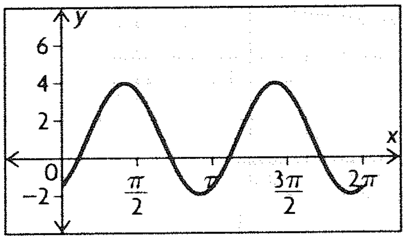
c) $f(x) = 3\cos\left(x - \frac{\pi}{2}\right)$

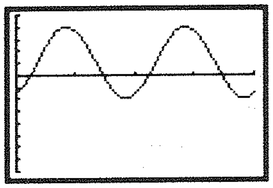


d) $f(x) = \cos\left(2\left(x + \frac{\pi}{2}\right)\right)$

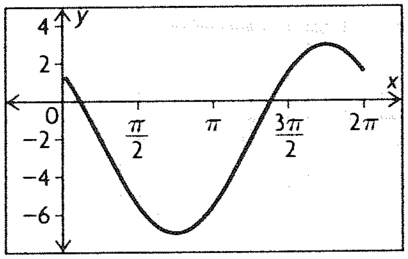


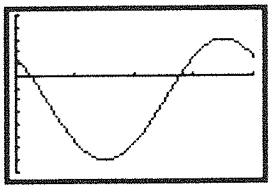
8. a)



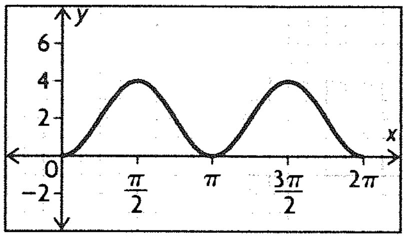


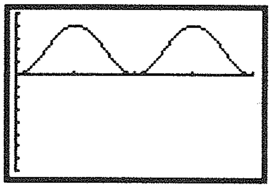
b)



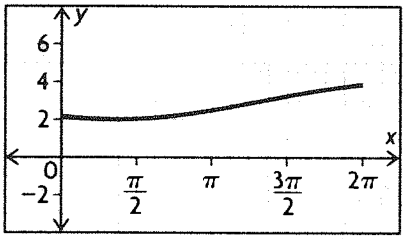


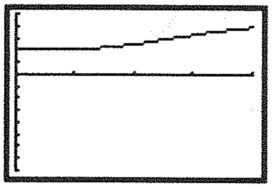
c)



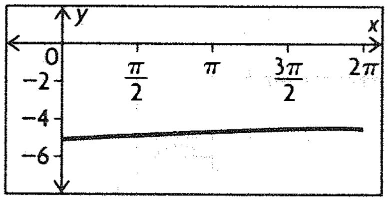


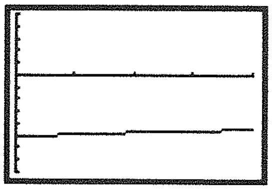
d)



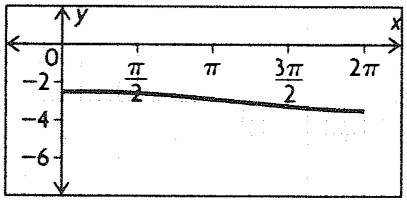


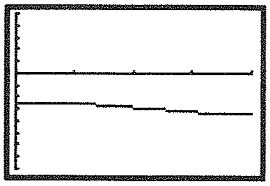
e)





f)





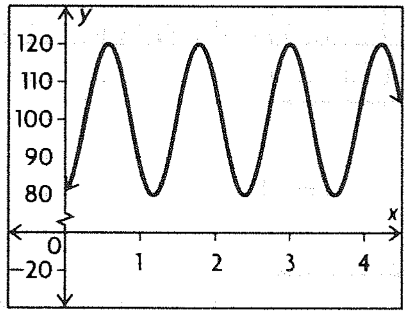
9. a) period: $\frac{2\pi}{|k|} = \frac{5\pi}{3}$

$$k = \frac{6}{5}$$

The period of the function is $\frac{6}{5}$.
This represents the time between one beat of a person's heart and the next beat.

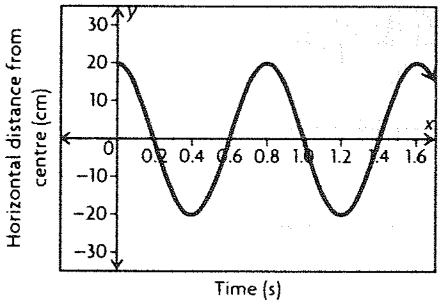
b) $P(60) = -20\cos\left(\frac{5\pi}{3}(60)\right) + 100 = 80$

c)



d) The range for the function is between 80 and 120. The range means the lowest blood pressure is 80 and the highest blood pressure is 120.

10. a)



b) There is a vertical stretch by a factor of 20. The period is 0.8 s.

$$\frac{2\pi}{k} = 0.8$$

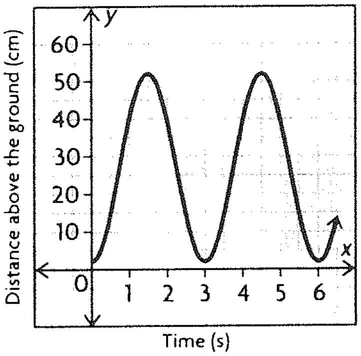
$$k = \frac{5\pi}{2}$$

There is a horizontal compression by a factor of $\frac{1}{|k|} = \frac{2}{5\pi}$.

There is a horizontal translation 0.2 to the left.

c) $y = 20\sin\left(\frac{5\pi}{2}(x + 0.2)\right)$

11. a)



b) vertical stretch by a factor of 25, reflection in the x-axis, vertical translation 27 units up; the period is 3 s.

$$\frac{2\pi}{k} = 3$$

$$k = \frac{2\pi}{3}$$

horizontal compression by a factor of $\frac{1}{|k|} = \frac{3}{2\pi}$

c) $y = -25\cos\left(\frac{2\pi}{3}x\right) + 27$

12. By looking at the difference in the x-values of the two maximums, $-\frac{5\pi}{7}$ and $-\frac{3\pi}{7}$, we see that the period is $\frac{2\pi}{7}$.

13. Answers may vary. For example, $\left(\frac{14\pi}{13}, 5\right)$. Since the maximum is 4 units above $y = 9$, the minimum would be at $y = 5$. If the period of the function is 2π, then the minimum would be at $\frac{\pi}{13} + \pi$ of $\frac{14\pi}{13}$.

14. a) This is a cosine function with amplitude = 1.

$$\text{period} = \frac{2\pi}{0.5} = 4\pi$$

$$y = \cos(4\pi x)$$

b) This is a sine function with a reflection in the x-axis and an amplitude = 2.

$$\text{period} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$y = -2\sin\left(\frac{\pi}{4}x\right)$$

c) The y-axis is $y = -1$ and the amplitude is 4. The function is shifted horizontally to the right by 10.

$$\text{period} = \frac{2\pi}{40} = \frac{\pi}{20}$$

$$y = 4\sin\left(\frac{\pi}{20}(x - 10)\right) - 1$$

15.

Start with graph of $y = \sin x$.

↓

Reflect in the x-axis and stretch vertically by a factor of 2 to produce graph of $y = -2\sin x$.

↓

Stretch horizontally by a factor of 2 to produce graph of $y = -2\sin(0.5x)$.

↓

Translate $\frac{\pi}{4}$ units to the right to produce graph of $y = -2\sin\left(0.5\left(x - \frac{\pi}{4}\right)\right)$.

↓

Translate 3 units up to produce graph of $y = -2\sin\left(0.5\left(x - \frac{\pi}{4}\right)\right) + 3$.

16. a) The car starts at the closest distance to the pole which is 100 m.

b) The centre of the track is 400 m from the pole because it is half the distance between the closest and furthest point.

c) The radius is 400 − 100 = 300 m.

d) The period of the function is 80 s. This is how long it takes to complete one lap.

e) $\frac{2\pi(300)}{80}$ m/s $\doteq$ 23.56194 m/s

Mid-Chapter Review, p. 349

1. a) $\frac{\cancel{\pi}}{8}\ \cancel{\text{radians}} \times \left(\frac{180^\circ}{\cancel{\pi\ \text{radians}}}\right) = 22.5^\circ$

b) $4\cancel{\pi\ \text{radians}} \times \left(\frac{180^\circ}{\cancel{\pi\ \text{radians}}}\right) = 720^\circ$

c) $5\ \cancel{\text{radians}} \times \left(\frac{180^\circ}{\pi\ \cancel{\text{radians}}}\right) \doteq 286.5^\circ$

d) $\frac{11\cancel{\pi}}{12}\ \cancel{\text{radians}} \times \left(\frac{180^\circ}{\cancel{\pi\ \text{radians}}}\right) = 165^\circ$

2. a) $125° = 125° \times \left(\dfrac{\pi \text{ radians}}{180°}\right) \doteq 2.2$ radians

b) $450° = 450° \times \left(\dfrac{\pi \text{ radians}}{180°}\right) \doteq 7.9$ radians

c) $5° = 5° \times \left(\dfrac{\pi \text{ radians}}{180°}\right) \doteq 0.1$ radians

d) $330° = 330° \times \left(\dfrac{\pi \text{ radians}}{180°}\right) \doteq 5.8$ radians

e) $215° = 215° \times \left(\dfrac{\pi \text{ radians}}{180°}\right) \doteq 3.8$ radians

f) $-140° = -140° \times \left(\dfrac{\pi \text{ radians}}{180°}\right) \doteq -2.4$ radians

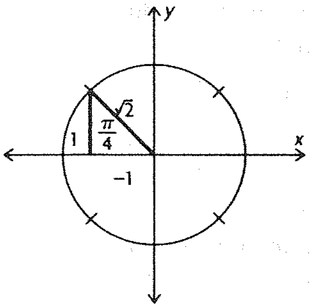
3. a) $10(2\pi) = 20\pi$

b) $\omega = \dfrac{20\pi}{5} = 4\pi$ radians/s

c) Circumference $= 2\pi(19) = 38\pi$

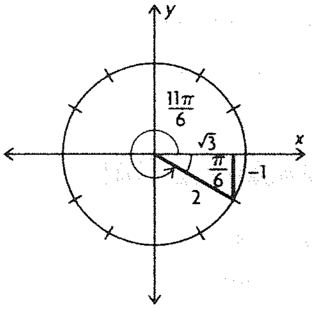
$38\pi \times 10$ revolutions $= 380\pi$ cm

4. a)



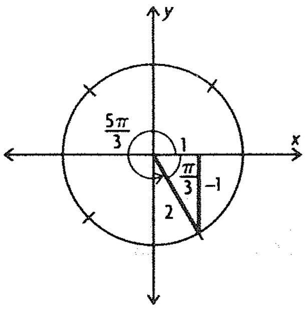
$$\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

b)



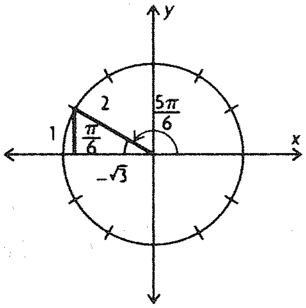
$$\sin\frac{11\pi}{6} = -\frac{1}{2}$$

c)



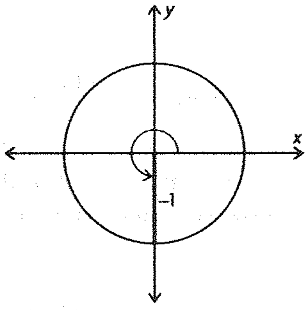
$$\tan\frac{5\pi}{3} = -\sqrt{3}$$

d)



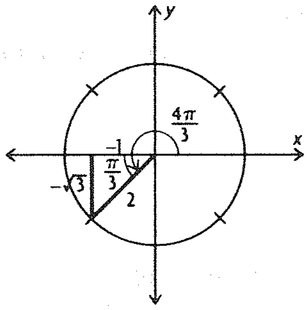
$$\tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

e)



$$\cos\frac{3\pi}{2} = \frac{0}{1} = 0$$

f)



$$\cos\frac{4\pi}{3} = -\frac{1}{2}$$

5. a) $(-3, 14)$ is in the second quadrant.

$\tan^{-1}\left(\frac{14}{-3}\right) \doteq -1.360$

$\theta \doteq \pi - 1.360 \doteq 1.78$

b) $(6, 7)$ is in the first quadrant.

$\tan^{-1}\left(\frac{7}{6}\right) \doteq 0.86$

c) $(1, 9)$ is in the first quadrant.

$\tan^{-1}\left(\frac{9}{1}\right) \doteq 1.46$

d) $(-5, -18)$ is in the third quadrant.

$\tan^{-1}\left(\frac{-18}{-5}\right) \doteq 1.30$

$\theta \doteq \pi + 1.30 \doteq 4.44$

e) $(2, 3)$ is in the first quadrant.

$\tan^{-1}\left(\frac{3}{2}\right) \doteq 0.98$

f) $(4, -20)$ is in the fourth quadrant.

$\tan^{-1}\left(\frac{-20}{4}\right) \doteq -1.373$

$\theta \doteq 2\pi - 1.360 \doteq 4.91$

6. a) This is in the second quadrant where sine is positive. Sine is also positive in the first quadrant. So, an equivalent expression would be $\sin\frac{\pi}{6}$.

b) This is in the fourth quadrant where cotangent is negative. Cotangent is also negative in the second quadrant. So, an equivalent expression would be $\cot\frac{3\pi}{4}$.

c) Secant is undefined at $-\frac{\pi}{2}$. It is also undefined at $\frac{\pi}{2}$. So, an equivalent expression would be $\sec\frac{\pi}{2}$.

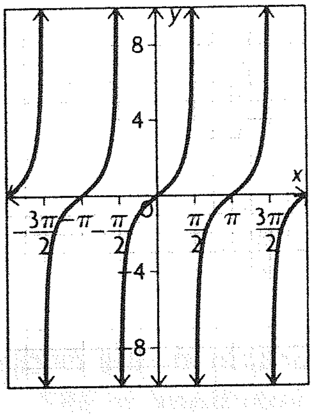
d) This is in the third quadrant where cosine is negative. Cosine is also negative in the second quadrant. So, an equivalent expression would be $\cos\frac{5\pi}{6}$.

7. a) $x = 0, \pm\pi, \pm 2\pi, \ldots; y = 0$

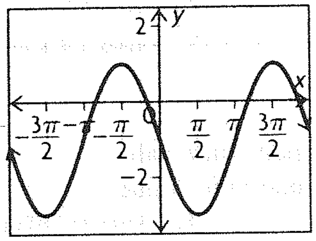
b) $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots; y = 1$

c) $x = 0, \pm\pi, \pm 2\pi, \ldots; y = 0$

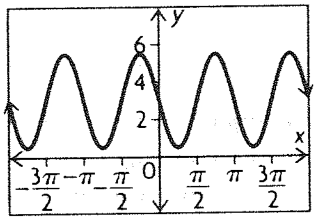
8. a)



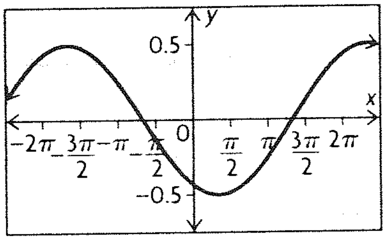
b)



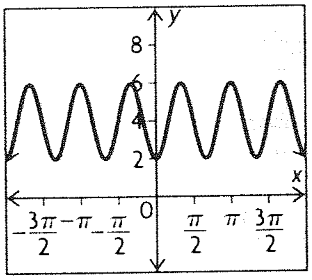
c)



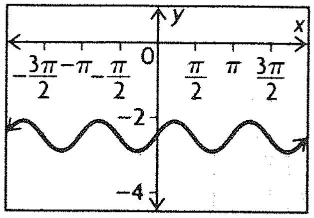
d)



e)



f)



9. $y = \frac{1}{3}\sin\left(-3\left(x + \frac{\pi}{8}\right)\right) - 23$

6.5 Exploring Graphs of the Reciprocal Trigonometric Functions, p. 353

1. a) The graph of $y = \csc x$ has vertical asymptotes at $0, \pm\pi, \pm 2\pi, \ldots$

$t_n = n\pi, n \in \mathbf{I}$

b) $y = \csc x$ has no maximum value.

c) $y = \csc x$ has no minimum value.

2. a) The graph of $y = \sec x$ has vertical asymptotes at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

$t_n = \frac{\pi}{2} + n\pi, n \in \mathbf{I}$

b) $y = \sec x$ has no maximum value.

c) $y = \sec x$ has no minimum value.

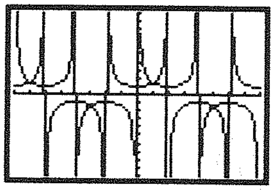
3. a) The graph of $y = \cot x$ has vertical asymptotes at $0, \pm\pi, \pm 2\pi, \ldots$

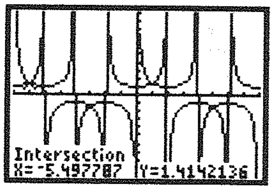
$t_n = n\pi, n \in \mathbf{I}$

b) The graph of $y = \cot x$ intersects the x-axis at $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}, \ldots$

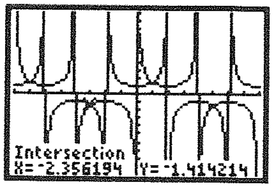
$t_n = \frac{\pi}{2} + n\pi, n \in \mathbf{I}$

4.

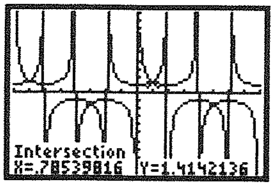




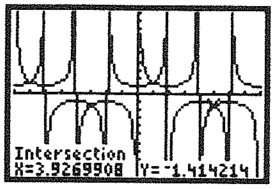
```
Intersection
X=-5.497787  Y=1.4142136
```



```
Intersection
X=-2.356194  Y=-1.414214
```



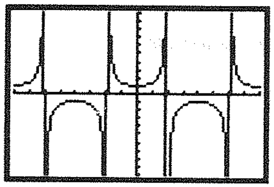
```
Intersection
X=.78539816  Y=1.4142136
```



```
Intersection
X=3.9269908  Y=-1.414214
```

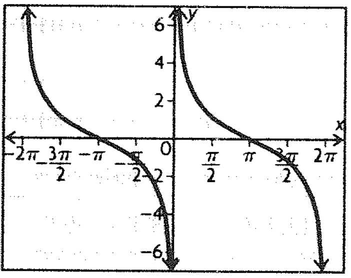
The values of x for which $y = \csc x$ and $y = \sec x$ intersect are $x = -5.50, -2.35, 0.79, 3.93$, the same values for which $y = \sin x$ and $y = \cos x$ were determined to intersect in Lesson 6.3.

5. Yes; the graphs of $y = \csc\left(x + \frac{\pi}{2}\right)$ and $y = \sec x$ are identical.

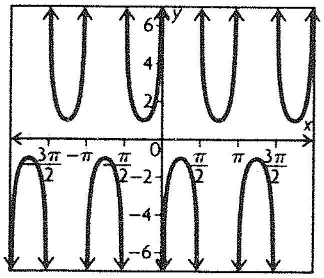


6. Answers may vary. For example, reflect the graph of $y = \tan x$ across the y-axis and then translate the graph $\frac{\pi}{2}$ units to the left.

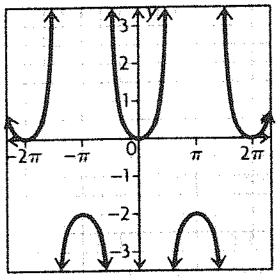
7. a) period $= 2\pi$



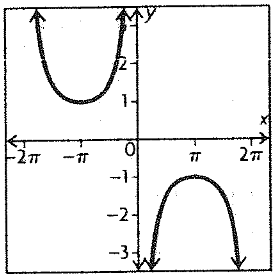
b) period $= \pi$



c) period $= 2\pi$



d) period $= 4\pi$



6.6 Modelling with Trigonometric Functions, p. 360–362

1. $y = 3\cos\left(\frac{2}{3}\left(x + \frac{\pi}{4}\right)\right) + 2$

2. For $x = \frac{\pi}{2}$,

$y = 3\cos\left(\frac{2}{3}\left(\frac{\pi}{2} + \frac{\pi}{4}\right)\right) + 2$

$y = 3\cos\left(\frac{2}{3}\left(\frac{3\pi}{4}\right)\right) + 2$

$y = 3\cos\left(\frac{\pi}{2}\right) + 2$

$y = 0 + 2$

$y = 2$

For $x = \frac{3\pi}{4}$

$y = 3\cos\left(\frac{2}{3}\left(\frac{3\pi}{4} + \frac{\pi}{4}\right)\right) + 2$

$y = 3\cos\left(\frac{2}{3}(\pi)\right) + 2$

$y = 3(-0.5) + 2$

$y = -1.5 + 2$

$y = 0.5$

For $x = \frac{11x}{6}$,

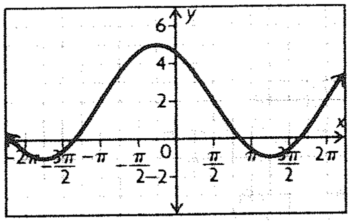
$y = 3\cos\left(\frac{2}{3}\left(\frac{11\pi}{6} + \frac{\pi}{4}\right)\right) + 2$

$y = 3\cos\left(\frac{2}{3}\left(\frac{25\pi}{12}\right)\right) + 2$

$y = 3\cos\left(\frac{25\pi}{18}\right) + 2$

$y \doteq 0.973\ 94$

3.



$x = 1.3$

4. amplitude and equation of the axis

5. a) The amplitude represents the radius of the circle in which the tip of the sparkler is moving.

b) The period represents the time it takes Mike to make one complete circle with the sparkler.

c) The equation of the axis represents the height above the ground of the centre of the circle in which the tip of the sparkler is moving.

d) A cosine function should be used because the starting point is at the highest point.

6. The amplitude of the function is 90 with the equation of the axis being $y = 30$.

$k = \frac{2\pi}{24} = \frac{\pi}{12}$

$y = 90\sin\left(\frac{\pi}{12}x\right) + 30$

7. The amplitude of the function is 250 with the equation of the axis being $y = 750$.

period = 3 seconds

$k = \frac{2\pi}{3}$

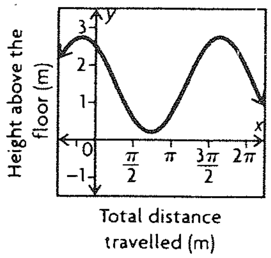
$y = 250\cos\left(\frac{2\pi}{3}x\right) + 750$

8. The amplitude of the function is 1.25 with the equation of the axis being $y = 1.5$. There is a reflection across the x-axis.

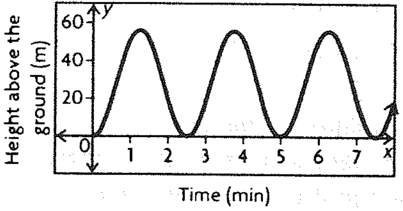
Circumference $= 2\pi(1.25) = 2.5\pi$

$k = \frac{2\pi}{2.5\pi} = \frac{4}{5}$

$y = -1.25\sin\left(\frac{4}{5}x\right) + 1.5$



9. 0.98 min < t < 1.52 min,
3.48 min < t < 4.02 min, 5.98 min < t < 6.52 min



10. a) The amplitude of the function is $\frac{15.7 - 8.3}{2} = 3.7$ with the equation of the axis being $y = \frac{8.3 + 15.7}{2}$ or $y = 12$.

period = 365 days

$$k = \frac{2\pi}{365}$$

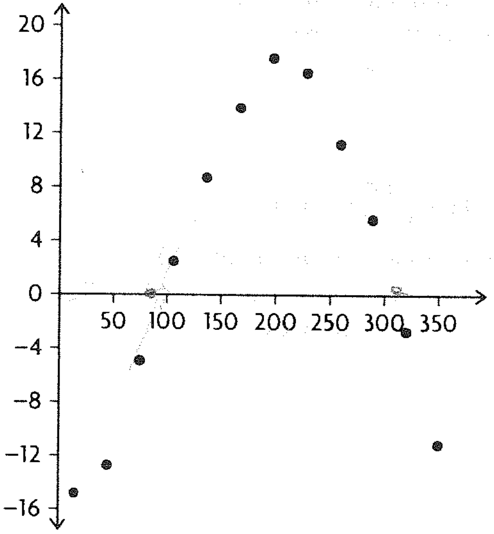
$$y = 3.7 \sin\left(\frac{2\pi}{365}x\right) + 12$$

b) For $x = 30$,

$$y = 3.7 \sin\left(\frac{2\pi}{365}(30)\right) + 12$$

$y \doteq 13.87$ hours

11.

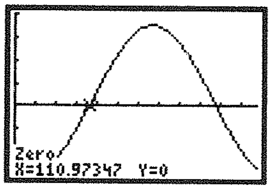


The axis is at $\frac{-14.8 + 17.6}{2} = 1.4$. The amplitude is 16.2. The period is 365 days.

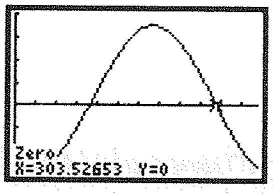
$$k = \frac{2\pi}{365}$$

$$T(t) = 16.2 \sin\left(\frac{2\pi}{365}(t - 116)\right) + 1.4$$

Graph the equation on a graphing calculator to determine when the temperature is below 0 °C.



Zero
X=110.97347 Y=0



Zero
X=303.52653 Y=0

$0 < t < 111$ and $304 < t < 365$

12. The student should graph the height of the nail above the ground as a function of the total distance travelled by the nail, because the nail would not be travelling at a constant speed. If the student graphed the height of the nail above the ground as a function of time, the graph would not be sinusoidal.

13. The axis is at 3 m = 300 cm. The amplitudes of the minute hand and the second hand is 15 and of the hour hand is 8.

The period of the minute hand is 60.

$$k = \frac{2\pi}{60} = \frac{\pi}{30}$$

minute hand: $D(t) = 15 \cos\left(\frac{\pi}{30}t\right) + 300$;

The period of the second hand is 1.

$$k = \frac{2\pi}{1} = 2\pi$$

second hand: $D(t) = 15 \cos(2\pi t) + 300$;

The period of the hour hand is 720.

$$k = \frac{2\pi}{720} = \frac{\pi}{360}$$

hour hand: $D(t) = 8 \cos\left(\frac{\pi}{360}\text{t}\right) + 300$

6.7 Rates of Change in Trigonometric Functions, pp. 369–373

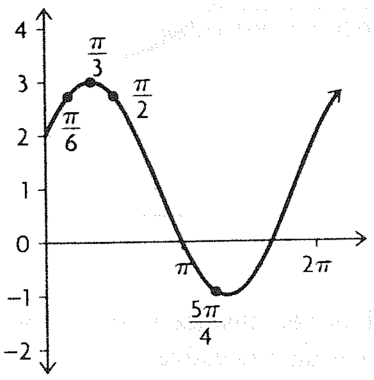
1. a) The average rate of change is zero in the intervals of $0 < x < \pi$, $\pi < x < 2\pi$.
b) The average rate of change is negative in the intervals of $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $\frac{3\pi}{2} < x < \frac{5\pi}{2}$.
c) The average rate of change is positive in the intervals of $\frac{\pi}{2} < x < \frac{3\pi}{2}$, $\frac{5\pi}{2} < x < 3\pi$.

2. a) Two points where the instantaneous rate of change is zero are $x = \frac{\pi}{4}$, $x = \frac{5\pi}{4}$.
b) Two points where the instantaneous rate of change is a negative value are $x = \frac{\pi}{2}$, $x = \frac{5\pi}{2}$.
c) Two points where the instantaneous rate of change is a positive value are $x = 0$, $x = 2\pi$.

3. Average rate of change for the interval $2 \le x \le 5$:

$$\left|\frac{0-0}{5-2}\right| = \left|\frac{0}{3}\right| = 0$$

4.



a) $0 \le x \le \frac{\pi}{2}$

$$\text{Average rate of change} = \frac{2.73 - 2}{\frac{\pi}{2} - 0} = \frac{0.73}{1.57} \doteq 0.465$$

b) $\frac{\pi}{6} \le x \le \frac{\pi}{2}$

$$\text{Average rate of change} = \frac{2.73 - 2.73}{\frac{\pi}{2} - \frac{\pi}{6}} = \frac{0}{1.047} = 0$$

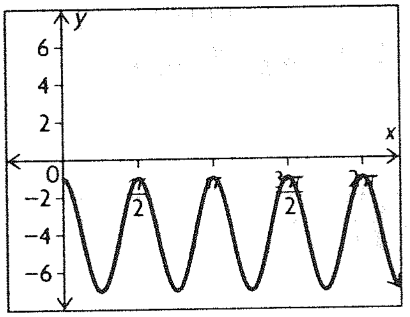
c) $\frac{\pi}{3} \le x \le \frac{\pi}{2}$

$$\text{Average rate of change} = \frac{2.73 - 3}{\frac{\pi}{2} - \frac{\pi}{3}} = \frac{-0.27}{0.5236} \doteq -0.5157$$

d) $\frac{\pi}{2} \le x \le \frac{5\pi}{4}$

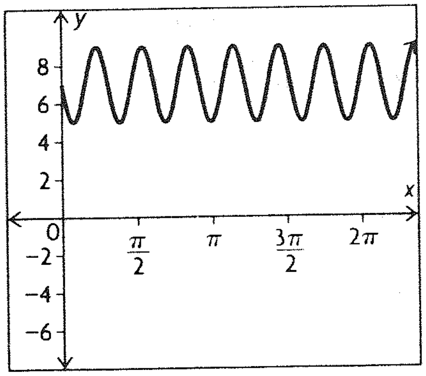
$$\text{Average rate of change} = \frac{-0.932 - 2.73}{\frac{5\pi}{4} - \frac{\pi}{2}} = \frac{-3.662}{2.356} \doteq -1.554$$

5.



a) The average rate of change is zero in the intervals of $0 < x < \frac{\pi}{2}$, $\pi < x < \frac{3\pi}{2}$.
b) The average rate of change is negative in the intervals of $0 < x < \frac{\pi}{4}$, $\pi < x < \frac{5\pi}{4}$.
c) The average rate of change is positive in the intervals of $\frac{\pi}{4} < x < \frac{\pi}{2}$, $\frac{5\pi}{4} < x < \frac{3\pi}{2}$.

6.



a) Two points where the instantaneous rate of change is zero are $x = \frac{1}{4}$, $x = \frac{3}{4}$.
b) Two points where the instantaneous rate of change is a negative value are $x = 0$, $x = 1$.
c) Two points where the instantaneous rate of change is a positive value are $x = \frac{1}{2}$, $x = \frac{3}{2}$.

7. a) $y = 6\cos(3x) + 2$ for $\frac{\pi}{4} \le x \le \pi$

For $x = \frac{\pi}{4}$,

$$y = 6\cos\left(3\left(\frac{\pi}{4}\right)\right) + 2$$

$y \doteq -2.2426$

For $x = \pi$,

$y = 6\cos(3(\pi)) + 2$

$y = -4$

$$\text{Average rate of change} = \frac{-2.2426 - (-4)}{\frac{\pi}{4} - \pi} = \frac{1.7574}{-2.3562} \doteq -0.7459$$

b) $y = -5\sin\left(\frac{1}{2}x\right) - 9$ for $\frac{\pi}{4} \le x \le \pi$

For $x = \frac{\pi}{4}$,

$$y = -5\sin\left(\frac{1}{2}\left(\frac{\pi}{4}\right)\right) - 9$$

$y \doteq -10.9134$

For $x = \pi$,

$$y = -5\sin\left(\frac{1}{2}\pi\right) - 9$$

$y = -14$

$$\text{Average rate of change} = \frac{-10.9134 - (-14)}{\frac{\pi}{4} - \pi} = \frac{3.0866}{-2.3562} \doteq -1.310$$

c) $y = \frac{1}{4}\cos(8x) + 6$ for $\frac{\pi}{4} \le x \le \pi$

For $x = \frac{\pi}{4}$,

$$y = \frac{1}{4}\cos\left(8\left(\frac{\pi}{4}\right)\right) + 6$$

$y = 6.25$

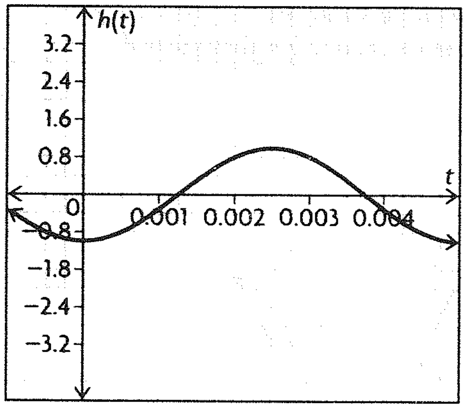
For $x = \pi$,

$$y = \frac{1}{4}\cos(8\pi) + 6$$

$y = 6.25$

$$\text{Average rate of change} = \frac{6.25 - 6.25}{\frac{\pi}{4} - \pi} = \frac{0}{-2.3562} = 0$$

8. The tip is at its minimum height at $t = 0$. Normally the sine function is 0 at 0, so the function in this case is translated to the right by $\frac{1}{4}$ of its period. The propeller makes 200 revolutions per second, so the period is $\frac{1}{200}$. The amplitude of the function is the length of the propeller, which is positive. Assume that it is 1 m for this exercise. Then the function that describes the height of the tip of the propeller is $h(t) = \sin(400\pi(t - \frac{1}{800}))$. The graph of this function is shown below.



$$t = \frac{1}{300} \doteq 0.0033$$

From the graph, it is clear that the instantaneous rate of change at $t = \frac{1}{300}$ is negative.

9. a) The axis is at 20.2. So, the equation of the axis is $y = 20.2$ and the amplitude is 4.5.

$$k = \frac{2\pi}{24} = \frac{\pi}{12}$$

$$R(t) = 4.5\cos\left(\frac{\pi}{12}t\right) + 20.2$$

b) fastest: $t = 6$ months, $t = 18$ months, $t = 30$ months, $t = 42$ months; slowest: $t = 0$ months, $t = 12$ months, $t = 24$ months, $t = 36$ months, $t = 48$ months

c) For $5 \le t \le 7$

$t = 5$

$$R(5) = 4.5\cos\left(\frac{\pi}{12}(5)\right) + 20.2$$

$R(5) \doteq 21.364$

$t = 7$

$R(7) = 4.5\cos\left(\frac{\pi}{12}(7)\right) + 20.2$

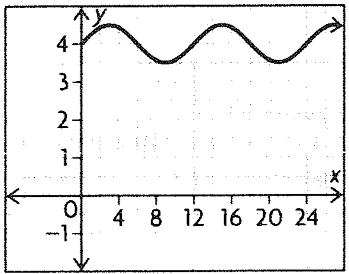
$R(7) \doteq 19.035$

Use the points (5, 21.364) and (7, 19.035).

$\frac{19.035 - 21.364}{7 - 5} \doteq \frac{-2.329}{2} \doteq -1.164$

$\doteq 1.164$ mice per owl/s

10. a)



The instantaneous rate of change appears to be at its greatest at 12 hours.

i) For $11 \le t \le 13$

$t = 11$

$y = 0.5\sin\left(\frac{\pi}{6}(11)\right) + 4$

$y = 3.75$

$t = 13$

$y = 0.5\sin\left(\frac{\pi}{6}(13)\right) + 4$

$y = 4.25$

Use the points (11, 3.75) and (13, 4.25).

$\frac{4.25 - 3.75}{13 - 11} = \frac{0.5}{2} = 0.25$ t/h

ii) For $11.5 \le t \le 12.5$

$t = 11.5$

$y = 0.5\sin\left(\frac{\pi}{6}(11.5)\right) + 4$

$y \doteq 3.8706$

$t = 12.5$

$y = 0.5\sin\left(\frac{\pi}{6}(12.5)\right) + 4$

$y \doteq 4.1294$

Use the points (11.5, 3.8706) and (12.5, 4.1294).

$\frac{4.1294 - 3.8706}{12.5 - 11.5} \doteq 0.2588$ t/h

iii) For $11.75 \le t \le 12.25$

$t = 11.75$

$y = 0.5\sin\left(\frac{\pi}{6}(11.75)\right) + 4$

$y \doteq 3.9347$

$t = 12.25$

$y = 0.5\sin\left(\frac{\pi}{6}(12.25)\right) + 4$

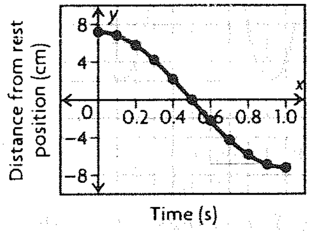
$y \doteq 4.0653$

Use the points (11.75, 3.9347 and (12.25, 4.0653).

$\frac{4.0653 - 3.9347}{12.25 - 11.75} = \frac{0.1306}{0.5} = 0.2612$ t/h

b) The estimate calculated in part iii) is the most accurate. The smaller the interval, the more accurate the estimate.

11. a)



b) half of one cycle

c) $\frac{-7.2 - 7.2}{1 - 0} = -14.4$ cm/s

d) The bob is moving the fastest when it passes through its rest position. You can tell because the images of the balls are farthest apart at this point.

e) The pendulum's rest position is halfway between the maximum and minimum values on the graph. Therefore, at this point, the pendulum's instantaneous rate of change is at its maximum.

12. $h(t) = \sin\left(\frac{\pi}{5}t\right)$

a) For $0 \le t \le 5$,

$h(0) = \sin\left(\frac{\pi}{5}(0)\right) = 0$

$h(5) = \sin\left(\frac{\pi}{5}(5)\right) = 0$

$\frac{0 - 0}{5 - 0} = 0$

b) For $5.5 \le t \le 6.5$

$t = 5.5$

$h(5.5) = \sin\left(\frac{\pi}{5}(5.5)\right)$

$h(5.5) = -0.309$

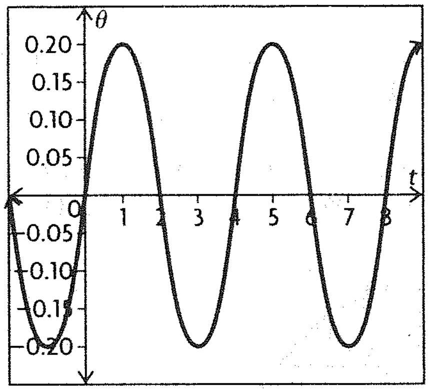
$h(6.5) = \sin\left(\frac{\pi}{5}(6.5)\right)$

$h(6.5) \doteq -0.809$

Use the points (5.5, −0.309) and (6.5, −0.809).

$$\frac{-0.809 - (-0.309)}{6.5 - 5.5} = \frac{-0.5}{1} = -0.5 \text{ m/s}$$

13. a)



b) When $t = 0$, $\theta = 0$. When $t = 1$, $\theta = 0.2$. The average rate of change is 0.2 radians/s.

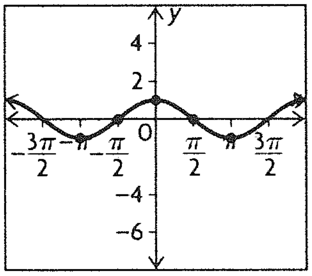
c) Answers may vary. For example, from the graph, it appears that the instantaneous rate of at $t = 1.5$ is about $-\frac{2}{3}$ radians/s.

d) The pendulum speed seems to be the greatest for $t = 0, 2, 4, 6$, and 8.

14. Answers may vary. For example, for $x = 0$, the instantaneous rate of change of $f(x) = \sin x$ is approximately 0.9003, while the instantaneous rate of change of $f(x) = 3 \sin x$ is approximately 2.7009. (The interval $-\frac{\pi}{4} < x < \frac{\pi}{4}$ was used.) Therefore, the instantaneous rate of change of $f(x) = 3 \sin x$ is at its maximum three times more than the instantaneous rate of change of $f(x) = \sin x$. However, there are points where the instantaneous rate of change is the same for the two functions. For example, at $x = \frac{\pi}{2}$, it is 0 for both functions.

15. a) By examining the graph of $f(x) = \sin x$, it appears that the instantaneous rate of change at the given values of x are −1, 0, 1, 0, and −1.

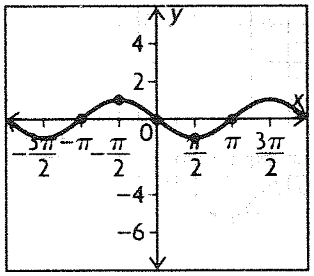
b)



The function is $f(x) = \cos x$. Based on this information, the derivative of $f(x) = \sin x$ is $\cos x$.

16. a) By examining the graph of $f(x) = \cos x$, it appears that the instantaneous rate of change at the given values of x are 0, 1, 0, −1, and 0.

b)



The function is $f(x) = -\sin x$. Based on this information, the derivative of $f(x) = \cos x$ is $-\sin x$.

Chapter Review, pp. 376–377

1. Circumference $= 2\pi r = 2\pi(16) = 32\pi$

$$\frac{33}{32\pi} = \frac{x}{2\pi}$$
$$(32\pi)(x) = (33)(2\pi)$$
$$(32\pi)(x) = 66\pi$$
$$x = \frac{66\pi}{32\pi}$$
$$x = \frac{33}{16}$$

2. Circumference $= 2\pi r = 2\pi(75) = 150\pi$

$$\frac{x}{150\pi} = \frac{\frac{14\pi}{15}}{2\pi}$$
$$(2\pi)(x) = (150\pi)\left(\frac{14\pi}{15}\right)$$
$$(2\pi)(x) = 140\pi^2$$
$$x = 70\pi$$

3. a) $20° = 20° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{\pi}{9}$ radians

b) $-50° = -50° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{-5\pi}{18}$ radians

c) $160° = 160° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{8\pi}{9}$ radians

d) $420° = 420° \times \left(\frac{\pi \text{ radians}}{180°}\right) = \frac{7\pi}{3}$ radians

4. a) $\frac{\pi}{4}$ radians;

$$\frac{\cancel{\pi}}{4} \cancel{\text{radians}} \times \left(\frac{180°}{\cancel{\pi \text{ radians}}}\right) = 45°$$

b) $-\frac{5\pi}{4}$ radians;

$$-\frac{5\pi}{4}\ \cancel{\text{radians}} \times \left(\frac{180^\circ}{\cancel{\pi\ \text{radians}}}\right) = -225^\circ$$

c) $\frac{8\pi}{3}$ radians;

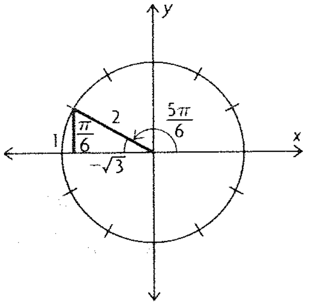
$$\frac{8\cancel{\pi}}{3}\ \cancel{\text{radians}} \times \left(\frac{180^\circ}{\cancel{\pi\ \text{radians}}}\right) = 480^\circ$$

d) $-\frac{2\pi}{3}$ radians;

$$-\frac{2\cancel{\pi}}{3}\ \cancel{\text{radians}} \times \left(\frac{180^\circ}{\cancel{\pi\ \text{radians}}}\right) = -120^\circ$$

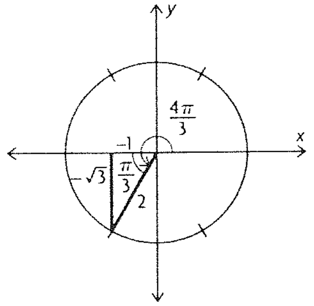
5. The functions must be located in the second or third quadrant, since $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$.

a)



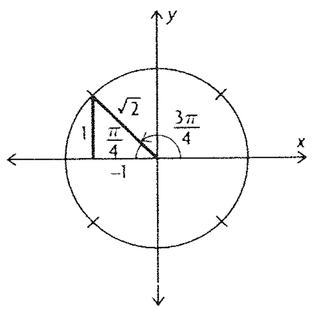
$$\sin^{-1}\frac{1}{2} = \frac{5\pi}{6}$$

b)



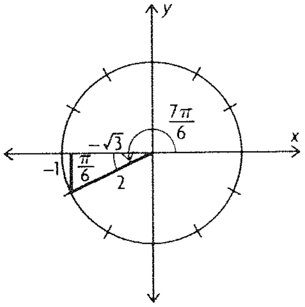
$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3}$$

c) $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$



$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

d)



$$\sin^{-1}\left(-\frac{1}{2}\right) = \frac{7\pi}{6}$$

6. $\cos\theta = \frac{x}{r} = \frac{-5}{13}$

$$13^2 = (-5)^2 + y^2$$
$$169 - 25 = y^2$$
$$144 = y^2$$
$$12 = y$$

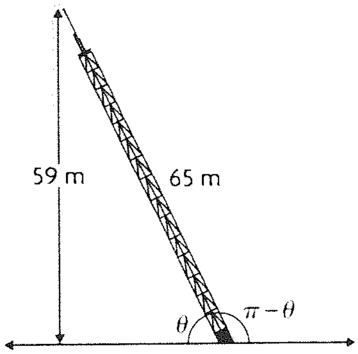
a) $\tan\theta = \frac{y}{x} = \frac{12}{13}$

b) $\sec\theta = \frac{r}{x} = -\frac{13}{5}$

c) Cosine is negative in the second and third quadrants.

$$\cos^{-1}\frac{-5}{13} \doteq 2.0 \text{ and } \pi + 2.0 \doteq 5.14$$

7.



$$\sin\theta = \frac{59}{65}$$

$$\theta = \sin^{-1}\frac{59}{65} = 1.14$$

$\pi - \theta = 2.00$

8. a) 2π radians

b) 2π radians

c) π radians

9. The axis is 2 and the amplitude is 5. It is shifted to the left $\frac{\pi}{3}$.

$$y = 5\sin\left(x + \frac{\pi}{3}\right) + 2$$

10. The axis is -1 and the amplitude is 3. It is shifted $\frac{\pi}{4}$ units to the left. It is reflected in the x-axis. The period is π.

$$k = \frac{2\pi}{\pi} = 2$$

$$y = -3\cos\left(2\left(x + \frac{\pi}{4}\right)\right) - 1$$

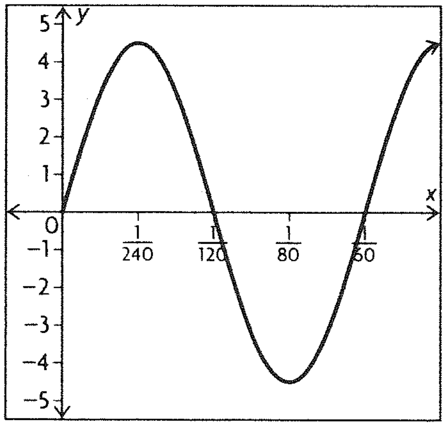
11. a) reflection in the x-axis, vertical stretch by a factor of 19, vertical translation 9 units down

b) horizontal compression by a factor of $\frac{1}{10}$, horizontal translation $\frac{\pi}{12}$ to the left

c) vertical compression by a factor of $\frac{10}{11}$, horizontal translation $\frac{\pi}{9}$ to the right, vertical translation 3 units up

d) reflection in the x-axis, reflection in the y-axis, horizontal translation π to the right

12. a)



b) period: $\frac{2\pi}{x} = \frac{120\pi}{1}$

$$(120\pi)(x) = 2\pi$$

$$x = \frac{2\pi}{120\pi}$$

$$x = \frac{1}{60}$$

c) The maximum occurs at $\left(\frac{1}{4}\right)\left(\frac{1}{60}\right) = \frac{1}{240}$.

d) The minimum occurs at $\left(\frac{3}{4}\right)\left(\frac{1}{60}\right) = \frac{3}{240} = \frac{1}{80}$.

13. a) 2π radians

b) 2π radians

c) π radians

14. a) The amplitude represents the radius of the circle in which the bumblebee is flying.

b) The period represents the time that the bumblebee takes to fly one complete circle.

c) The equation of the axis represents the height, above the ground, of the centre of the circle in which the bumblebee is flying.

d) Since the bumblebee is at it lowest point at $t = 0$, the cosine function should be used to represent the function.

15. The axis is at $\frac{15\,000 + 500}{2} = 7750$. So, the amplitude is $7750 - 500 = 7250$. The period is 12 months.

$$k = \frac{2\pi}{12} = \frac{\pi}{6}$$

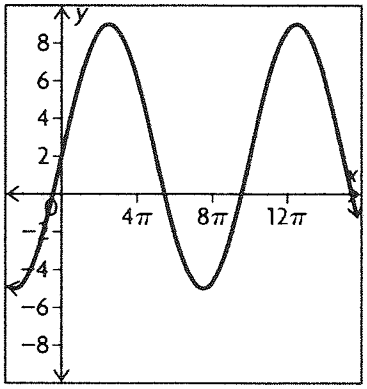
$$P(m) = 7250\cos\left(\frac{\pi}{6}m\right) + 7750$$

16. The axis is at $\frac{180 + 120}{2} = 150$. So, the amplitude is $150 - 120 = 30$. It is shifted to the right $\frac{\pi}{2}$. The period is 1.2 seconds.

$$k = \frac{2\pi}{1.2} = \frac{5\pi}{3}$$

$$h(t) = 30\sin\left(\frac{5\pi}{3}t - \frac{\pi}{2}\right) + 150$$

17.

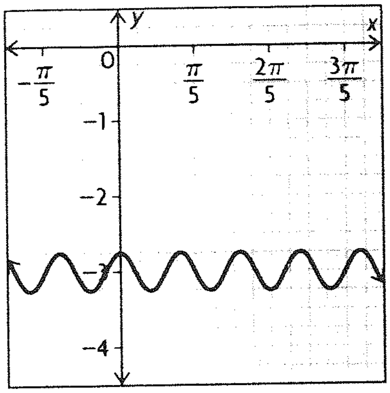


a) $0 < x < 5\pi$, $10\pi < x < 15\pi$

b) $2.5\pi < x < 7.5\pi$, $12.5\pi < x < 17.5\pi$

c) $0 < x < 2.5\pi$, $7.5\pi < x < 12.5\pi$

18.



a) $x = 0, x = \frac{1}{2}$

b) $x = \frac{1}{8}, x = \frac{5}{8}$

c) $x = \frac{3}{8}, x = \frac{7}{8}$

19. a) period: $\frac{2\pi}{x} = \frac{8\pi}{3}$

$$x = \frac{3}{4}\text{ s}$$

b) The period represents the time between one beat of a person's heart and the next beat.

c) For $t = 0.2$,

$$P(0.2) = 100 - 20\cos\left(\frac{8\pi}{3}(0.2)\right)$$

$P(0.2) = 102$

For $t = 0.3$,

$$P(0.3) = 100 - 20\cos\left(\frac{8\pi}{3}(0.3)\right)$$

$$P(0.3) \doteq 116$$

$$\frac{116 - 102}{0.3 - 0.2} = \frac{14}{0.1} = 140$$

d) For $t = 0.4$,

$$P(0.4) = 100 - 20\cos\left(\frac{8\pi}{3}(0.4)\right)$$

$P(0.4) \doteq 119.6$

For $t = 0.6$,

$$P(0.6) = 100 - 20\cos\left(\frac{8\pi}{3}(0.6)\right)$$

$$P(0.6) \doteq 93.8$$

$$\frac{93.8 - 119.6}{0.6 - 0.4} = \frac{-25.8}{0.2} = -129$$

Chapter Self-Test, p. 378

1. $y = \sec x$

2. $\sin\frac{3\pi}{2} = -1$

$\cos\pi = -1$

$\tan\frac{7\pi}{4} = -1$

$\csc\frac{3\pi}{2} = -1$

$\sec 2\pi = 1$

$\cot\frac{3\pi}{4} = -1$

$\sec 2\pi$ has a different value.

3. $y = -12\cos\left(\frac{5}{3}\left(x + \frac{\pi}{6}\right)\right) + 100$

$y = -12\cos\left(\frac{5}{3}\left(\frac{5\pi}{4} + \frac{\pi}{6}\right)\right) + 100$

$y \doteq 108.5$

4. For $d = 52$ (Feb 21)

$T(52) = -20\cos\left(\frac{2\pi}{365}(52 - 10)\right) + 25$

$T(52) \doteq 10.0$

For $d = 128$ (May 8),

$T(128) = -20\cos\left(\frac{2\pi}{365}(128 - 10)\right) + 25$

$T(128) \doteq 33.9$

$\frac{33.9 - 10.0}{128 - 52} = \frac{23.9}{76} \doteq 0.31$ °C per day

5. $\frac{5\pi}{8}$ radians;

$\frac{5\cancel{\pi}}{8}\ \cancel{\text{radians}} \times \left(\frac{180^\circ}{\cancel{\pi\text{ radians}}}\right) = 112.5^\circ$

$\frac{2\pi}{3}$ radians;

$\frac{2\cancel{\pi}}{3}\ \cancel{\text{radians}} \times \left(\frac{180^\circ}{\cancel{\pi\text{ radians}}}\right) = 120^\circ$

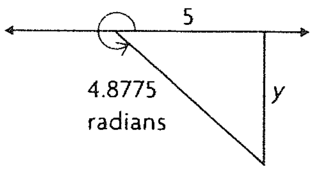
$\frac{3\pi}{5}$ radians;

$\frac{3\cancel{\pi}}{5}\ \cancel{\text{radians}} \times \left(\frac{180^\circ}{\cancel{\pi\text{ radians}}}\right) = 108^\circ$

So, from smallest to largest, the angles are $\frac{3\pi}{5}$, 110°, $\frac{5\pi}{8}$, 113°, and $\frac{2\pi}{3}$.

6. $y = \sin\left(x + \frac{5\pi}{8}\right)$

7.



$2\pi - 4.8755 = 1.4077$

$\tan(1.4077) = \frac{y}{5}$

$y = 5\tan(1.4077)$

$y \doteq -30$

8. a) $-3\cos\left(\frac{\pi}{12}x\right) + 22$

b) For $t = 0$ (sunrise)

$T(0) = -3\cos\left(\frac{\pi}{12}(0)\right) + 22$

$T(0) = 19$

For $t = 6$,

$T(6) = -3\cos\left(\frac{\pi}{12}(6)\right) + 22$

$T(6) = 22$

$\frac{22 - 19}{6 - 0} = \frac{3}{6} \doteq 0.5$ °C per hour

c) $11 \le t \le 13$

For $t = 11$ (5 p.m.)

$T(11) = -3\cos\left(\frac{\pi}{12}(11)\right) + 22$

$T(11) = 24.9$

For $t = 13$ (7 p.m.),

$T(13) = -3\cos\left(\frac{\pi}{12}(13)\right) + 22$

$T(13) = 24.9$

$\frac{24.9 - 24.9}{13 - 11} = \frac{0}{2} \doteq 0$ °C per hour

Chapters 4–6 Cumulative Review pp. 380–383

1.

$$
\begin{aligned}
x^4 + 3x^3 &= 4x^2 + 12x \\
x^4 + 3x^3 - 4x^2 - 12x &= 0 \\
(x^4 + 3x^3) + (-4x^2 - 12x) &= 0 \\
x^3(x + 3) - 4x(x + 3) &= 0 \\
(x + 3)(x^3 - 4x) &= 0 \\
x(x + 3)(x^2 - 4) &= 0 \\
x(x + 3)(x + 2)(x - 2) &= 0 \\
x &= 0, -3, -2, 2
\end{aligned}
$$

The correct answer is **d)**.

2. $f(x) = a(x + 1)(x - 1)(x - 4)$

$36 = a(2 + 1)(2 - 1)(2 - 4)$

$36 = a(3)(1)(-2)$

$36 = -6a$

$-6 = a$

So, the equation is

$$
\begin{aligned}
f(x) &= -6(x + 1)(x - 1)(x - 4) \\
&= -6(x^2 - 1)(x - 4) \\
&= -6(x^3 - 4x^2 - x + 4) \\
&= -6x^3 + 24x^2 + 6x - 24
\end{aligned}
$$

The correct answer is **b)**.

3. $2 - 3x < x - 5$

$2 - 4x < -5$

$-4x < -7$

$x > \frac{7}{4}$

-2 is not greater than $\frac{7}{4}$.

The correct answer is **a)**.

4. $-10 \le 3x + 5 \le 8$

$-15 \le 3x \le 3$

$-5 \le x \le 1$

The correct answer is **c)**.

5. Using the graph $f(x) < g(x)$ from $x = 2$ to ∞.

So, $x > 2$.

The correct answer is **a)**.

6. $h(t) = -5t^2 + 3.5t + 10$

$-5t^2 + 3.5t + 10 > 10$

$-5t^2 + 3.5t > 0$

$t(-5t + 3.5) > 0$

The critical points are $t = 0$ and $t = 0.7$.

Test -1: $(-1)(-5(-1) + 3.5) > 0$

$(-1)(5 + 3.5) > 0$ FALSE

Test 0.5: $(0.5)(-5(0.5) + 3.5) > 0$

$(0.5)(-2.5 + 3.5) > 0$ TRUE

Test 1: $(1)(-5(1) + 3.5) > 0$

$(1)(-5 + 3.5) > 0$ FALSE

So, the solution is $t \in (0, 0.7)$.

The correct answer is **b)**.

7. From the given information, the function must have a maximum at $x = 0$ and a minimum at $x = 2$. Choice **a)** is not a possible set of zeros for the function. If the function has zeros only at $x = 0$ and $x = 1$, then the function has to increase to $(0, 0)$ and turn there and begin to decrease. Sometime before $x = 1$, the function must turn again and increase to get back to the x-axis before $x = 1$. But this contradicts the given information that the function is decreasing for $0 < x < 2$.

The correct answer is **a)**.

8. $f(x) = 2x^3 - 4x^2 + 6x$

$$f(-0.05) = 2(-0.05)^3 - 4(-0.05)^2 + 6(-0.05)$$
$$= -0.310\,25$$

Find the rate of change using the points $(0, 0)$ and $(-0.05, -0.310\,25)$.

$$m = (-0.310\,25 - 0) \div (-0.05 - 0)$$
$$= 6.2$$

So, the correct answer is **c)**.

9. The graph has vertical asymptotes at $x^2 - 3x = 0$

$x(x - 3) = 0$

$x = 0$ and $x = 3$

Test $x = -1$: The function will be $(+) \div (-)(-) = (+)$

So, the function is positive to the left of zero.

Test $x = 1$: The function will be $(+) \div (+)(-) = (-)$

So, the function is negative between 0 and 3.

Test $x = 4$: The function will be $(+) \div (+)(+) = (+)$

So, the function is positive to the right of 3.

The function that matches this is **c)**.

10. The function will have a horizontal asymptote of 0 and vertical asymptotes of $x = -5$ and $x = 2$. So the correct answer is **c)**.

11. To have an oblique asymptote, the degree of the numerator must be greater than the degree of the denominator by exactly 1. Neither choice **a)** nor choice **c)** meets this condition. While it seems that choice **b)** meets the condition, it does not because the numerator can be factored and the function simplified. $g(x) = \frac{(x + 3)(x - 3)}{(x - 3)} = x + 3.$

The correct answer is **d)**.

12. Only functions a and b are undefined at $x = 3$. Examine the behaviour of these functions for $-2 < x < 3$.

For choice a: Test $x = 0$: $(+) \div (+) = (+)$

For choice b: Test $x = 0$: $(+) \div (-) = (-)$

So, choice a is positive for $-2 < x < 3$.

The correct answer is **a)**.

13. Choose a value that is very close to and to the left of $\frac{3}{5}$.

Try 0.599:

$(2 - 3(0.599)) \div (5(0.599) - 3) = -40.6$

So, the function approaches $-\infty$.

The correct answer is **d)**.

14. $\frac{3 - 2x}{x + 2} = 3x$

$$3 - 2x = 3x(x + 2)$$
$$3 - 2x = 3x^2 + 6x$$
$$0 = 3x^2 + 8x - 3$$
$$0 = (3x - 1)(x + 3)$$
$$x = \frac{1}{3} \text{ and } -3$$

The correct answer is **c)**.

15. Any of the steps listed can be used to begin solving the rational equation.

The correct answer is **d)**.

16. $2x - 3 \le \frac{2}{x}$

$$2x - 3 - \frac{2}{x} \le 0$$
$$\frac{2x^2}{x} - \frac{3x}{x} - \frac{2}{x} \le 0$$
$$\frac{2x^2 - 3x - 2}{x} \le 0$$
$$\frac{(2x + 1)(x - 2)}{x} \le 0$$

The correct answer is **a)**.

17. $x - 3 > \frac{6}{x - 2}$

$$x - 3 - \frac{6}{x - 2} > 0$$
$$\frac{x(x - 2)}{x - 2} - \frac{3(x - 2)}{x - 2} - \frac{6}{x - 2} > 0$$
$$\frac{x^2 - 2x - 3x + 6 - 6}{x - 2} > 0$$
$$\frac{x^2 - 5x}{x - 2} > 0$$
$$\frac{x(x - 5)}{x - 2} > 0$$

The critical points are 0, 5 and 2.

Test -1: $(-)(-) \div (-) = (-)$ FALSE

Test 1: $(+)(-) \div (-) = (+)$ TRUE

Test 3: $(+)(-) \div (+) = (-)$ FALSE

Test 6: $(+)(+) \div (+) = (+)$ TRUE

So, the solution is $(0, 2)$ and $(5, \infty)$.

The correct answer is **d)**.

18. Let $y = f(x)$.

$$\frac{G(a + h) - G(a)}{h} = \frac{G(1.001) - G(1)}{0.001}$$
$$= \frac{0.9985 - 1}{0.001}$$
$$\doteq -1.5$$

So, the correct answer is **b)**.

19. $\frac{s(a+h)-s(a)}{h} = \frac{s(3.001)-s(3)}{0.001}$
$= \frac{-7.009\,009-(-7)}{0.001}$
$\doteq -9$

So, the correct answer is **b)**.

20. $P = 3\left(\frac{5\pi}{12}\right) + 2(3)$
$= \frac{5\pi}{4} + 6$ m

The correct answer is **b)**.

21. $20\left(\frac{\pi}{180}\right) = \frac{\pi}{9}$
$135\left(\frac{\pi}{180}\right) = \frac{3\pi}{4}$
$-270\left(\frac{\pi}{180}\right) = -\frac{3\pi}{2}$

Each of the pairs of angles are equivalent.
The correct answer is **d)**.

22. $\tan^{-1}\left(\frac{7}{4}\right) \doteq 1.0517$
$\pi - 1.0517 \doteq 2.09$

The correct answer is **c)**.

23. Since $\sin\theta = -\frac{\sqrt{3}}{2}$, the value for r is 2 and the value for $a = \sqrt{3}$.
So, find b.
$(\sqrt{3})^2 + b^2 = 2^2$
$3 + b^2 = 4$
$b^2 = 1$
$b = 1$

The angle is in either in the third or fourth quadrant since the sine is negative.
So, $\cos\theta = \frac{1}{2}$ or $-\frac{1}{2}$.
$\tan\theta = \sqrt{3}$ when $\cos\theta = -\frac{1}{2}$
or $-\sqrt{3}$ when $\cos\theta = \frac{1}{2}$

The correct answer is **a)**.

24. $x = \sin^{-1} 0.5$
$x = \frac{\pi}{6}$ and $\frac{5\pi}{6}$

The correct answer is **d)**.

25. The graph shown has been stretched vertically by a factor of 3, since the minimum is -4 and the maximum is 2, a difference of 6. The graph has also been compressed horizontally by a factor of $\frac{1}{2}$, as the period shown is only π. The graph has been translated down 1 unit. The equation for the graph shown is $y = 3\sin(2x) - 1$.
The correct answer is **b)**.

26. The equation shows a horizontal stretch by a factor of 3 and a horizontal translation of 2π units to the left.
The correct answer is **d)**.

27. The maximum height represented by choice **b)** is $41 - 5 = 36$, not 41 as required.
The maximum height represented by choice **c)** is $18 - 23 = -5$, not 41 as required.
The maximum height represented by choice **d)** is $41 - 36 = 5$, not 41 as required.
The maximum height represented by choice **a)** is $18 + 23 = 41$, as required. The minimum height represented by choice **a)** is $-18 + 23 = 5$, as required.
The correct answer is **a)**.

28. Using a graphing calculator, the function is decreasing in both intervals which means the instantaneous rate of change is negative in both intervals.
The correct answer is **c)**.

29. $P(0) = 23.7\cos\left(\frac{\pi}{6}(0-7) + 24.1\right)$
$\doteq 3.575$

$P(4) = 23.7\cos\left(\frac{\pi}{6}(4-7) + 24.1\right)$
$\doteq 24.1$

So, the rate of change is $(24.1 - 3.575) \div (4 - 0)$ or 5.131 25.

$P(1) = 23.7\cos\left(\frac{\pi}{6}(1-7)\right) + 24.1$
$\doteq 0.4$

$P(7) = 23.7\cos\left(\frac{\pi}{6}(0)\right) + 24.1$
$\doteq 47.8$

So, the rate of change is $(47.8 - 0.4) \div (7 - 1)$ or 7.9.

$P(16) = 23.7\cos\left(\frac{\pi}{6}(16-7)\right) + 24.1$
$\doteq 24.1$

So, the rate of change is $(24.1 - 47.8) \div (16 - 7)$ or -2.63.

$P(10) = 23.7\cos\left(\frac{\pi}{6}(10-7)\right) + 24.1$
$\doteq 24.1$

$P(18) = 23.7\cos\left(\frac{\pi}{6}(18 - 7)\right) + 24.1$

$\doteq 44.62$

So, the rate of change is $(44.62 - 24.1) \div (18 - 10)$ or 2.565.

The correct answer is **b)**.

30. a) length $= 50 - 2x$

width $= 40 - 2x$

height $= x$

$V = lwh$

$= (50 - 2x)(40 - 2x)x$

b) $6000 = (50 - 2x)(40 - 2x)x$

$6000 = (2000 - 80x - 100x + 4x^2)x$

$6000 = 2000x - 180x^2 + 4x^3$

$0 = 4x^3 - 180x^2 + 2000x - 6000$

$0 = x^3 - 45x^2 + 500x - 1500$

The possible reasonable solutions are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 10, \pm 12, \pm 15, \pm 20$.

Use synthetic division to determine the solution.

$$\begin{array}{r|rrrr} 3 & 1 & -45 & 500 & -1500 \\ & \downarrow & 3 & -126 & 1122 \\ \hline & 1 & -42 & 374 & -378 \end{array}$$

$$\begin{array}{r|rrrr} 5 & 1 & -45 & 500 & -1500 \\ & \downarrow & 5 & -200 & 1500 \\ \hline & 1 & -40 & 300 & 0 \end{array}$$

So, $x = 5$.

Now solve $x^2 - 40x + 300 = 0$

$(x - 10)(x - 30) = 0$

So, $x = 10$ or 30, but 30 does not make sense in the context of the problem. So, $x = 5$ or 10.

c) Using a graphing calculator, find the relative maximum value. It occurs when $x \doteq 7.4$ cm.

d) Using a graphing calculator, the range is $3 < x < 12.8$.

31. a) $f(x)$: $0 = x^2 - 5x + 6$

$0 = (x - 3)(x - 2)$

$x = 3$ and $x = 2$

$g(x)$: $0 = x - 3$

$x = 3$

$\frac{f(x)}{g(x)}$: $0 = \frac{x^2 - 5x + 6}{x - 3}$

The function is undefined at $x = 3$, so the zero is $x = 2$.

$\frac{g(x)}{f(x)}$: $0 = \frac{x - 3}{x^2 - 5x + 6}$

The function is undefined at 2 and 3, so there are no zeros.

b) $\frac{f(x)}{g(x)}$: hole at $x = 3$, no asymptotes

$\frac{g(x)}{f(x)}$: hole at $x = 3$, asymptotes at $x = 2$ and $y = 0$

c) tangent at $x = 1$

$\frac{f(x)}{g(x)}$: $y = x - 2$

$\frac{g(x)}{f(x)}$: $y = -x$

32. a) Vertical compressions and stretches do not affect location of zeros; maximum and minimum values are multiplied by the scale factor, but locations are unchanged; instantaneous rates of change are multiplied by the scale factor; Horizontal compressions and stretches move locations of zeros, maximums, and minimums toward or away from the y-axis by the reciprocal of the scale factor; instantaneous rates of change are multiplied by the reciprocal of scale factor; Vertical translations change location of zeros or remove them; maximum and minimum values are increased or decreased by the amount of the translation, but locations are unchanged; instantaneous rates of change are unchanged; Horizontal translations move location of zeros by the same amount as the translation; maximum and minimum values are unchanged, but locations are moved by the same amount as the translation; instantaneous rates of change are unchanged, but locations are moved by the same amount as the translation.

b) $y = \cos x$: Vertical compressions and stretches do not affect location of zeros; maximum and minimum values are multiplied by the scale factor, but locations are unchanged; instantaneous rates of change are multiplied by the scale factor; Horizontal compressions and stretches move locations of zeros, maximums, and minimums toward or away from the y-axis by the reciprocal of the scale factor; instantaneous rates of change are multiplied by the reciprocal of scale factor; Vertical translations change location of zeros or remove them; maximum and minimum values are increased or decreased by the amount of the translation, but locations are unchanged; instantaneous rates of change are unchanged; Horizontal translations move location of zeros by the same amount as the translation; maximum and minimum values are unchanged, but locations are moved by the same amount as the translation; instantaneous rates of change are unchanged, but locations are moved by the same amount as the translation.

$y = \tan x$: Vertical compressions and stretches do not affect location of zeros; instantaneous rates of change are multiplied by the scale factor; Horizontal compressions and stretches move locations of zeros toward or away from the y-axis by the reciprocal of the scale factor; instantaneous rates of change are multiplied by the reciprocal of scale factor; Vertical translations change location of zeros or remove them; instantaneous rates of change are unchanged; Horizontal translations move location of zeros by the same amount as the translation; instantaneous rates of change are unchanged, but locations are moved by the same amount as the translation.

CHAPTER 7

Trigonometric Identities and Equations

Getting Started, p. 386

1. a) Do the following to isolate and solve for x:

$$3x - 7 = 5 - 9x$$
$$3x - 7 + 9x = 5 - 9x + 9x$$
$$12x - 7 = 5$$
$$12x - 7 + 7 = 5 + 7$$
$$12x = 12$$
$$\frac{12x}{12} = \frac{12}{12}$$
$$x = 1$$

b) Do the following to isolate and solve for x:

$$2(x + 3) - \frac{x}{4} = \frac{1}{2}$$
$$2x + 6 - \frac{x}{4} = \frac{1}{2}$$
$$2x + 6 - \frac{x}{4} - 6 = \frac{1}{2} - 6$$
$$2x - \frac{x}{4} = -\frac{11}{2}$$
$$\frac{7x}{4} = -\frac{11}{2}$$
$$\frac{7x}{4} \times \frac{4}{7} = -\frac{11}{2} \times \frac{4}{7}$$
$$x = -\frac{44}{14}$$
$$x = -\frac{22}{7}$$

c) Factor the left side of the equation and solve for x as follows:

$$x^2 - 5x - 24 = 0$$
$$(x - 8)(x + 3) = 0$$
$$x - 8 = 0 \text{ or } x + 3 = 0$$
$$x - 8 + 8 = 0 + 8 \text{ or } x + 3 - 3 = 0 - 3$$
$$x = 8 \text{ or } x = -3$$

d) Move all terms to the left side of the equation, factor, and solve for x as follows:

$$6x^2 + 11x = 10$$
$$6x^2 + 11x - 10 = 10 - 10$$
$$6x^2 + 11x - 10 = 0$$
$$(3x - 2)(2x + 5) = 0$$
$$3x - 2 = 0 \quad \text{or} \quad 2x + 5 = 0$$
$$3x \quad 2 + 2 = 0 + 2 \text{ or } 2x + 5 - 5 = 0 - 5$$
$$3x = 2 \text{ or } 2x = -5$$
$$\frac{3x}{3} = \frac{2}{3} \text{ or } \frac{2x}{2} = -\frac{5}{2}$$
$$x = \frac{2}{3} \text{ or } x = -\frac{5}{2}$$

e) Use the quadratic formula to solve for x as follows:

$$x^2 + 2x - 1 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-1)}}{2(1)}$$
$$x = \frac{-2 \pm \sqrt{4 + 4}}{2}$$
$$x = \frac{-2 \pm \sqrt{8}}{2}$$
$$x = \frac{-2 \pm 2\sqrt{2}}{2}$$
$$x = -1 \pm \sqrt{2}$$

f) Move all terms to the left side of the equation and use the quadratic formula to solve for x as follows:

$$3x^2 = 3x + 1$$
$$3x^2 - 3x - 1 = 3x + 1 - 3x - 1$$
$$3x^2 - 3x - 1 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(3)(-1)}}{2(3)}$$
$$x = \frac{3 \pm \sqrt{9 + 12}}{6}$$
$$x = \frac{3 \pm \sqrt{21}}{6}$$

2. To show that $AB = CD$, the distance formula should be used as follows:

First for AB:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$d = \sqrt{(2 - 1)^2 + \left(\frac{1}{2} - 0\right)^2}$$

$$d = \sqrt{(1)^2 + \left(\frac{1}{2}\right)^2}$$

$$d = \sqrt{1 + \frac{1}{4}}$$

$$d = \sqrt{\frac{5}{4}}$$

$$d = \frac{\sqrt{5}}{2}$$

Now for CD:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{\left(0 - \left(-\frac{1}{2}\right)\right)^2 + (6 - 5)^2}$$

$$d = \sqrt{\left(\frac{1}{2}\right)^2 + (1)^2}$$

$$d = \sqrt{\frac{1}{4} + 1}$$

$$d = \sqrt{\frac{5}{4}}$$

$$d = \frac{\sqrt{5}}{2}$$

So, $AB = CD$.

3. a) $\sin A = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{8}{17};$

$\cos A = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{15}{17};$

$\tan A = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{8}{15};$

$\csc A = \dfrac{\text{hypotenuse}}{\text{opposite side}} = \dfrac{17}{8};$

$\sec A = \dfrac{\text{hypotenuse}}{\text{adjacent side}} = \dfrac{17}{15};$

$\cot A = \dfrac{\text{adjacent side}}{\text{opposite side}} = \dfrac{15}{8}$

b) To determine the measure of $\angle A$ in radians, any of the ratios found in part (a) can be used. In this case, $\sin A$ will be used:

$$\sin A = \frac{8}{17}$$

$$\sin^{-1}(\sin A) = \sin^{-1}\frac{8}{17}$$

$$m\angle A = \sin^{-1}\frac{8}{17}$$

$$m\angle A = 0.5 \text{ radians}$$

c) From the figure:

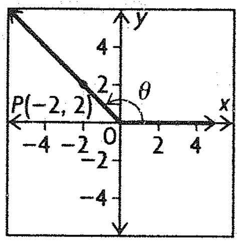
$$\sin B = \frac{15}{17}$$

$$\sin^{-1}(\sin B) = \sin^{-1}\left(\frac{15}{17}\right)$$

$$m\angle B = \sin^{-1}\left(\frac{15}{17}\right)$$

$$m\angle B = 61.9°$$

4. a)



b) Drawing a segment from $(-2, 2)$ down to the negative x-axis forms a right triangle with two legs of length 2. The tangent of the related acute angle is $\frac{2}{2}$ or 1.

The measure in radians of the acute angle that has a tangent equal to 1 is $\frac{\pi}{4}$, so the value of the related acute angle is $\frac{\pi}{4}$ radians.

c) Since the point at $(-2, 2)$ is in the second quadrant, the terminal arm of the angle θ must also be in the second quadrant. Since the angle with a terminal arm in the second quadrant and a tangent of -1 measures $\frac{3\pi}{4}$ radians, $\theta = \frac{3\pi}{4}$ radians.

5. a) The x-coordinate of point A is $\cos\frac{\pi}{4}$, while the y-coordinate is $\sin\frac{\pi}{4}$. Therefore, the coordinates of point A are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. The x-coordinate of point B is $\cos\frac{\pi}{3}$, while the y-coordinate is $\sin\frac{\pi}{3}$. Therefore, the coordinates of point B are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The x-coordinate of point C is $\cos\frac{2\pi}{3}$, while the y-coordinate is $\sin\frac{2\pi}{3}$. Therefore, the coordinates of point C are $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The x-coordinate of point D the is $\cos\frac{5\pi}{6}$, while the y-coordinate is $\sin\frac{5\pi}{6}$. Therefore, coordinates of point D are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. The x-coordinate of point E is $\cos\frac{7\pi}{6}$, while the y-coordinate is $\sin\frac{7\pi}{6}$. Therefore, the coordinates of point E are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. The x-coordinate of point F is $\cos\frac{5\pi}{4}$, while the

y-coordinate is $\sin \frac{5\pi}{4}$. Therefore, the coordinates of point F are $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. The x-coordinate of point G is $\cos \frac{4\pi}{3}$, while the y-coordinate is $\sin \frac{4\pi}{3}$. Therefore, the coordinates of point G are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The x-coordinate of point H is $\cos \frac{5\pi}{3}$, while the y-coordinate is $\sin \frac{5\pi}{3}$. Therefore, the coordinates of point H are $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The x-coordinate of point I is $\cos \frac{7\pi}{4}$, while the y-coordinate is $\sin \frac{7\pi}{4}$. Therefore, the coordinates of point I are $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. The x-coordinate of point J is $\cos \frac{11\pi}{6}$, while the y-coordinate is $\sin \frac{11\pi}{6}$. Therefore, the coordinates of point J are $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

b) i) $\cos \frac{3\pi}{4}$ is equal to the x-coordinate of the point at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, so $\cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$.

ii) $\sin \frac{11\pi}{6}$ is equal to the y-coordinate of point J, so $\sin \frac{11\pi}{6} = -\frac{1}{2}$.

iii) $\cos \pi$ is equal to the x-coordinate of the point at $(-1, 0)$, so $\cos \pi = -1$.

iv) $\csc \frac{\pi}{6}$ is equal to the reciprocal of the y-coordinate of the point at $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, so $\csc \frac{\pi}{6} = \frac{1}{\frac{1}{2}} = 2$.

6. a) Since $\tan x = -\frac{3}{4}$, the leg opposite angle x in a right triangle has a length of 3, while the leg adjacent to angle x has a length of 4. For this reason, the length of the hypotenuse can be calculated as follows:

$$3^2 + 4^2 = z^2$$
$$9 + 16 = z^2$$
$$25 = z^2$$
$$z = 5$$

Therefore, if the angle x is in the second quadrant, the other five trigonometric ratios are as follows:

$$\sin x = \frac{\text{opposite leg}}{\text{hypotenuse}} = \frac{3}{5};$$
$$\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}} = -\frac{4}{5};$$
$$\csc x = \frac{\text{hypotenuse}}{\text{opposite leg}} = \frac{5}{3};$$
$$\sec x = \frac{\text{hypotenuse}}{\text{adjacent leg}} = -\frac{5}{4};$$
$$\cot x = \frac{\text{adjacent leg}}{\text{opposite leg}} = -\frac{4}{3}$$

If the angle x is in the fourth quadrant, the other five trigonometric ratios are as follows:

$$\sin x = \frac{\text{opposite leg}}{\text{hypotenuse}} = -\frac{3}{5};$$
$$\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{4}{5};$$
$$\csc x = \frac{\text{hypotenuse}}{\text{opposite leg}} = -\frac{5}{3};$$
$$\sec x = \frac{\text{hypotenuse}}{\text{adjacent leg}} = \frac{5}{4};$$
$$\cot x = \frac{\text{adjacent leg}}{\text{opposite leg}} = -\frac{4}{3}$$

b) To determine the value of the angle x, any of the ratios found in part (a) can be used. If x is in the second quadrant, $\sin x = \frac{3}{5}$, so x can be solved for as follows:

$$\sin x = \frac{3}{5}$$
$$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{3}{5}\right)$$
$$x = \sin^{-1}\left(\frac{3}{5}\right)$$
$$x = 2.5$$

If x is in the fourth quadrant, $\sin x = -\frac{3}{5}$, so x can be solved for as follows:

$$\sin x = -\frac{3}{5}$$
$$\sin^{-1}(\sin x) = \sin^{-1}\left(-\frac{3}{5}\right)$$
$$x = \sin^{-1}\left(-\frac{3}{5}\right)$$
$$x = 5.6$$

7. a) $\tan \theta = \frac{\sin \theta}{\cos \theta}$, $\cos \theta \neq 0$ is true.

b) $\sin^2 \theta + \cos^2 \theta = 1$ is true.

c) $\sec \theta = \frac{1}{\sin \theta}$, $\sin \theta \neq 0$ is false, since $\sec \theta$ is the reciprocal of $\cos \theta$, not $\sin \theta$.

d) $\cos^2 \theta = \sin^2 \theta - 1$ is false. This can be shown by performing the following operations:

$$\cos^2 \theta = \sin^2 \theta - 1$$
$$\cos^2 \theta + 1 = \sin^2 \theta - 1 + 1$$
$$\cos^2 \theta + 1 = \sin^2 \theta$$
$$\cos^2 \theta + 1 - \cos^2 \theta = \sin^2 \theta - \cos^2 \theta$$
$$\sin^2 \theta - \cos^2 \theta = 1$$

Since $\sin^2\theta + \cos^2\theta = 1$ is true, $\sin^2\theta - \cos^2\theta = 1$ must be false.

e) $1 + \tan^2\theta = \sec^2\theta$ is true

f) $\cot\theta = \frac{\cos\theta}{\sin\theta}$, $\sin\theta \neq 0$ is true.

8. The sinusoidal function $y = \sin x$ is transformed into the function $y = a\sin k(x - d) + c$ by vertically stretching or compressing the function $y = \sin x$ by a factor of a, horizontally stretching or compressing it by a factor of $\frac{1}{k}$, reflecting it in the x-axis if $a < 0$, reflecting it in the y-axis is $k < 0$, vertically translating it c units up or down, and horizontally translating it d units to the right or the left. Therefore, an appropriate flow chart would be as follows:

Perform a vertical stretch/compression by a factor of $|a|$.

↓

Use $\left|\frac{1}{k}\right|$ to determine the horizontal stretch/compression.

↓

Use a and k to determine whether the function is reflected in the y-axis or the x-axis.

↓

Perform a vertical translation of c units up or down.

↓

Perform a horizontal translation of d units to the right or the left.

7.1 Exploring Equivalent Trigonometric Functions, pp. 392–393

1. **a)** Answers may vary. For example: The graph is that of the trigonometric function $y = \cos\theta$. Since the period of $y = \cos\theta$ is 2π radians, the graph repeats itself every 2π radians. Therefore, three possible equivalent expressions for the graph are $y = \cos(\theta + 2\pi)$, $y = \cos(\theta + 4\pi)$, and $y = \cos(\theta - 2\pi)$.

b) The trigonometric functions $y = \cos\theta$ and $y = \sin\left(\theta + \frac{\pi}{2}\right)$ are equivalent. Since the period of $y = \sin\left(\theta + \frac{\pi}{2}\right)$ is 2π radians, the graph repeats itself every 2π radians. Therefore, three possible equivalent expressions for the graph using the sine function are $y = \sin\left(\theta + \frac{\pi}{2}\right)$, $y = \sin\left(\theta - \frac{3\pi}{2}\right)$, and $y = \sin\left(\theta + \frac{5\pi}{2}\right)$.

2. a) The graph of the trigonometric function $y = \csc\theta$ is symmetric with respect to the origin, so $y = \csc\theta$ is an odd function. Therefore, $\csc(-\theta) = -\csc\theta$. The graph of the trigonometric function $y = \sec\theta$ is symmetric with respect to the y-axis, so $y = \sec\theta$ is an even function. Therefore, $\sec(-\theta) = \sec\theta$. The graph of the trigonometric function $y = \cot\theta$ is symmetric with respect to the origin, so $y = \cot\theta$ is an odd function. Therefore, $\cot(-\theta) = -\cot\theta$.

b) If the graph of the trigonometric function $y = \csc\theta$ is reflected in the y-axis, the equation of the resulting graph is $y = \csc(-\theta)$. Also, if the graph of $y = \csc\theta$ is reflected in the x-axis, the equation of the resulting graph is $y = -\csc\theta$. Since the graph of $y = \csc\theta$ is symmetric with respect to the origin, a reflection in the y-axis is the same as a reflection in the x-axis. Therefore, the equation $\csc(-\theta) = -\csc\theta$ must be true. If the graph of the trigonometric function $y = \sec\theta$ is reflected in the y-axis, the equation of the resulting graph is $y = \sec(-\theta)$. Since the graph of $y = \sec\theta$ is symmetric with respect to the y-axis, a reflection of the function in the y-axis results in the function itself. Therefore, the equation $\sec(-\theta) = \sec\theta$ must be true. If the graph of the trigonometric function $y = \cot\theta$ is reflected in the y-axis, the equation of the resulting graph is $y = \cot(-\theta)$. Also, if the graph of $y = \cot\theta$ is reflected in the x-axis, the equation of the resulting graph is $y = -\cot\theta$. Since the graph of $y = \cot\theta$ is symmetric with respect to the origin, a reflection in the y-axis is the same as a reflection in the x-axis. Therefore, the equation $\cot(-\theta) = -\cot\theta$ must be true.

3. a) Since $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$,

$$\sin\frac{\pi}{6} = \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \cos\left(\frac{3\pi}{6} - \frac{\pi}{6}\right)$$

$$= \cos\frac{2\pi}{6} = \cos\frac{\pi}{3}$$

b) Since $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$,

$$\cos\frac{5\pi}{12} = \sin\left(\frac{\pi}{2} - \frac{5\pi}{12}\right)$$

$$= \sin\left(\frac{6\pi}{12} - \frac{5\pi}{12}\right) = \sin\frac{\pi}{12}$$

c) Since $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$,

$$\tan\frac{3\pi}{8} = \cot\left(\frac{\pi}{2} - \frac{3\pi}{8}\right)$$

$$= \cot\left(\frac{4\pi}{8} - \frac{3\pi}{8}\right) = \cot\frac{\pi}{8}$$

d) Since $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$,

$$\cos\frac{5\pi}{16} = \sin\left(\frac{\pi}{2} - \frac{5\pi}{16}\right)$$

$$= \sin\left(\frac{8\pi}{16} - \frac{5\pi}{16}\right) = \sin\frac{3\pi}{16}$$

e) Since $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$,

$$\sin\frac{\pi}{8} = \cos\left(\frac{\pi}{2} - \frac{\pi}{8}\right)$$

$$= \cos\left(\frac{4\pi}{8} - \frac{\pi}{8}\right)$$

$$= \cos\frac{3\pi}{8}$$

f) Since $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$,

$$\tan\frac{\pi}{6} = \cot\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \cot\left(\frac{3\pi}{6} - \frac{\pi}{6}\right)$$

$$= \cot\frac{2\pi}{6} = \cot\frac{\pi}{3}$$

4. a) Since $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$,

$$\frac{1}{\sin\theta} = \frac{1}{\cos\left(\frac{\pi}{2} - \theta\right)}$$

Therefore, $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$.

Since $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$, $\frac{1}{\cos\theta} = \frac{1}{\sin\left(\frac{\pi}{2} - \theta\right)}$.

Therefore, $\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$.

Since $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$, $\frac{1}{\tan\theta} = \frac{1}{\cot\left(\frac{\pi}{2} - \theta\right)}$.

Therefore, $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$.

b) The trigonometric function $y = \sec\left(\frac{\pi}{2} - \theta\right)$ can also be written as $y = \sec\left(-\left(\theta - \frac{\pi}{2}\right)\right)$. If the graph of the trigonometric function $y = \sec\left(-\left(\theta - \frac{\pi}{2}\right)\right)$ is reflected in the y-axis, the equation of the resulting graph is $y = \sec\left(\theta - \frac{\pi}{2}\right)$. Since the graph of $y = \sec\left(-\left(\theta - \frac{\pi}{2}\right)\right)$ is symmetric with respect to the y-axis, a reflection of the function in the y-axis results in the function itself. For this reason, $\sec\left(\frac{\pi}{2} - \theta\right) = \sec\left(\theta - \frac{\pi}{2}\right)$, so $\csc\theta = \sec\left(\theta - \frac{\pi}{2}\right)$. Therefore, $\frac{1}{\csc\theta} = \frac{1}{\sec\left(\theta - \frac{\pi}{2}\right)}$, or $\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$. This is a known identity, so the cofunction identity $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$ must be true. The trigonometric function $y = \csc\left(\frac{\pi}{2} - \theta\right)$ can also be written as $y = \csc\left(-\left(\theta - \frac{\pi}{2}\right)\right)$. If the graph of the trigonometric function $y = \csc\left(-\left(\theta - \frac{\pi}{2}\right)\right)$ is reflected in the y-axis, the equation of the resulting resulting graph is $y = \csc\left(\theta - \frac{\pi}{2}\right)$. Since the graph of $y = \csc\left(-\left(\theta - \frac{\pi}{2}\right)\right)$ is symmetric with respect to the origin, a reflection in the y-axis is the same as a reflection in the x-axis. For this reason,

$$-\csc\left(\frac{\pi}{2} - \theta\right) = \csc\left(\theta - \frac{\pi}{2}\right), \text{ or}$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \csc\left(\theta - \frac{\pi}{2} + \pi\right)$$

$$= \csc\left(\theta + \frac{\pi}{2}\right), \text{ so}$$

$$\sec\theta = \csc\left(\theta + \frac{\pi}{2}\right)$$

Therefore, $\frac{1}{\sec\theta} = \frac{1}{\csc\left(\theta + \frac{\pi}{2}\right)}$, or

$$\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$$

This is a known identity, so the cofunction identity $\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$ must be true.

The trigonometric function $y = \tan\left(\frac{\pi}{2} - \theta\right)$ can also be written as $y = \tan\left(-\left(\theta - \frac{\pi}{2}\right)\right)$. If the graph of the trigonometric function $y = \tan\left(-\left(\theta - \frac{\pi}{2}\right)\right)$ is reflected in the y-axis, the equation of the resulting graph is $y = \tan\left(-\left(\theta - \frac{\pi}{2}\right)\right)$.

Since the graph of $y = \tan\left(-\left(\theta - \frac{\pi}{2}\right)\right)$ is symmetric with respect to the origin, a reflection in the y-axis is the same as a reflection in the x-axis. For this reason, $-\tan\left(\frac{\pi}{2} - \theta\right) = \tan\left(\theta - \frac{\pi}{2}\right)$, or $\tan\left(\frac{\pi}{2} - \theta\right) = \tan\left(\theta - \frac{\pi}{2} + \pi\right) = \tan\left(\theta + \frac{\pi}{2}\right)$, so $\cot\theta = \tan\left(\theta + \frac{\pi}{2}\right)$. Therefore, $\frac{1}{\cot\theta} = \frac{1}{\tan\left(\theta + \frac{\pi}{2}\right)}$, or $\tan\theta = \cot\left(\theta + \frac{\pi}{2}\right)$. This is a known identity, so the cofunction identity $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$ must be true.

5. a) Since $\sin\theta = \sin(\pi - \theta)$,

$$\sin\frac{7\pi}{8} = \sin\left(\pi - \frac{7\pi}{8}\right) = \sin\frac{\pi}{8}$$

b) Since $\cos\theta = -\cos(\pi + \theta)$,

$$\begin{aligned}\cos\frac{13\pi}{12} &= -\cos\left(\pi + \frac{13\pi}{12}\right)\\ &= -\cos\left(\frac{12\pi}{12} + \frac{13\pi}{12}\right) = -\cos\frac{25\pi}{12}\\ &= -\cos\left(\frac{25\pi}{12} - 2\pi\right)\\ &= -\cos\left(\frac{25\pi}{12} - \frac{24\pi}{12}\right) = -\cos\frac{\pi}{12}\end{aligned}$$

c) Since $\tan\theta = \tan(\pi + \theta)$,

$$\begin{aligned}\tan\frac{5\pi}{4} &= \tan\left(\frac{4\pi}{4} + \frac{5\pi}{4}\right) = \tan\frac{9\pi}{4}\\ &= \tan\left(\frac{9\pi}{4} - 2\pi\right)\\ &= \tan\left(\frac{9\pi}{4} - \frac{8\pi}{4}\right) = \tan\frac{\pi}{4}\end{aligned}$$

d) Since $\cos\theta = \cos(2\pi - \theta)$,

$$\begin{aligned}\cos\frac{11\pi}{6} &= \cos\left(2\pi - \frac{11\pi}{6}\right)\\ &= \cos\left(\frac{12\pi}{6} - \frac{11\pi}{6}\right) = \cos\frac{\pi}{6}\end{aligned}$$

e) Since $\sin\theta = -\sin(2\pi - \theta)$,

$$\begin{aligned}\sin\frac{13\pi}{8} &= -\sin\left(2\pi - \frac{13\pi}{8}\right)\\ &= -\sin\left(\frac{16\pi}{8} - \frac{13\pi}{8}\right) = -\sin\frac{3\pi}{8}\end{aligned}$$

f) Since $\tan\theta = -\tan(2\pi - \theta)$,

$$\begin{aligned}\tan\frac{5\pi}{3} &= -\tan\left(2\pi - \frac{5\pi}{3}\right)\\ &= -\tan\left(\frac{6\pi}{3} - \frac{5\pi}{3}\right) = -\tan\frac{\pi}{3}\end{aligned}$$

6. a) Assume the circle is a unit circle. Let the coordinates of Q be (x, y). Since P and Q are reflections of each other in the line $y = x$, the coordinates of P are (y, x). Draw a line from P to the positive x-axis. The hypotenuse of the new right triangle makes an angle of $\left(\frac{\pi}{2} - \theta\right)$ with the positive x-axis. Since the x-coordinate of P is y, $\cos\left(\frac{\pi}{2} - \theta\right) = y$. Also, since the y-coordinate of Q is y, $\sin\theta = y$. Therefore, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

b) Assume the circle is a unit circle. Let the coordinates of the vertex on the circle of the right triangle in the first quadrant be (x, y). Then $\sin\theta = y$, so $-\sin\theta = -y$. The point on the circle that results from rotating the vertex by $\frac{\pi}{2}$ counterclockwise about the origin has coordinates $(-y, x)$, so $\cos\left(\frac{\pi}{2} + \theta\right) = -y$. Therefore, $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$.

7. a) true; the period of cosine is 2π

b) false; Answers may vary. For example: Let $\theta = \frac{\pi}{2}$. Then the left side is $\sin\frac{\pi}{2}$, or 1. The right side is $-\sin\frac{\pi}{2}$ or -1.

c) false; Answers may vary. For example: Let $\theta = \pi$. Then the left side is $\cos\pi$, or -1. The right side is $-\cos 5\pi$, or 1.

d) false; Answers may vary. For example: Let $\theta = \frac{\pi}{4}$. Then the left side is $\tan\frac{3\pi}{4}$, or $-\frac{\sqrt{2}}{2}$. The right side is $\tan\frac{\pi}{4}$, or $\frac{\sqrt{2}}{2}$.

e) false; Answers may vary. For example: Let $\theta = \pi$. Then the left side is $\cot\frac{3\pi}{4}$, or -1. The right side is $\tan\frac{\pi}{4}$, or 1.

f) false; Answers may vary. For example: Let $\theta = \frac{\pi}{2}$. Then the left side is $\sin \frac{5\pi}{2}$, or 1. The right side is $\sin\left(-\frac{\pi}{2}\right)$, or −1.

7.2 Compound Angle Formulas, pp. 400–401

1. a) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$, $\sin a \cos 2a + \cos a \sin 2a = \sin(a + 2a) = \sin 3a$.

b) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\cos 4x \cos 3x - \sin 4x \sin 3x = \cos(4x + 3x)$$
$$= \cos 7x$$

2. a) Since $\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$,

$$\frac{\tan 170^\circ - \tan 110^\circ}{1 + \tan 170^\circ \tan 110^\circ} = \tan(170^\circ - 110^\circ)$$
$$= \tan 60^\circ = \sqrt{3}$$

b) Since $\cos(a - b) = \cos a \cos b + \sin a \sin b$,

$$\cos\frac{5\pi}{12}\cos\frac{\pi}{12} + \sin\frac{5\pi}{12}\sin\frac{\pi}{12} = \cos\left(\frac{5\pi}{12} - \frac{\pi}{12}\right)$$
$$= \cos\frac{4\pi}{12} = \cos\frac{\pi}{3} = \frac{1}{2}$$

3. a) Two angles from the special triangles are 30° and 45°, so $75^\circ = 30^\circ + 45^\circ$.

b) Two angles from the special triangles are 30° and 45°, so $-15^\circ = 30^\circ - 45^\circ$.

c) Two angles from the special triangles are $\frac{\pi}{6}$ and $\frac{\pi}{3}$, so $-\frac{\pi}{6} = \frac{\pi}{6} - \frac{2\pi}{6} = \frac{\pi}{6} - \frac{\pi}{3}$.

d) Two angles from the special triangles are $\frac{\pi}{6}$ and $\frac{\pi}{4}$, so $\frac{\pi}{12} = \frac{3\pi}{12} - \frac{2\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$.

e) Two angles from the special triangles are 45° and 60°, so $105^\circ = 45^\circ + 60^\circ$.

f) Two angles from the special triangles are $\frac{\pi}{3}$ and $\frac{\pi}{2}$, so $\frac{5\pi}{6} = \frac{2\pi}{6} + \frac{3\pi}{6} = \frac{\pi}{3} + \frac{\pi}{2}$.

4. a) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\sin 75^\circ = \sin(30^\circ + 45^\circ)$$
$$= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$$
$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$$
$$= \frac{\sqrt{2} + \sqrt{6}}{4}$$

b) Since $\cos(a - b) = \cos a \cos b + \sin a \sin b$,

$$\cos 15^\circ = \cos(45^\circ - 30^\circ)$$
$$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{2} + \sqrt{6}}{4}$$

c) Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\tan\frac{5\pi}{12} = \tan\left(\frac{2\pi}{12} + \frac{3\pi}{12}\right) = \tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$$
$$= \frac{\tan\frac{\pi}{6} + \tan\frac{\pi}{4}}{1 - \tan\frac{\pi}{6}\tan\frac{\pi}{4}} = \frac{\frac{\sqrt{3}}{3} + 1}{1 - \left(\frac{\sqrt{3}}{3}\right)(1)}$$
$$= \frac{\frac{\sqrt{3}}{3} + \frac{3}{3}}{\frac{3}{3} - \frac{\sqrt{3}}{3}} = \frac{\frac{\sqrt{3} + 3}{3}}{\frac{3 - \sqrt{3}}{3}}$$
$$= \frac{\sqrt{3} + 3}{3} \times \frac{3}{3 - \sqrt{3}} = \frac{\sqrt{3} + 3}{3 - \sqrt{3}}$$
$$= \frac{(\sqrt{3} + 3)(3 + \sqrt{3})}{(3 - \sqrt{3})(3 + \sqrt{3})}$$
$$= \frac{3\sqrt{3} + 9 + 3 + 3\sqrt{3}}{9 - 3\sqrt{3} + 3\sqrt{3} - 3}$$
$$= \frac{12 + 6\sqrt{3}}{6} = 2 + \sqrt{3}$$

d) Since $\sin(a - b) = \sin a \cos b - \cos a \sin b$,

$$\sin\left(-\frac{\pi}{12}\right) = \sin\left(\frac{3\pi}{12} - \frac{4\pi}{12}\right) = \sin\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$$
$$= \sin\frac{\pi}{4}\cos\frac{\pi}{3} - \cos\frac{\pi}{4}\sin\frac{\pi}{3}$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$
$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

e) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\cos 105^\circ = \cos(45^\circ + 60^\circ)$$
$$= \cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$

$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

f) Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\tan \frac{23\pi}{12} = \tan\left(\frac{9\pi}{12} + \frac{14\pi}{12}\right) = \tan\left(\frac{3\pi}{4} + \frac{7\pi}{6}\right)$$

$$= \frac{\tan \frac{3\pi}{4} + \tan \frac{7\pi}{6}}{1 - \tan \frac{3\pi}{4} \tan \frac{7\pi}{6}} = \frac{-1 + \frac{\sqrt{3}}{3}}{1 - (-1)\left(\frac{\sqrt{3}}{3}\right)}$$

$$= \frac{-\frac{3}{3} + \frac{\sqrt{3}}{3}}{\frac{3}{3} - \left(-\frac{\sqrt{3}}{3}\right)} = \frac{\frac{\sqrt{3} - 3}{3}}{\frac{3 + \sqrt{3}}{3}}$$

$$= \frac{\sqrt{3} - 3}{3} \times \frac{3}{3 + \sqrt{3}} = \frac{\sqrt{3} - 3}{3 + \sqrt{3}}$$

$$= \frac{(\sqrt{3} - 3)(3 - \sqrt{3})}{(3 + \sqrt{3})(3 - \sqrt{3})}$$

$$= \frac{3\sqrt{3} - 9 - 3 + 3\sqrt{3}}{9 + 3\sqrt{3} - 3\sqrt{3} - 3}$$

$$= \frac{-12 + 6\sqrt{3}}{6}$$

$$= -2 + \sqrt{3}$$

5. a) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\sin\left(\pi + \frac{\pi}{6}\right) = \sin \pi \cos \frac{\pi}{6} + \cos \pi \sin \frac{\pi}{6}$$

$$= (0)\left(\frac{\sqrt{3}}{2}\right) + (-1)\left(\frac{1}{2}\right)$$

$$= 0 + \left(-\frac{1}{2}\right) = -\frac{1}{2}$$

b) Since $\cos(a - b) = \cos a \cos b + \sin a \sin b$,

$$\cos\left(\pi - \frac{\pi}{4}\right) = \cos \pi \cos \frac{\pi}{4} + \sin \pi \sin \frac{\pi}{4}$$

$$= (-1)\left(\frac{\sqrt{2}}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right)$$

$$= -\frac{\sqrt{2}}{2} + 0$$

$$= -\frac{\sqrt{2}}{2}$$

c) Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\tan\left(\frac{\pi}{4} + \pi\right) = \frac{\tan \frac{\pi}{4} + \tan \pi}{1 - \tan \frac{\pi}{4} \tan \pi}$$

$$= \frac{1 + 0}{1 - (1)(0)}$$

$$= \frac{1 + 0}{1 - 0}$$

$$= \frac{1}{1} = 1$$

d) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\sin\left(-\frac{\pi}{2} + \frac{\pi}{3}\right) = \sin\left(-\frac{\pi}{2}\right) \cos \frac{\pi}{3}$$

$$+ \cos\left(-\frac{\pi}{2}\right) \sin \frac{\pi}{3}$$

$$= (-1)\left(\frac{1}{2}\right) + (0)\left(\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{1}{2} + 0$$

$$= -\frac{1}{2}$$

e) Since $\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$,

$$\tan\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\tan \frac{\pi}{3} - \tan \frac{\pi}{6}}{1 + \tan \frac{\pi}{3} \tan \frac{\pi}{6}}$$

$$= \frac{\sqrt{3} - \frac{\sqrt{3}}{3}}{1 + (\sqrt{3})\left(\frac{\sqrt{3}}{3}\right)}$$

$$= \frac{\frac{3\sqrt{3}}{3} - \frac{\sqrt{3}}{3}}{1 + \frac{3}{3}}$$

$$= \frac{\frac{3\sqrt{3} - \sqrt{3}}{3}}{1 + 1}$$

$$= \frac{\frac{2\sqrt{3}}{3}}{2}$$

$$= \frac{2\sqrt{3}}{3} \times \frac{1}{2}$$

$$= \frac{2\sqrt{3}}{6}$$

$$= \frac{\sqrt{3}}{3}$$

f) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = \cos \frac{\pi}{2} \cos \frac{\pi}{3} - \sin \frac{\pi}{2} \sin \frac{\pi}{3}$$

$$= (0)\left(\frac{1}{2}\right) - (1)\left(\frac{\sqrt{3}}{2}\right)$$

$$\begin{aligned} &= 0 - \frac{\sqrt{3}}{2} \\ &= -\frac{\sqrt{3}}{2} \end{aligned}$$

6. a) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\begin{aligned} \sin(\pi + x) &= \sin \pi \cos x + \cos \pi \sin x \\ &= (0)(\cos x) + (-1)(\sin x) \\ &= 0 + (-\sin x) \\ &= 0 - \sin x \\ &= -\sin x \end{aligned}$$

b) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\begin{aligned} \cos\left(x + \frac{3\pi}{2}\right) &= \cos x \cos \frac{3\pi}{2} - \sin x \sin \frac{3\pi}{2} \\ &= (\cos x)(0) - (\sin x)(-1) \\ &= 0 - (-\sin x) = \sin x \end{aligned}$$

c) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\begin{aligned} \cos\left(x + \frac{\pi}{2}\right) &= \cos x \cos \frac{\pi}{2} - \sin x \sin \frac{\pi}{2} \\ &= (\cos x)(0) - (\sin x)(1) \\ &= 0 - \sin x \\ &= -\sin x \end{aligned}$$

d) Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\begin{aligned} \tan(x + \pi) &= \frac{\tan x + \tan \pi}{1 - \tan x \tan \pi} \\ &= \frac{\tan x + 0}{1 - (\tan x)(0)} \\ &= \frac{\tan x + 0}{1 - 0} \\ &= \frac{\tan x}{1} \\ &= \tan x \end{aligned}$$

e) Since $\sin(a - b) = \sin a \cos b - \cos a \sin b$,

$$\begin{aligned} \sin(x - \pi) &= \sin x \cos \pi - \cos x \sin \pi \\ &= (\sin x)(-1) - (\cos x)(0) \\ &= -\sin x - 0 \\ &= -\sin x \end{aligned}$$

f) Since $\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$,

$$\begin{aligned} \tan(2\pi - x) &= \frac{\tan 2\pi - \tan x}{1 + \tan 2\pi \tan x} \\ &= \frac{0 - \tan x}{1 + (0)(\tan x)} \\ &= \frac{-\tan x}{1 + 0} \\ &= \frac{-\tan x}{1} \\ &= -\tan x \end{aligned}$$

7. a) Since $\sin(\pi + x) = \sin(x + \pi)$, $\sin(\pi + x)$ is equivalent to $\sin x$ translated π units to the left, which is equivalent to $-\sin x$.

b) $\cos\left(x + \frac{3\pi}{2}\right)$ is equivalent to $\cos x$ translated $\frac{3\pi}{2}$ units to the left, which is equivalent to $\sin x$.

c) $\cos\left(x + \frac{\pi}{2}\right)$ is equivalent to $\cos x$ translated $\frac{\pi}{2}$ units to the left, which is equivalent to $-\sin x$.

d) $\tan(x + \pi)$ is equivalent to $\tan x$ translated π units to the left, which is equivalent to $\tan x$.

e) $\sin(x - \pi)$ is equivalent to $\sin x$ translated π units to the right, which is equivalent to $-\sin x$.

f) Since $\tan(2\pi - x) = \tan(-x + 2\pi)$, and since the period of the tangent function is 2π, $\tan(2\pi - x)$ is equivalent to $\tan(-x)$, which is equivalent to $\tan x$ reflected in the y-axis, which is equivalent to $-\tan x$.

8. a) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\begin{aligned} \cos 75^\circ &= \cos(30^\circ + 45^\circ) \\ &= \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ \\ &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

b) Since $\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$,

$$\begin{aligned} \tan(-15^\circ) &= \tan(30^\circ - 45^\circ) \\ &= \frac{\tan 30^\circ - \tan 45^\circ}{1 + \tan 30^\circ \tan 45^\circ} \\ &= \frac{\frac{\sqrt{3}}{3} - 1}{1 + \left(\frac{\sqrt{3}}{3}\right)(1)} = \frac{\frac{\sqrt{3}}{3} - \frac{3}{3}}{\frac{3}{3} + \frac{\sqrt{3}}{3}} \\ &= \frac{\frac{\sqrt{3} - 3}{3}}{\frac{3 + \sqrt{3}}{3}} = \frac{\sqrt{3} - 3}{3} \times \frac{3}{3 + \sqrt{3}} \\ &= \frac{\sqrt{3} - 3}{3 + \sqrt{3}} = \frac{\left(\sqrt{3} - 3\right)\left(3 - \sqrt{3}\right)}{\left(3 + \sqrt{3}\right)\left(3 - \sqrt{3}\right)} \\ &= \frac{3\sqrt{3} - 9 - 3 + 3\sqrt{3}}{9 + 3\sqrt{3} - 3\sqrt{3} - 3} \\ &= \frac{-12 + 6\sqrt{3}}{6} \\ &= -2 + \sqrt{3} \end{aligned}$$

c) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\cos\frac{11\pi}{12} = \cos\left(\frac{2\pi}{12} + \frac{9\pi}{12}\right)$$
$$= \cos\left(\frac{\pi}{6} + \frac{3\pi}{4}\right)$$
$$= \cos\frac{\pi}{6}\cos\frac{3\pi}{4} - \sin\frac{\pi}{6}\sin\frac{3\pi}{4}$$
$$= \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$
$$= \frac{-\sqrt{6} - \sqrt{2}}{4}$$

d) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\sin\frac{13\pi}{12} = \sin\left(\frac{3\pi}{12} + \frac{10\pi}{12}\right)$$
$$= \sin\left(\frac{\pi}{4} + \frac{5\pi}{6}\right)$$
$$= \sin\frac{\pi}{4}\cos\frac{5\pi}{6} + \cos\frac{\pi}{4}\sin\frac{5\pi}{6}$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

e) Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\tan\frac{7\pi}{12} = \tan\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right)$$
$$= \tan\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$$
$$= \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{3}}{1 - \tan\frac{\pi}{4}\tan\frac{\pi}{3}}$$
$$= \frac{1 + \sqrt{3}}{1 - (1)(\sqrt{3})}$$
$$= \frac{1 + \sqrt{3}}{1 - \sqrt{3}}$$
$$= \frac{(1 + \sqrt{3})(1 + \sqrt{3})}{(1 - \sqrt{3})(1 + \sqrt{3})}$$
$$= \frac{1 + \sqrt{3} + \sqrt{3} + 3}{1 - \sqrt{3} + \sqrt{3} - 3}$$
$$= \frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3}$$

f) Since $\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$,

$$\tan\left(-\frac{5\pi}{12}\right) = \tan\left(\frac{4\pi}{12} - \frac{9\pi}{12}\right) = \tan\left(\frac{\pi}{3} - \frac{3\pi}{4}\right)$$
$$= \frac{\tan\frac{\pi}{3} - \tan\frac{3\pi}{4}}{1 + \tan\frac{\pi}{3}\tan\frac{3\pi}{4}}$$
$$= \frac{\sqrt{3} - (-1)}{1 + (\sqrt{3})(-1)}$$
$$= \frac{\sqrt{3} + 1}{1 - \sqrt{3}}$$
$$= \frac{1 + \sqrt{3}}{1 - \sqrt{3}}$$
$$= \frac{(1 + \sqrt{3})(1 + \sqrt{3})}{(1 - \sqrt{3})(1 + \sqrt{3})}$$
$$= \frac{1 + \sqrt{3} + \sqrt{3} + 3}{1 - \sqrt{3} + \sqrt{3} - 3}$$
$$= \frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3}$$

9. a) Since $\sin x = \frac{4}{5}$, the leg opposite the angle x in a right triangle has a length of 4, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 4^2 = 5^2$$
$$x^2 + 16 = 25$$
$$x^2 + 16 - 16 = 25 - 16$$
$$x^2 = 9$$
$$x = 3, \text{ in quadrant I}$$

Since $\cos x = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos x = \frac{3}{5}$. In addition, since $\sin y = -\frac{12}{13}$, the leg opposite the angle y in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 + 144 - 144 = 169 - 144$$

$x^2 = 25$

$x = 5$, in quadrant IV

Since $\cos y = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos y = \frac{5}{13}$. Therefore, since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$= \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)$$
$$= \frac{15}{65} - \left(-\frac{48}{65}\right)$$
$$= \frac{15}{65} + \frac{48}{65}$$
$$= \frac{63}{65}$$

b) Since $\sin x = \frac{4}{5}$, the leg opposite the angle x in a right triangle has a length of 4, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 4^2 = 5^2$$
$$x^2 + 16 = 25$$
$$x^2 + 16 - 16 = 25 - 16$$
$$x^2 = 9$$

$x = 3$, in quadrant I

Since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos x = \frac{3}{5}$. In addition, since $\sin y = -\frac{12}{13}$, the leg opposite the angle y in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 + 144 - 144 = 169 - 144$$
$$x^2 = 25$$

$x = 5$, in quadrant IV

Since $\cos y = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos y = \frac{5}{13}$. Therefore, since $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$= \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right)$$
$$= \frac{20}{65} + \left(-\frac{36}{65}\right)$$
$$= \frac{20}{65} - \frac{36}{65}$$
$$= -\frac{16}{65}$$

c) Since $\sin x = \frac{4}{5}$, the leg opposite the angle x in a right triangle has a length of 4, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 4^2 = 5^2$$
$$x^2 + 16 = 25$$
$$x^2 + 16 - 16 = 25 - 16$$
$$x^2 = 9$$

$x = 3$, in quadrant I

Since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos x = \frac{3}{5}$. In addition, since $\sin y = -\frac{12}{13}$, the leg opposite the angle y in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 + 144 - 144 = 169 - 144$$
$$x^2 = 25$$

$x = 5$, in quadrant IV

Since $\cos y = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos y = \frac{5}{13}$. Therefore, since $\cos(a - b) = \cos a \cos b + \sin a \sin b$,

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
$$= \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) = \frac{15}{65} + \left(-\frac{48}{65}\right)$$
$$= \frac{15}{65} - \frac{48}{65} = -\frac{33}{65}$$

d) Since $\sin x = \frac{4}{5}$, the leg opposite the angle x in a right triangle has a length of 4, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 4^2 = 5^2$$
$$x^2 + 16 = 25$$
$$x^2 + 16 - 16 = 25 - 16$$
$$x^2 = 9$$

$x = 3$, in quadrant I

Since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos x = \frac{3}{5}$. In addition, since $\sin y = -\frac{12}{13}$, the leg opposite the angle y in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$

$$x^2 + 144 - 144 = 169 - 144$$
$$x^2 = 25$$
$$x = 5, \text{ in quadrant IV}$$

Since $\cos y = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos y = \frac{5}{13}$. Therefore, since $\sin(a - b) = \sin a \cos b - \cos a \sin b$,

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$= \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) = \frac{20}{65} - \left(-\frac{36}{65}\right)$$
$$= \frac{20}{65} + \frac{36}{65} = \frac{56}{65}$$

e) Since $\sin x = \frac{4}{5}$, the leg opposite the angle x in a right triangle has a length of 4, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 4^2 = 5^2$$
$$x^2 + 16 = 25$$
$$x^2 + 16 - 16 = 25 - 16$$
$$x^2 = 9$$
$$x = 3, \text{ in quadrant I}$$

Since $\tan x = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan x = \frac{4}{3}$. In addition, since $\sin y = -\frac{12}{13}$, the leg opposite the angle y in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 + 144 - 144 = 169 - 144$$
$$x^2 = 25$$
$$x = 5, \text{ in quadrant IV}$$

Since $\tan y = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan y = -\frac{12}{5}$. (The reason the sign is negative is because angle y is in the fourth quadrant.) Therefore, since

$$\tan(\mathrm{a} + \mathrm{b}) = \frac{\tan a + \tan b}{1 - \tan a \tan b},$$
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$= \frac{\frac{4}{3} + \left(-\frac{12}{5}\right)}{1 - \left(\frac{4}{3}\right)\left(-\frac{12}{5}\right)} = \frac{\frac{20}{15} + \left(-\frac{36}{15}\right)}{1 - \left(-\frac{48}{15}\right)}$$
$$= \frac{\frac{20}{15} - \frac{36}{15}}{1 + \frac{48}{15}} = \frac{-\frac{16}{15}}{\frac{63}{15}}$$
$$= -\frac{16}{15} \times \frac{15}{63} = -\frac{16}{63}$$

f) Since $\sin x = \frac{4}{5}$, the leg opposite the angle x in a right triangle has a length of 4, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 4^2 = 5^2$$
$$x^2 + 16 = 25$$
$$x^2 + 16 - 16 = 25 - 16$$
$$x^2 = 9$$
$$x = 3, \text{ in quadrant I}$$

Since $\tan x = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan x = \frac{4}{3}$. In addition, since $\sin y = -\frac{12}{13}$, the leg opposite the angle y in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 + 144 - 144 = 169 - 144$$
$$x^2 = 25$$
$$x = 5, \text{ in quadrant IV}$$

Since $\tan y = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan y = -\frac{12}{5}$. (The reason the sign is negative is because angle y is in the fourth quadrant.) Therefore, since

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b},$$
$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$
$$= \frac{\frac{4}{3} - \left(-\frac{12}{5}\right)}{1 + \left(\frac{4}{3}\right)\left(-\frac{12}{5}\right)} = \frac{\frac{20}{15} + \frac{36}{15}}{1 + \left(-\frac{48}{15}\right)} = \frac{\frac{56}{15}}{-\frac{33}{15}}$$
$$= \frac{56}{15} \times \left(-\frac{15}{33}\right) = -\frac{56}{33}$$

10. Since $\sin \alpha = \frac{7}{25}$, the leg opposite the angle α in a right triangle has a length of 7, while the hypotenuse of the right triangle has a length of 25. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 7^2 = 25^2$$
$$x^2 + 49 = 625$$
$$x^2 + 49 - 49 = 625 - 49$$
$$x^2 = 576$$
$$x = 24, \text{ in quadrant I}$$

Since $\cos \alpha = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos \alpha = \frac{24}{25}$. Also, since $\tan \alpha = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan \alpha = \frac{7}{24}$. In addition, since

$\cos\beta = \frac{5}{13}$, the leg adjacent to the angle β in a right triangle has a length of 5, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$5^2 + y^2 = 13^2$$
$$25 + y^2 = 169$$
$$25 + y^2 - 25 = 169 - 25$$
$$y^2 = 144$$
$$x = 12, \text{ in quadrant I}$$

Since $\sin\beta = \frac{\text{opposite leg}}{\text{hypotenuse}}$, $\sin\beta = \frac{12}{13}$. Also, since $\tan\beta = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan\beta = \frac{12}{5}$. Therefore, since $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$= \left(\frac{7}{25}\right)\left(\frac{5}{13}\right) + \left(\frac{24}{25}\right)\left(\frac{12}{13}\right)$$
$$= \frac{35}{325} + \frac{288}{325} = \frac{323}{325}$$

Also, since $\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$= \frac{\frac{7}{24} + \frac{12}{5}}{1 - \left(\frac{7}{24}\right)\left(\frac{12}{5}\right)}$$
$$= \frac{\frac{35}{120} + \frac{288}{120}}{1 - \frac{84}{120}} = \frac{\frac{323}{120}}{\frac{36}{120}}$$
$$= \frac{323}{120} \times \frac{120}{36} = \frac{323}{36}$$

11. a) Since $\cos(a - b) = \cos a \cos b + \sin a \sin b$,

$$\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$$
$$= (0)(\cos x) + (1)(\sin x)$$
$$= 0 + \sin x = \sin x$$

b) Since $\sin(a - b) = \sin a \cos b - \cos a \sin b$,

$$\sin\left(\frac{\pi}{2} - x\right) = \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x$$
$$= (1)(\cos x) - (0)(\sin x)$$
$$= \cos x - 0 = \cos x$$

12. a) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$, and since $\sin(a - b) = \sin a \cos b - \cos a \sin b$, the expression $\sin(\pi + x) + \sin(\pi - x)$ can be simplified as follows:

$$\sin(\pi + x) + \sin(\pi - x)$$
$$= \sin\pi\cos x + \cos\pi\sin x + \sin\pi\cos x - \cos\pi\sin x$$
$$= 2\sin\pi\cos x = (2)(0)(\sin x) = 0$$

b) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$, and since $\sin(a + b) = \sin a \cos b + \cos a \sin b$, the expression $\cos\left(x + \frac{\pi}{3}\right) - \sin\left(x + \frac{\pi}{6}\right)$ can be simplified as follows:

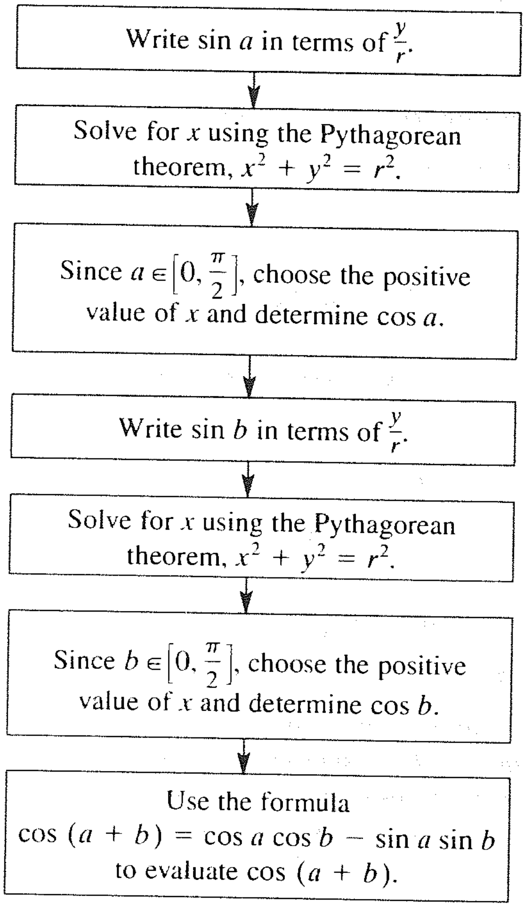
$$\cos x\cos\frac{\pi}{3} - \sin x\sin\frac{\pi}{3} - \left(\sin x\cos\frac{\pi}{6} + \cos x\sin\frac{\pi}{6}\right)$$
$$= \cos x\cos\frac{\pi}{3} - \sin x\sin\frac{\pi}{3} - \sin x\cos\frac{\pi}{6} - \cos x\sin\frac{\pi}{6}$$
$$= \left(\frac{1}{2}\right)(\cos x) - \left(\frac{\sqrt{3}}{2}\right)(\sin x) - \left(\frac{\sqrt{3}}{2}\right)(\sin x) - \left(\frac{1}{2}\right)(\cos x)$$
$$= -\sqrt{3}\sin x$$

13. Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$, since $\sin(a - b) = \sin a \cos b - \cos a \sin b$, since $\cos(a + b) = \cos a \cos b - \sin a \sin b$, and since $\cos(a - b) = \cos a \cos b + \sin a \sin b$, the expression $\frac{\sin(f + g) + \sin(f - g)}{\cos(f + g) + \cos(f - g)}$ can be simplified as follows:

$$\frac{\sin(f + g) + \sin(f - g)}{\cos(f + g) + \cos(f - g)}$$
$$= \frac{\sin f\cos g + \cos f\sin g + \sin f\cos g - \cos f\sin g}{\cos f\cos g - \sin f\sin g + \cos f\cos g + \sin f\sin g}$$
$$= \frac{2\sin f\cos g}{2\cos f\cos g} = \frac{\sin f}{\cos f}$$
$$= \tan f,\ \cos f \neq 0,\ \cos g \neq 0$$

14. If $\sin a$ and $\sin b$ are written as $\frac{y}{r}$, where y is the side opposite angles a and b in a right triangle, and r is the radius of the right triangle, the side x adjacent to angles a and b can be found with the formula $x^2 + y^2 = r^2$. Once x is determined, $\cos a$ and $\cos b$ can be written as $\frac{x}{r}$, and since the terminal arms of angles a and b lie in the first quadrant, $\cos a$ and $\cos b$ are positive. With $\sin a$, $\sin b$, $\cos a$, and $\cos b$ known, $\cos(a + b)$ can be found with the formula $\cos(a + b) = \cos a \cos b - \sin a \sin b$.

Therefore, an appropriate flow chart would be as follows:



15. The compound angle formulas used in this lesson are as follows:

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

$\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\cos(x - y) = \cos x \cos y + \sin x \sin y$

$\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

$\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

The two sine formulas are the same, except for the operators. The operator for $\sin(x + y)$ is $+$, while the operator for $\sin(x - y)$ is $-$. Remembering that the same operator is used on both the left and right sides in both equations will help you remember the formulas. Similarly, the two cosine formulas are the same, except for the operators. The operator for $\cos(x + y)$ is $-$, while the operator for $\cos(x - y)$ is $+$. Remembering that the operator on the left side is the opposite of the operator on the right side in both equations will help you remember the formulas. The two tangent formulas are the same, except for the operators in the numerator and the denominator on the right side. The operators for $\tan(x + y)$ are $+$ in the numerator and $-$ in the denominator, while the operators for $\tan(x - y)$ are $-$ in the numerator and $+$ in the denominator. Remembering that the operators in the numerator and the denominator are opposite in both equations, and that the operator in the numerator is the same as the operator on the left side, will help you remember the formulas.

16. Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$, and since $\cos(a - b) = \cos a \cos b + \sin a \sin b$, the formula

$\sin C + \sin D = 2 \sin\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right)$ can be developed as follows:

$$2 \sin\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right)$$

$$= (2)\left(\left(\sin\frac{C}{2}\right)\left(\cos\frac{D}{2}\right) + \left(\cos\frac{C}{2}\right)\left(\sin\frac{D}{2}\right)\right)$$

$$\times\left(\left(\cos\frac{C}{2}\right)\left(\cos\frac{D}{2}\right) + \left(\sin\frac{C}{2}\right)\left(\sin\frac{D}{2}\right)\right)$$

$$= (2)\left(\left(\sin\frac{C}{2}\right)\left(\cos\frac{C}{2}\right)\left(\cos^2\frac{D}{2}\right) + \left(\sin\frac{D}{2}\right)\right.$$

$$\times\left(\cos\frac{D}{2}\right)\left(\cos^2\frac{C}{2}\right) + \left(\sin\frac{D}{2}\right)\left(\cos\frac{D}{2}\right)$$

$$\left.\times\left(\sin^2\frac{C}{2}\right) + \left(\sin\frac{C}{2}\right)\left(\cos\frac{C}{2}\right)\left(\sin^2\frac{D}{2}\right)\right)$$

$$= 2\left(\sin\frac{C}{2}\right)\left(\cos\frac{C}{2}\right)\left(\cos^2\frac{D}{2} + \sin^2\frac{D}{2}\right)$$

$$+ 2\left(\sin\frac{D}{2}\right)\left(\cos\frac{D}{2}\right)\left(\cos^2\frac{C}{2} + \sin^2\frac{C}{2}\right)$$

$$= 2\left(\sin\frac{C}{2}\right)\left(\cos\frac{C}{2}\right) + 2\left(\sin\frac{D}{2}\right)\left(\cos\frac{D}{2}\right)$$

$$= \sin\left(2\left(\frac{C}{2}\right)\right) + \sin\left(2\left(\frac{D}{2}\right)\right)$$

$$= \sin C + \sin D$$

17. Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$, $\cot(x + y)$ in terms of $\cot x$ and $\cot y$ can be determined as follows:

$\cot(x + y)$

$$= \frac{1}{\tan(x + y)} = \frac{1}{\frac{\tan x + \tan y}{1 - \tan x \tan y}} = \frac{1 - \tan x \tan y}{\tan x + \tan y}$$

$$= \frac{1 - \frac{1}{\cot x \cot y}}{\frac{1}{\cot x} + \frac{1}{\cot y}} = \frac{\frac{\cot x \cot y}{\cot x \cot y} - \frac{1}{\cot x \cot y}}{\frac{\cot y}{\cot x \cot y} + \frac{\cot x}{\cot x \cot y}}$$

$$= \frac{\frac{\cot x \cot y - 1}{\cot x \cot y}}{\frac{\cot x + \cot y}{\cot x \cot y}} = \frac{\cot x \cot y - 1}{\cot x \cot y} \times \frac{\cot x \cot y}{\cot x + \cot y}$$

$$= \frac{\cot x \cot y - 1}{\cot x + \cot y}$$

18. Let $C = x + y$ and let $D = x - y$.

$\cos C + \cos D = \cos(x + y) + \cos(x - y)$

$= \cos x \cos y - \sin x \sin y$

$+ \cos x \cos y + \sin x \sin y = 2 \cos x \cos y$

$\frac{C + D}{2} = \frac{x + y + x - y}{2} = x$

$\frac{C - D}{2} = \frac{x + y - x + y}{2} = y$

So, $\cos C + \cos D = 2 \cos\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right)$.

19. Let $C = x + y$ and let $D = x - y$.

$\cos C - \cos D = \cos(x + y) - \cos(x - y)$

$= \cos x \cos y - \sin x \sin y$

$- (\cos x \cos y - \sin x \sin y) = -2 \sin x \sin y$

$\frac{C + D}{2} = \frac{x + y + x - y}{2} = x$

$\frac{C - D}{2} = \frac{x + y - x + y}{2} = y$

So, $\cos C - \cos D = -2 \sin\left(\frac{C + D}{2}\right)\sin\left(\frac{C - D}{2}\right)$.

7.3 Double Angle Formulas, pp. 407–408

1. a) Since $\sin 2\theta = 2 \sin \theta \cos \theta$,

$2 \sin 5x \cos 5x = \sin 2(5x) = \sin 10x$.

b) Since $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$,

$\cos^2 \theta - \sin^2 \theta = \cos 2\theta$.

c) Since $\cos 2\theta = 1 - 2 \sin^2 \theta$,

$1 - 2 \sin^2 3x = \cos 2(3x) = \cos 6x$.

d) Since $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$,

$\frac{2 \tan 4x}{1 - \tan^2 4x} = \tan 2(4x) = \tan 8x$.

e) Since $\sin 2\theta = 2 \sin \theta \cos \theta$,

$4 \sin \theta \cos \theta = (2)(2 \sin \theta \cos \theta) = 2 \sin 2\theta$.

f) Since $\cos 2\theta = 2 \cos^2 \theta - 1$,

$2 \cos^2 \frac{\theta}{2} - 1 = \cos 2\left(\frac{\theta}{2}\right) = \cos \theta$.

2. a) Since $\sin 2\theta = 2 \sin \theta \cos \theta$,

$2 \sin 45° \cos 45° = \sin 2(45°) = \sin 90° = 1$

b) Since $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$,

$\cos^2 30° - \sin^2 30° = \cos 2(30°) = \cos 60° = \frac{1}{2}$

c) Since $\sin 2\theta = 2 \sin \theta \cos \theta$,

$2 \sin \frac{\pi}{12} \cos \frac{\pi}{12} = \sin 2\left(\frac{\pi}{12}\right) = \sin \frac{\pi}{6} = \frac{1}{2}$

d) Since $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$,

$\cos^2 \frac{\pi}{12} - \sin^2 \frac{\pi}{12} = \cos 2\left(\frac{\pi}{12}\right) = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

e) Since $\cos 2\theta = 1 - 2 \sin^2 \theta$,

$1 - 2 \sin^2 \frac{3\pi}{8} = \cos 2\left(\frac{3\pi}{8}\right) = \cos \frac{6\pi}{8}$

$= \cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$

f) Since $\sin 2\theta = 2 \sin \theta \cos \theta$,

$2 \tan 60° \cos^2 60° = (2)\left(\frac{\sin 60°}{\cos 60°}\right)(\cos^2 60°)$

$= 2 \sin 60° \cos 60° = \sin 2(60°)$

$= \sin 120° = \frac{\sqrt{3}}{2}$

3. a) Since $\sin 2\theta = 2 \sin \theta \cos \theta$,

$\sin 4\theta = 2 \sin 2\theta \cos 2\theta$

b) Since $\cos 2\theta = 2 \cos^2 \theta - 1$,

$\cos 3x = 2 \cos^2(1.5x) - 1$

c) Since $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$,

$\tan x = \frac{2 \tan(0.5x)}{1 - \tan^2(0.5x)}$

d) Since $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$,

$\cos 6\theta = \cos^2 3\theta - \sin^2 3\theta$

e) Since $\sin 2\theta = 2 \sin \theta \cos \theta$,

$\sin x = 2 \sin(0.5x) \cos(0.5x)$

f) Since $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$,

$\tan 5\theta = \frac{2 \tan(2.5\theta)}{1 - \tan^2(2.5\theta)}$

4. Since $\cos \theta = \frac{3}{5}$, the leg adjacent to the angle θ in a right triangle has a length of 3, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$3^2 + y^2 = 5^2$

$9 + y^2 = 25$

$9 + y^2 - 9 = 25 - 9$

$$y^2 = 16$$
$$y = 4, \text{ in quadrant I}$$

Since $\sin\theta = \dfrac{\text{opposite leg}}{\text{hypotenuse}}$, $\sin\theta = \frac{4}{5}$.
Therefore, since $\sin 2\theta = 2\sin\theta\cos\theta$,

$$\sin 2\theta = (2)\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25}.$$

Also, since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$$\cos 2\theta = \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$$
$$= \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}.$$

Finally, since $\tan\theta = \dfrac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan\theta = \frac{4}{3}$.

Since $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$,

$$\tan 2\theta = \frac{(2)(\frac{4}{3})}{1-(\frac{4}{3})^2}$$
$$= \frac{\frac{8}{3}}{1-\frac{16}{9}} = \frac{\frac{8}{3}}{\frac{9}{9}-\frac{16}{9}} = \frac{\frac{8}{3}}{-\frac{7}{9}}$$
$$= \frac{8}{3} \times -\frac{9}{7} = -\frac{24}{7}$$

5. Since $\tan\theta = -\frac{7}{24}$, the leg opposite the angle θ in a right triangle has a length of 7, while the leg adjacent to the angle θ has a length of 24. For this reason, the hypotenuse of the right triangle can be calculated as follows:

$$7^2 + 24^2 = c^2$$
$$49 + 576 = c^2$$
$$625 = c^2$$
$$c = 25, \text{ in quadrant II}$$

Since $\sin\theta = \dfrac{\text{opposite leg}}{\text{hypotenuse}}$, $\sin\theta = \frac{7}{25}$, and since $\cos\theta = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos\theta = -\frac{24}{25}$. (The reason the sign is negative is because angle θ is in the second quadrant.) Therefore, since $\sin 2\theta = 2\sin\theta\cos\theta$, $\sin 2\theta = (2)(\frac{7}{25})(-\frac{24}{25}) = -\frac{336}{625}$. Also, since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$$\cos 2\theta = \left(-\frac{24}{25}\right)^2 - \left(\frac{7}{25}\right)^2$$
$$= \frac{576}{625} - \frac{49}{625} = \frac{527}{625}$$

Finally, since $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$,

$$\tan 2\theta = \frac{(2)(-\frac{7}{24})}{1-(-\frac{7}{24})^2} = \frac{-\frac{7}{12}}{1-\frac{49}{576}}$$
$$= \frac{-\frac{7}{12}}{\frac{576}{576}-\frac{49}{576}} = \frac{-\frac{7}{12}}{\frac{527}{576}}$$
$$= -\frac{7}{12} \times \frac{576}{527} = -\frac{336}{527}$$

6. Since $\sin\theta = -\frac{12}{13}$, the leg opposite the angle θ in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 + 144 - 144 = 169 - 144$$
$$x^2 = 25$$
$$x = 5, \text{ in quadrant IV}$$

Since $\cos\theta = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos\theta = \frac{5}{13}$. Therefore, since $\sin 2\theta = 2\sin\theta\cos\theta$, $\sin 2\theta = (2)(-\frac{12}{13})(\frac{5}{13}) = -\frac{120}{169}$. Also, since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$\cos 2\theta = (\frac{5}{13})^2 - (-\frac{12}{13})^2 = \frac{25}{169} - \frac{144}{169} = -\frac{119}{169}$. Finally, since $\tan\theta = \dfrac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan\theta = -\frac{12}{5}$. (The reason the sign is negative is because angle θ is in the fourth quadrant.) Since

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta},$$

$$\tan 2\theta = \frac{(2)(-\frac{12}{5})}{1-(-\frac{12}{5})^2} = \frac{-\frac{24}{5}}{1-\frac{144}{25}} = \frac{-\frac{24}{5}}{\frac{25}{25}-\frac{144}{25}} = \frac{-\frac{24}{5}}{-\frac{119}{25}}$$
$$= \frac{-\frac{24}{5}}{-\frac{119}{25}} = -\frac{24}{5} \times -\frac{25}{119} = \frac{120}{119}$$

7. Since $\cos\theta = -\frac{4}{5}$, the leg adjacent to the angle θ in a right triangle has a length of 4, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$4^2 + y^2 = 5^2$$
$$16 + y^2 = 25$$
$$16 + y^2 - 16 = 25 - 16$$
$$y^2 = 9$$
$$y = 3, \text{ in quadrant II}$$

Since $\sin\theta = \dfrac{\text{opposite leg}}{\text{hypotenuse}}$, $\sin\theta = \frac{3}{5}$.

Therefore, since $\sin 2\theta = 2\sin\theta\cos\theta$,

$\sin 2\theta = (2)(\frac{3}{5})(-\frac{4}{5}) = -\frac{24}{25}$.

Also, since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$$\cos 2\theta = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2$$
$$= \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$

Finally, since $\tan\theta = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan\theta = -\frac{3}{4}$. (The reason the sign is negative is because angle θ is in the second quadrant.) Since $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$,

$$\tan 2\theta = \frac{(2)(-\frac{3}{4})}{1 - (-\frac{3}{4})^2} = \frac{-\frac{3}{2}}{1 - \frac{9}{16}} = \frac{-\frac{3}{2}}{\frac{16}{16} - \frac{9}{16}}$$
$$= \frac{-\frac{3}{2}}{\frac{7}{16}} = -\frac{3}{2} \times \frac{16}{7} = -\frac{24}{7}$$

8. To determine the value of a, first rearrange the equation as follows:

$$2\tan x - \tan 2x + 2a = 1 - \tan 2x \tan^2 x$$
$$2\tan x - \tan 2x + 2a + \tan 2x$$
$$= 1 - \tan 2x \tan^2 x + \tan 2x$$
$$2\tan x + 2a = 1 - \tan 2x \tan^2 x + \tan 2x$$
$$2\tan x + 2a - 1 = 1 - \tan 2x \tan^2 x + \tan 2x - 1$$
$$2\tan x + 2a - 1 = \tan 2x - \tan 2x \tan^2 x$$
$$2\tan x + 2a - 1 = (\tan 2x)(1 - \tan^2 x)$$
$$\frac{2\tan x + 2a - 1}{1 - \tan^2 x} = \frac{(\tan 2x)(1 - \tan^2 x)}{1 - \tan^2 x}$$
$$\tan 2x = \frac{2\tan x + 2a - 1}{1 - \tan^2 x}$$

Since $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$, the value of $2a - 1$ must equal 0. Therefore, a can be solved for as follows:

$$2a - 1 = 0$$
$$2a - 1 + 1 = 0 + 1$$
$$2a = 1$$
$$\frac{2a}{2} = \frac{1}{2}$$
$$a = \frac{1}{2}$$

9. Jim can find the sine of $\frac{\pi}{8}$ by using the formula $\cos 2x = 1 - 2\sin^2 x$ and isolating $\sin x$ on one side of the equation as follows:

$$\cos 2x = 1 - 2\sin^2 x$$
$$\cos 2x + 2\sin^2 x = 1 - 2\sin^2 x + 2\sin^2 x$$
$$\cos 2x + 2\sin^2 x = 1$$
$$\cos 2x + 2\sin^2 x - \cos 2x = 1 - \cos 2x$$
$$2\sin^2 x = 1 - \cos 2x$$
$$\frac{2\sin^2 x}{2} = \frac{1 - \cos 2x}{2}$$
$$\sin^2 x = \frac{1 - \cos 2x}{2}$$
$$\sin x = \pm\sqrt{\frac{1 - \cos 2x}{2}}$$

The cosine of $\frac{\pi}{4}$ is $\frac{\sqrt{2}}{2}$, so

$$\sin\frac{\pi}{8} = \pm\sqrt{\frac{1 - \cos((2)(\frac{\pi}{8}))}{2}}$$
$$= \pm\sqrt{\frac{1 - \cos\frac{\pi}{4}}{2}} = \pm\sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}}$$
$$= \pm\sqrt{\frac{\frac{2}{2} - \frac{\sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{\frac{2 - \sqrt{2}}{2}}{2}}$$
$$= \pm\sqrt{\frac{2 - \sqrt{2}}{2} \times \frac{1}{2}}$$
$$= \pm\sqrt{\frac{2 - \sqrt{2}}{4}} = \pm\frac{\sqrt{2 - \sqrt{2}}}{2}$$

Since $\frac{\pi}{8}$ is in the first quadrant, the sign of $\sin\frac{\pi}{8}$ is positive. Therefore, $\sin\frac{\pi}{8} = \frac{\sqrt{2 - \sqrt{2}}}{2}$.

10. Marion can find the cosine of $\frac{\pi}{12}$ by using the formula $\cos 2x = 2\cos^2 x - 1$ and isolating $\cos x$ on one side of the equation as follows:

$$\cos 2x = 2\cos^2 x - 1$$
$$\cos 2x + 1 = 2\cos^2 x - 1 + 1$$
$$\cos 2x + 1 = 2\cos^2 x$$
$$\frac{\cos 2x + 1}{2} = \frac{2\cos^2 x}{2}$$
$$\cos^2 x = \frac{1 + \cos 2x}{2}$$
$$\cos x = \pm\sqrt{\frac{1 + \cos 2x}{2}}$$

The cosine of $\frac{\pi}{6}$ is $\frac{\sqrt{3}}{2}$, so

$$\cos\frac{\pi}{12} = \pm\sqrt{\frac{1 + \cos((2)(\frac{\pi}{12}))}{2}}$$
$$= \pm\sqrt{\frac{1 + \cos\frac{\pi}{6}}{2}} = \pm\sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}}$$
$$= \pm\sqrt{\frac{\frac{2}{2} + \frac{\sqrt{3}}{2}}{2}} = \pm\sqrt{\frac{\frac{2 + \sqrt{3}}{2}}{2}}$$
$$= \pm\sqrt{\frac{2 + \sqrt{3}}{2} \times \frac{1}{2}}$$
$$= \pm\sqrt{\frac{2 + \sqrt{3}}{4}} = \pm\frac{\sqrt{2 + \sqrt{3}}}{2}$$

Since $\frac{\pi}{12}$ is in the first quadrant, the sign of $\cos\frac{\pi}{12}$ is positive. Therefore, $\cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$.

11. a) Since $\sin 2\theta = 2\sin\theta\cos\theta$,

$\sin 4x = 2\sin 2x\cos 2x$

$= (2)(2\sin x\cos x)(\cos 2x)$. At this point, either the formula $\cos 2\theta = 2\cos^2\theta - 1$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$, or $\cos 2\theta = 1 - 2\sin^2\theta$ can be used to simplify for $\cos 2x$. If the formula $\cos 2\theta = 2\cos^2\theta - 1$ is used, the formula for $\sin 4x$ can be developed as follows:

$$\begin{aligned}\sin 4x &= (2)(2\sin x\cos x)(\cos 2x)\\ &= (2)(2\sin x\cos x)(2\cos^2 x - 1)\\ &= (4\sin x\cos x)(2\cos^2 x - 1)\\ &= 8\cos^3 x\sin x - 4\sin x\cos x\end{aligned}$$

If the formula $\cos 2\theta = \cos^2\theta - \sin^2\theta$ is used, the formula for $\sin 4x$ can be developed as follows:

$$\begin{aligned}\sin 4x &= (2)(2\sin x\cos x)(\cos 2x)\\ &= (2)(2\sin x\cos x)(\cos^2 x - \sin^2 x)\\ &= (4\sin x\cos x)(\cos^2 x - \sin^2 x)\\ &= 4\cos^3 x\sin x - 4\sin^3 x\cos x\end{aligned}$$

If the formula $\cos 2\theta = 1 - 2\sin^2\theta$ is used, the formula for $\sin 4x$ can be developed as follows:

$$\begin{aligned}\sin 4x &= (2)(2\sin x\cos x)(\cos 2x)\\ &= (2)(2\sin x\cos x)(1 - 2\sin^2 x)\\ &= (4\sin x\cos x)(1 - 2\sin^2 x)\\ &= 4\sin x\cos x - 8\sin^3 x\cos x\end{aligned}$$

b) The value of $\sin\frac{2\pi}{3}$ is $\frac{\sqrt{3}}{2}$. Using the formula $\sin 4x = 4\sin x\cos x - 8\sin^3 x\cos x$, $\sin\frac{2\pi}{3} = \sin\frac{8\pi}{3}$ can be verified as follows:

$$\sin 4x = 4\sin x\cos x - 8\sin^3 x\cos x$$

$$\sin 4\left(\frac{2\pi}{3}\right) = 4\sin\frac{2\pi}{3}\cos\frac{2\pi}{3} - 8\sin^3\frac{2\pi}{3}\cos\frac{2\pi}{3}$$

$$\sin\frac{8\pi}{3} = (4)\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) - (8)\left(\frac{\sqrt{3}}{2}\right)^3\left(-\frac{1}{2}\right)$$

$$\sin\frac{8\pi}{3} = -\frac{4\sqrt{3}}{4} - (-4)\left(\frac{3\sqrt{3}}{8}\right)$$

$$\sin\frac{8\pi}{3} = -\frac{4\sqrt{3}}{4} - \left(-\frac{3\sqrt{3}}{2}\right)$$

$$\sin\frac{8\pi}{3} = -\frac{4\sqrt{3}}{4} - \left(-\frac{6\sqrt{3}}{4}\right)$$

$$\sin\frac{8\pi}{3} = -\frac{4\sqrt{3}}{4} + \frac{6\sqrt{3}}{4}$$

$$\sin\frac{8\pi}{3} = \frac{2\sqrt{3}}{4}$$

$$\sin\frac{8\pi}{3} = \frac{\sqrt{3}}{2}$$

12. a) Since $\sin(a + b) = \sin a\cos b + \cos a\sin b$, $\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$. Since $\sin 2\theta = 2\sin\theta\cos\theta$,

$\sin 3\theta = (2\sin\theta\cos\theta)(\cos\theta) + \cos 2\theta\sin\theta$

$= 2\cos^2\theta\sin\theta + \cos 2\theta\sin\theta$. At this point, either the formula $\cos 2\theta = 2\cos^2\theta - 1$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$, or $\cos 2\theta = 1 - 2\sin^2\theta$ can be used to simplify for $\cos 2\theta$. If the formula $\cos 2\theta = 2\cos^2\theta - 1$ is used, the formula for $\sin 3\theta$ can be developed as follows:

$$\sin 3\theta = 2\cos^2\theta\sin\theta + \cos 2\theta\sin\theta$$
$$\sin 3\theta = 2\cos^2\theta\sin\theta + (2\cos^2\theta - 1)(\sin\theta)$$
$$\sin 3\theta = 2\cos^2\theta\sin\theta + 2\cos^2\theta\sin\theta - \sin\theta$$
$$\sin 3\theta = 4\cos^2\theta\sin\theta - \sin\theta$$

If the formula $\cos 2\theta = \cos^2\theta - \sin^2\theta$ is used, the formula for $\sin 3\theta$ can be developed as follows:

$$\sin 3\theta = 2\cos^2\theta\sin\theta + \cos 2\theta\sin\theta$$
$$\sin 3\theta = 2\cos^2\theta\sin\theta + (\cos^2\theta - \sin^2\theta)(\sin\theta)$$
$$\sin 3\theta = 2\cos^2\theta\sin\theta + \cos^2\theta\sin\theta - \sin^3\theta$$
$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

If the formula $\cos 2\theta = 1 - 2\sin^2\theta$ is used, the formula for $\sin 3\theta$ can be developed as follows:

$$\sin 3\theta = 2\cos^2\theta\sin\theta + \cos 2\theta\sin\theta$$
$$\sin 3\theta = 2\cos^2\theta\sin\theta + (1 - 2\sin^2\theta)(\sin\theta)$$
$$\sin 3\theta = 2\cos^2\theta\sin\theta + \sin\theta - 2\sin^3\theta$$

b) Since $\cos(a + b) = \cos a\cos b - \sin a\sin b$, $\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta\cos\theta - \sin\theta\sin 2\theta$. Since $\sin 2\theta = 2\sin\theta\cos\theta$,

$\cos 3\theta = \cos 2\theta\cos\theta - (\sin\theta)(2\sin\theta\cos\theta)$

$= \cos 2\theta\cos\theta - 2\sin^2\theta\cos\theta$. At this point, either the formula $\cos 2\theta = 2\cos^2\theta - 1$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$, or $\cos 2\theta = 1 - 2\sin^2\theta$ can be used to simplify for $\cos 2\theta$. If the formula $\cos 2\theta = 2\cos^2\theta - 1$ is used, the formula for $\cos 3\theta$ can be developed as follows:

$$\cos 3\theta = \cos 2\theta\cos\theta - 2\sin^2\theta\cos\theta$$
$$\cos 3\theta = (2\cos^2\theta - 1)(\cos\theta) - 2\sin^2\theta\cos\theta$$
$$\cos 3\theta = 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$

If the formula $\cos 2\theta = \cos^2\theta - \sin^2\theta$ is used, the formula for $\cos 3\theta$ can be developed as follows:

$$\cos 3\theta = \cos 2\theta\cos\theta - 2\sin^2\theta\cos\theta$$
$$\cos 3\theta = (\cos^2\theta - \sin^2\theta)(\cos\theta) - 2\sin^2\theta\cos\theta$$
$$\cos 3\theta = \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta$$
$$\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$$

If the formula $\cos 2\theta = 1 - 2\sin^2\theta$ is used, the formula for $\cos 3\theta$ can be developed as follows:

$$\cos 3\theta = \cos 2\theta \cos\theta - 2\sin^2\theta\cos\theta$$
$$\cos 3\theta = (1 - 2\sin^2\theta)(\cos\theta) - 2\sin^2\theta\cos\theta$$
$$\cos 3\theta = \cos\theta - 2\sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta$$
$$\cos 3\theta = \cos\theta - 4\sin^2\theta\cos\theta$$

c) Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\tan 3\theta = \tan(2\theta + \theta) = \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta \tan\theta}$$

Since $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$,

$$\tan 3\theta = \frac{\dfrac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)(\tan\theta)}$$
$$= \frac{\dfrac{2\tan\theta}{1-\tan^2\theta} + \dfrac{\tan\theta - \tan^3\theta}{1-\tan^2\theta}}{1 - \dfrac{2\tan^2\theta}{1-\tan^2\theta}}$$
$$= \frac{\dfrac{2\tan\theta}{1-\tan^2\theta} + \dfrac{\tan\theta - \tan^3\theta}{1-\tan^2\theta}}{\dfrac{1-\tan^2\theta}{1-\tan^2\theta} - \dfrac{2\tan^2\theta}{1-\tan^2\theta}}$$
$$= \frac{\dfrac{2\tan\theta + \tan\theta - \tan^3\theta}{1-\tan^2\theta}}{\dfrac{1-\tan^2\theta - 2\tan^2\theta}{1-\tan^2\theta}}$$
$$= \frac{\dfrac{3\tan\theta - \tan^3\theta}{1-\tan^2\theta}}{\dfrac{1-3\tan^2\theta}{1-\tan^2\theta}}$$
$$= \frac{3\tan\theta - \tan^3\theta}{1-\tan^2\theta} \times \frac{1-\tan^2\theta}{1-3\tan^2\theta}$$
$$= \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

13. a) Since $\sin^2 x = \frac{8}{9}$, and since the angle x is in the second quadrant, $\sin x = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{\sqrt{9}} = \frac{2\sqrt{2}}{3}$.

Since $\sin x = \frac{2\sqrt{2}}{3}$, the leg opposite the angle x in a right triangle has a length of $2\sqrt{2}$, while the hypotenuse of the right triangle has a length of 3. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + (2\sqrt{2})^2 = 3^2$$
$$x^2 + 8 = 9$$
$$x^2 + 8 - 8 = 9 - 8$$
$$x^2 = 1$$
$$x = -1, \text{ in quadrant II}$$

Since $\cos x = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, and since the angle x is in the second quadrant, $\cos x = -\frac{1}{3}$.

Since $\sin 2\theta = 2\sin\theta\cos\theta$,

$$\sin 2x = 2\sin x\cos x = (2)\left(\frac{2\sqrt{2}}{3}\right)\left(-\frac{1}{3}\right) = -\frac{4\sqrt{2}}{9}$$

b) Since $\sin^2 x = \frac{8}{9}$, and since the angle x is in the second quadrant, $\sin x = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{\sqrt{9}} = \frac{2\sqrt{2}}{3}$.

Since $\sin x = \frac{2\sqrt{2}}{3}$, the leg opposite the angle x in a right triangle has a length of $2\sqrt{2}$, while the hypotenuse of the right triangle has a length of 3. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + (2\sqrt{2})^2 = 3^2$$
$$x^2 + 8 = 9$$
$$x^2 + 8 - 8 = 9 - 8$$
$$x^2 = 1$$
$$x = -1, \text{ in quadrant II}$$

Since $\cos x = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, and since the angle x is in the second quadrant, $\cos x = -\frac{1}{3}$. Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= \left(-\frac{1}{3}\right)^2 - \left(\frac{2\sqrt{2}}{3}\right)^2$$
$$= \frac{1}{9} - \frac{8}{9} = -\frac{7}{9}$$

(The formulas $\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 2\theta = 1 - 2\sin^2\theta$ could also have been used.)

c) Since $\sin^2 x = \frac{8}{9}$, and since the angle x is in the second quadrant, $\sin x = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{\sqrt{9}} = \frac{2\sqrt{2}}{3}$.

Since $\sin x = \frac{2\sqrt{2}}{3}$, the leg opposite the angle x in a right triangle has a length of $2\sqrt{2}$, while the hypotenuse of the right triangle has a length of 3. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + (2\sqrt{2})^2 = 3^2$$
$$x^2 + 8 = 9$$
$$x^2 + 8 - 8 = 9 - 8$$
$$x^2 = 1$$
$$x = -1, \text{ in quadrant II}$$

Since $\cos x = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, and since the angle x is in the second quadrant, $\cos x = -\frac{1}{3}$. Since $\cos 2\theta = 2\cos^2\theta - 1$, $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$, and since $\cos x = -\frac{1}{3}$, $-\frac{1}{3} = 2\cos^2\frac{\theta}{2} - 1$. The value of $\cos\frac{\theta}{2}$ can now be determined as follows:

$$-\frac{1}{3} = 2\cos^2\frac{\theta}{2} - 1$$

$$-\frac{1}{3} + 1 = 2\cos^2\frac{\theta}{2} - 1 + 1$$

$$\frac{2}{3} = 2\cos^2\frac{\theta}{2}$$

$$\frac{\frac{2}{3}}{2} = \frac{2\cos^2\frac{\theta}{2}}{2}$$

$$\cos^2\frac{\theta}{2} = \frac{2}{3} \times \frac{1}{2}$$

$$\cos^2\frac{\theta}{2} = \frac{2}{6} = \frac{1}{3}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1}{3}} = \frac{\sqrt{1}}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

d) Since $\sin^2 x = \frac{8}{9}$, and since the angle x is in the second quadrant, $\sin x = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{\sqrt{9}} = \frac{2\sqrt{2}}{3}$. Since $\sin x = \frac{2\sqrt{2}}{3}$, the leg opposite the angle x in a right triangle has a length of $2\sqrt{2}$, while the hypotenuse of the right triangle has a length of 3. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + (2\sqrt{2})^2 = 3^2$$

$$x^2 + 8 = 9$$

$$x^2 + 8 - 8 = 9 - 8$$

$$x^2 = 1$$

$$x = -1, \text{ in quadrant II}$$

Since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, and since the angle x is in the second quadrant, $\cos x = -\frac{1}{3}$. Since $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$,

$$\sin 3x = 3\cos^2 x\sin x - \sin^3 x$$

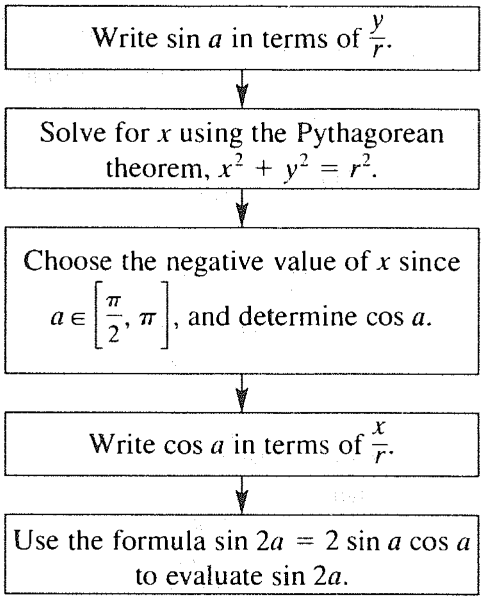
$$= (3)\left(-\frac{1}{3}\right)^2\left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{2\sqrt{2}}{3}\right)^3$$

$$= (3)\left(\frac{1}{9}\right)\left(\frac{2\sqrt{2}}{3}\right) - \frac{16\sqrt{2}}{27}$$

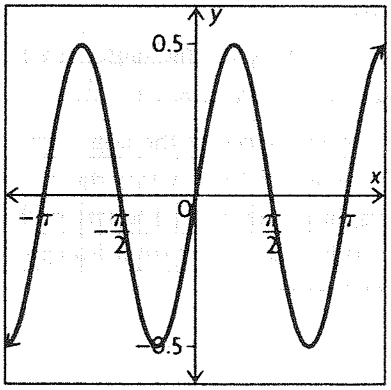
$$= \frac{6\sqrt{2}}{27} - \frac{16\sqrt{2}}{27}$$

$$= -\frac{10\sqrt{2}}{27}$$

14. If $\sin a$ is written as $\frac{y}{r}$, where y is the side opposite angle a in a right triangle, and r is the radius of the right triangle, the side x adjacent to angle a can be found with the formula $x^2 + y^2 = r^2$. Once x is determined, $\cos a$ can be written as $\frac{x}{r}$, and since the terminal arm of angle a lies in the second quadrant, $\cos a$ is negative. With $\sin a$ and $\cos a$ known, $\sin 2a$ can be found with the formula $\sin 2\theta = 2\sin\theta\cos\theta$, Therefore, an appropriate flow chart would be as follows:

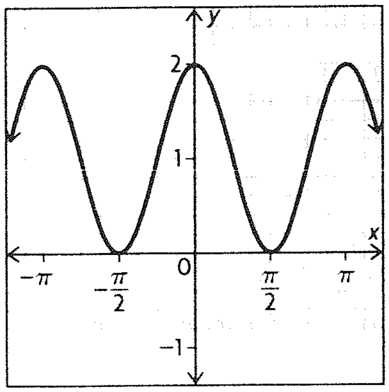


15. a) Since $\sin 2\theta = 2\sin\theta\cos\theta$, $\sin 2x = 2\sin x\cos x$. For this reason, $\frac{\sin 2x}{2} = \frac{2\sin x\cos x}{2} = \sin x\cos x$. Therefore, the graph of $f(x) = \sin x\cos x$ is the same as that of $f(x) = \frac{\sin 2x}{2}$. The graph of $f(x) = \frac{\sin 2x}{2}$ can be obtained by vertically compressing $f(x) = \sin x$ by a factor of $\frac{1}{2}$ and horizontally compressing it by a factor of $\frac{1}{2}$. The graph is shown below:



b) Since $\cos 2\theta = 2\cos^2\theta - 1$, $\cos 2x = 2\cos^2 x - 1$. For this reason, $\cos 2x + 1 = 2\cos^2 x - 1 + 1 = 2\cos^2 x$.

Therefore, the graph of $f(x) = 2\cos^2 x$ is the same as that of $f(x) = \cos 2x + 1$. The graph of $f(x) = \cos 2x + 1$ can be obtained by horizontally compressing $f(x) = \cos x$ by a factor of $\frac{1}{2}$ and vertically translating it 1 unit up. The graph is shown below:

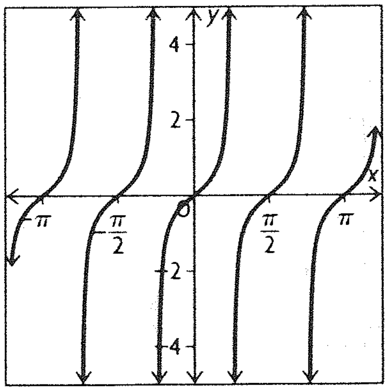


c) Since $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$, $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$.

For this reason, $\frac{\tan 2x}{2} = \frac{\frac{2\tan x}{1-\tan^2 x}}{2}$

$= \frac{2\tan x}{1-\tan^2 x} \times \frac{1}{2} = \frac{\tan x}{1-\tan^2 x}$. Therefore, the graph of $f(x) = \frac{\tan x}{1-\tan^2 x}$ is the same as that of $f(x) = \frac{\tan 2x}{2}$.

The graph of $f(x) = \frac{\tan 2x}{2}$ can be obtained by vertically compressing $f(x) = \tan x$ by a factor of $\frac{1}{2}$ and horizontally compressing it by a factor of $\frac{1}{2}$. The graph is shown below:



16. a) To eliminate A from the equations $x = \tan 2A$ and $y = \tan A$ to find an equation that relates x to y, first take the $\tan^{-1}$ of both sides of both equations and solve for A as follows:

$$x = \tan 2A$$
$$\tan^{-1} x = \tan^{-1}(\tan 2A)$$
$$\tan^{-1} x = 2A$$
$$A = \frac{\tan^{-1} x}{2}$$
$$y = \tan A$$
$$\tan^{-1} y = \tan^{-1}(\tan A)$$
$$A = \tan^{-1} y$$

Since $A = \frac{\tan^{-1} x}{2}$ and $A = \tan^{-1} y$,

$$\frac{\tan^{-1} x}{2} = \tan^{-1} y$$

b) To eliminate A from the equations $x = \cos 2A$ and $y = \cos A$ to find an equation that relates x to y, first take the $\cos^{-1}$ of both sides of both equations and solve for A as follows:

$$x = \cos 2A$$
$$\cos^{-1} x = \cos^{-1}(\cos 2A)$$
$$\cos^{-1} x = 2A$$
$$A = \frac{\cos^{-1} x}{2}$$
$$y = \cos A$$
$$\cos^{-1} y = \cos^{-1}(\cos A)$$
$$A = \cos^{-1} y$$

Since $A = \frac{\cos^{-1} x}{2}$ and $A = \cos^{-1} y$,

$$\frac{\cos^{-1} x}{2} = \cos^{-1} y$$

c) To eliminate A from the equations $x = \cos 2A$ and $y = \csc A$ to find an equation that relates x to y, first take the $\cos^{-1}$ of both sides of the first equation and the $\csc^{-1}$ of both sides of the second equation and solve for A as follows:

$$x = \cos 2A$$
$$\cos^{-1} x = \cos^{-1}(\cos 2A)$$
$$\cos^{-1} x = 2A$$
$$A = \frac{\cos^{-1} x}{2}$$
$$A = \csc A$$
$$\csc^{-1} y = \csc^{-1}(\csc A)$$
$$A = \csc^{-1} y$$

Since $A = \frac{\cos^{-1} x}{2}$ and $A = \cos^{-1} y$,

$$\frac{\cos^{-1} x}{2} = \csc^{-1} y, \text{ or } \frac{\cos^{-1} x}{2} = \sin^{-1}\left(\frac{1}{y}\right)$$

d) To eliminate A from the equations $x = \sin 2A$ and $y = \sec 4A$ to find an equation that relates x to y, first take the $\sin^{-1}$ of both sides of the first equation and the $\sec^{-1}$ of both sides of the second equation and solve for A as follows:

$$\begin{aligned}
x &= \sin 2A\\
\sin^{-1} x &= \sin^{-1}(\sin 2A)\\
\sin^{-1} x &= 2A\\
A &= \frac{\sin^{-1} x}{2}\\
y &= \sec 4A\\
\sec^{-1} y &= \sec^{-1}(\sec 4A)\\
\sec^{-1} y &= 4A\\
A &= \frac{\sec^{-1} y}{4}
\end{aligned}$$

Since $A = \dfrac{\sin^{-1} x}{2}$ and $A = \dfrac{\sec^{-1} y}{4}$,

$\dfrac{\sin^{-1} x}{2} = \dfrac{\sec^{-1} y}{4}$, or $\dfrac{\sin^{-1} x}{2} = \dfrac{\cos^{-1}\left(\frac{1}{y}\right)}{4}$

17. a) Since $\cos 2\theta = 1 - 2\sin^2\theta$, the equation $\cos 2x = \sin x$ can be rewritten $1 - 2\sin^2 x = \sin x$. This equation can be solved as follows:

$$\begin{aligned}
1 - 2\sin^2 x &= \sin x\\
1 - 2\sin^2 x + 2\sin^2 x - 1 &= \sin x + 2\sin^2 x - 1\\
2\sin^2 x + \sin x - 1 &= 0\\
(2\sin x - 1)(\sin x + 1) &= 0\\
2\sin x - 1 &= 0\\
2\sin x - 1 + 1 &= 0 + 1\\
2\sin x &= 1\\
\frac{2\sin x}{2} &= \frac{1}{2}\\
\sin x &= \frac{1}{2}\\
x &= \frac{\pi}{6} \text{ or } \frac{5\pi}{6}\\
\text{or } \sin x + 1 &= 0\\
\sin x + 1 - 1 &= 0 - 1\\
\sin x &= -1\\
x &= \frac{3\pi}{2}
\end{aligned}$$

Therefore, $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$, or $\dfrac{3\pi}{2}$.

b) Since $\sin 2\theta = 2\sin\theta\cos\theta$, and since $\cos 2\theta = 2\cos^2\theta - 1$, the equation $\sin 2x - 1 = \cos 2x$ can be rewritten $2\sin x\cos x - 1 = 2\cos^2 x - 1$. This equation can be solved as follows:

$$\begin{aligned}
2\sin x\cos x - 1 &= 2\cos^2 x - 1\\
2\sin x\cos x - 1 + 1 &= 2\cos^2 x - 1 + 1\\
2\sin x\cos x &= 2\cos^2 x
\end{aligned}$$

$$2\sin x\cos x - 2\sin x\cos x$$
$$= 2\cos^2 x - 2\sin x\cos x$$

$$\begin{aligned}
2\cos^2 x - 2\sin x\cos x &= 0\\
(2\cos x)(\cos x - \sin x) &= 0\\
2\cos x &= 0\\
\frac{2\cos x}{2} &= \frac{0}{2}\\
\cos x &= 0\\
x &= \frac{\pi}{2} \text{ or } \frac{3\pi}{2}\\
\text{or } \cos x - \sin x &= 0\\
\cos x - \sin x + \sin x &= 0 + \sin x\\
\cos x &= \sin x\\
x &= \frac{\pi}{4} \text{ or } \frac{5\pi}{4}
\end{aligned}$$

Therefore, $x = \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{5\pi}{4}$, or $\dfrac{3\pi}{2}$.

18. First write $\sin\theta$ and $\cos\theta$ in terms of $\tan\theta$.

$$\begin{aligned}
\sin\theta &= \frac{\sin\theta}{\cos\theta} \times \cos\theta\\
&= \frac{\tan\theta}{\sec\theta}\\
&= \frac{\tan\theta}{\sqrt{1 + \tan^2\theta}}
\end{aligned}$$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, so

$$\begin{aligned}
\cos\theta &= \frac{\sin\theta}{\tan\theta}\\
&= \frac{\frac{\tan\theta}{\sqrt{1 + \tan^2\theta}}}{\tan\theta}\\
&= \frac{1}{\sqrt{1 + \tan^2\theta}}
\end{aligned}$$

a)
$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta\\
&= 2\left(\frac{\tan\theta}{\sqrt{1 + \tan^2\theta}}\right)\left(\frac{1}{\sqrt{1 + \tan^2\theta}}\right)\\
&= \frac{2\tan\theta}{1 + \tan^2\theta}
\end{aligned}$$

b)
$$\begin{aligned}
\cos 2\theta &= \cos^2\theta - \sin^2\theta\\
&= \left(\frac{1}{\sqrt{1 + \tan^2\theta}}\right)^2 - \left(\frac{\tan\theta}{\sqrt{1 + \tan^2\theta}}\right)^2\\
&= \frac{1 - \tan^2\theta}{1 + \tan^2\theta}
\end{aligned}$$

c) Use the results from parts **a)** and **b)**

$$\frac{\sin 2\theta}{1 + \cos 2\theta} = \frac{\frac{2\tan\theta}{1 + \tan^2\theta}}{1 + \frac{1 - \tan^2\theta}{1 + \tan^2\theta}}$$

$$= \frac{\frac{2\tan\theta}{1+\tan^2\theta}}{\frac{1+\tan^2\theta+1-\tan^2\theta}{1+\tan^2\theta}}$$

$$= \frac{\frac{2\tan\theta}{1+\tan^2\theta}}{\frac{2}{1+\tan^2\theta}}$$

$$= \frac{2\tan\theta}{1+\tan^2\theta}\times\frac{1+\tan^2\theta}{2}$$

$$= \tan\theta$$

d) Use the results from parts **a)** and **b)**

$$\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{1-\frac{1-\tan^2\theta}{1+\tan^2\theta}}{\frac{2\tan\theta}{1+\tan^2\theta}}$$

$$= \frac{\frac{1+\tan^2\theta-1+\tan^2\theta}{1+\tan^2\theta}}{\frac{2\tan\theta}{1+\tan^2\theta}}$$

$$= \frac{\frac{2\tan^2\theta}{1+\tan^2\theta}}{\frac{2\tan\theta}{1+\tan^2\theta}}$$

$$= \frac{2\tan^2\theta}{1+\tan^2\theta}\times\frac{1+\tan^2\theta}{2\tan\theta}$$

$$= \tan\theta$$

Mid-Chapter Review, p. 411

1. a) Since $\cos(2\pi-\theta)=\cos\theta$, $\cos\theta=\cos(2\pi-\theta)$. Therefore,

$$\cos\frac{\pi}{16}=\cos\left(2\pi-\frac{\pi}{16}\right)$$

$$=\cos\left(\frac{32\pi}{16}-\frac{\pi}{16}\right)$$

$$=\cos\frac{31\pi}{16}$$

b) Since $\sin(\pi-\theta)=\sin\theta$, $\sin\theta=\sin(\pi-\theta)$.

Therefore, $\sin\frac{7\pi}{9}=\sin\left(\pi-\frac{7\pi}{9}\right)$

$$=\sin\left(\frac{9\pi}{9}-\frac{7\pi}{9}\right)=\sin\frac{2\pi}{9}$$

c) Since $\tan(\pi+\theta)=\tan\theta$, $\tan\theta=\tan(\pi+\theta)$.

Therefore, $\tan\frac{9\pi}{10}=\tan\left(\pi+\frac{9\pi}{10}\right)$

$$=\tan\left(\frac{10\pi}{10}+\frac{9\pi}{10}\right)=\tan\frac{19\pi}{10}$$

d) Since $-\cos\theta=\cos(\pi+\theta)$,

$$-\cos\frac{2\pi}{5}=\cos\left(\pi+\frac{2\pi}{5}\right)$$

$$=\cos\left(\frac{5\pi}{5}+\frac{2\pi}{5}\right)=\cos\frac{7\pi}{5}$$

e) Since $-\sin\theta=\sin(\pi+\theta)$,

$$-\sin\frac{9\pi}{7}=\sin\left(\pi+\frac{9\pi}{7}\right)$$

$$=\sin\left(\frac{7\pi}{7}+\frac{9\pi}{7}\right)=\sin\frac{16\pi}{7}$$

Since $\sin\theta=\sin(\theta-2\pi)$,

$$\sin\frac{16\pi}{7}=\sin\left(\frac{16\pi}{7}-2\pi\right)$$

$$=\sin\left(\frac{16\pi}{7}-\frac{14\pi}{7}\right)$$

$$=\sin\frac{2\pi}{7}$$

Therefore, $-\sin\frac{9\pi}{7}=\sin\frac{2\pi}{7}$.

f) Since $\tan(\pi+\theta)=\tan\theta$, $\tan\theta=\tan(\pi+\theta)$.

Therefore, $\tan\frac{3\pi}{4}=\tan\left(\pi+\frac{3\pi}{4}\right)$

$$=\tan\left(\frac{4\pi}{4}+\frac{3\pi}{4}\right)=\tan\frac{7\pi}{4}$$

2. Since $\cos\theta=\sin\left(\theta+\frac{\pi}{2}\right)$, the equation $y=-6\cos\left(x+\frac{\pi}{2}\right)+4$ can be rewritten $y=-6\sin\left(x+\frac{\pi}{2}+\frac{\pi}{2}\right)+4$ $=-6\sin(x+\pi)+4$. Since a horizontal translation of π to the left or right is equivalent to a reflection in the x-axis, the equation $y=-6\sin(x+\pi)+4$ can be rewritten $y=6\sin x+4$. Therefore, the equation $y=-6\cos\left(x+\frac{\pi}{2}\right)+4$ can be rewritten $y=6\sin x+4$.

3. a) Since $\cos(a+b)=\cos a\cos b-\sin a\sin b$,

$$\cos\left(x+\frac{5\pi}{3}\right)=\cos x\cos\frac{5\pi}{3}-\sin x\sin\frac{5\pi}{3}$$

$$=(\cos x)\left(\frac{1}{2}\right)-(\sin x)\left(-\frac{\sqrt{3}}{2}\right)$$

$$=\frac{1}{2}\cos x+\frac{\sqrt{3}}{2}\sin x$$

b) Since $\sin(a + b) \sin a \cos b + \cos a \sin b$,

$$\sin\left(x + \frac{5\pi}{6}\right) = (\sin x)\left(\cos\frac{5\pi}{6}\right) + (\cos x)\left(\sin\frac{5\pi}{6}\right)$$
$$= (\sin x)\left(-\frac{\sqrt{3}}{2}\right) + (\cos x)\left(\frac{1}{2}\right)$$
$$= \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$$

c) Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\tan\left(x + \frac{5\pi}{4}\right) = \frac{\tan x + \tan\frac{5\pi}{4}}{1 - \tan x \tan\frac{5\pi}{4}}$$
$$= \frac{\tan x + 1}{1 - (\tan x)(1)}$$
$$= \frac{1 + \tan x}{1 - \tan x}$$

d) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\cos\left(x + \frac{4\pi}{3}\right) = \cos x \cos\frac{4\pi}{3} - \sin x \sin\frac{4\pi}{3}$$
$$= (\cos x)\left(-\frac{1}{2}\right) - (\sin x)\left(-\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x$$

4. a) Since $\sin(a - b) = \sin a \cos b - \cos a \cos b$,

$$\sin\left(x - \frac{11\pi}{6}\right) = \sin x \cos\frac{11\pi}{6} - \cos x \sin\frac{11\pi}{6}$$
$$= (\sin x)\left(\frac{\sqrt{3}}{2}\right) - (\cos x)\left(-\frac{1}{2}\right)$$
$$= \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$$

b) Since $\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$,

$$\tan\left(x - \frac{\pi}{3}\right) = \frac{\tan x - \tan\frac{\pi}{3}}{1 + \tan x \tan\frac{\pi}{3}}$$
$$= \frac{\tan x - \sqrt{3}}{1 + (\tan x)(\sqrt{3})} = \frac{\tan x - \sqrt{3}}{1 + \sqrt{3}\tan x}$$

c) Since $\cos(a - b) = \cos a \cos b + \sin a \sin b$,

$$\cos\left(x - \frac{7\pi}{4}\right) = (\cos x)\left(\cos\frac{7\pi}{4}\right) + (\sin x)\left(\sin\frac{7\pi}{4}\right)$$
$$= (\cos x)\left(\frac{\sqrt{2}}{2}\right) + (\sin x)\left(-\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x$$

d) Since $\sin(a - b) - \sin a \cos b - \cos a \sin b$,

$$\sin\left(x - \frac{2\pi}{3}\right) = (\sin x)\left(\cos\frac{2\pi}{3}\right) - (\cos x)\left(\sin\frac{2\pi}{3}\right)$$
$$= (\sin x)\left(-\frac{1}{2}\right) - (\cos x)\left(\frac{\sqrt{3}}{2}\right)$$
$$= -\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x$$

5. a) Since $\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$,

$$\frac{\tan\frac{8\pi}{9} - \tan\frac{5\pi}{9}}{1 + \tan\frac{8\pi}{9}\tan\frac{5\pi}{9}} = \tan\left(\frac{8\pi}{9} - \frac{5\pi}{9}\right)$$
$$= \tan\frac{3\pi}{9} = \tan\frac{\pi}{3} = \sqrt{3}$$

b) Since $\sin(a - b) = \sin a \cos b - \cos a \sin b$,

$$\sin\frac{299\pi}{298}\cos\frac{\pi}{298} - \cos\frac{299\pi}{298}\sin\frac{\pi}{298}$$
$$= \sin\left(\frac{299\pi}{298} - \frac{\pi}{298}\right) = \sin\frac{298\pi}{298}$$
$$= \sin\pi = 0$$

c) Since $\sin(a - b) = \sin a \cos b - \cos a \sin b$,

$$\sin 50° \cos 20° - \cos 50° \sin 20°$$
$$= \sin(50° - 20°) = \sin 30° = \frac{1}{2}$$

d) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\sin\frac{3\pi}{8}\cos\frac{\pi}{8} + \cos\frac{3\pi}{8}\sin\frac{\pi}{8} = \sin\left(\frac{3\pi}{8} + \frac{\pi}{8}\right)$$
$$= \sin\frac{4\pi}{8} = \sin\frac{\pi}{2} = 1$$

6. a) Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\frac{2\tan x}{1 - \tan^2 x} = \frac{\tan x + \tan x}{1 - (\tan x)(\tan x)}$$
$$= \tan(x + x) = \tan 2x$$

b) Since $\sin(a + b) = \sin a \cos b + \cos a \sin b$,

$$\sin\frac{x}{5}\cos\frac{4x}{5} + \cos\frac{x}{5}\sin\frac{4x}{5} = \sin\left(\frac{x}{5} + \frac{4x}{5}\right)$$
$$= \sin\frac{5x}{5} = \sin x$$

c) $$\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$$
$$= (0)\cos x + (1)\sin x = \sin x$$

d) $$\sin\left(\frac{\pi}{2} + x\right) = \sin\frac{\pi}{2}\cos x + \cos\frac{\pi}{2}\sin x$$
$$= (1)\cos x + (0)\sin x = \cos x$$

e) $\cos\left(\frac{\pi}{4} + x\right) + \cos\left(\frac{\pi}{4} + x\right) = 2\cos\left(\frac{\pi}{4} + x\right)$

$= 2\left[\cos\frac{\pi}{4}\cos x - \sin\frac{\pi}{4}\sin x\right]$

$= 2\left[\left(\frac{\sqrt{2}}{2}\right)\cos x - \left(\frac{\sqrt{2}}{2}\right)\sin x\right]$

$= \sqrt{2}(\cos x - \sin x)$

f) $\tan\left(x - \frac{\pi}{4}\right) = \dfrac{\tan x - \tan\left(\frac{\pi}{4}\right)}{1 + \tan x \tan\left(\frac{\pi}{4}\right)} = \dfrac{\tan x - 1}{1 + \tan x}$

7. $a = \sqrt{3}$ and $b = -3$, so

$R = \sqrt{(\sqrt{3})^2 + (-3)^2} = \sqrt{12} = 2\sqrt{3}$

$\cos\alpha = \dfrac{\sqrt{3}}{2\sqrt{3}} = \dfrac{1}{2}$

$\sin\alpha = \dfrac{-3}{2\sqrt{3}} = \dfrac{-\sqrt{3}}{2}$

Since $\cos\alpha$ is positive and $\sin\alpha$ is negative, α is in the fourth quadrant. $\alpha = -\frac{\pi}{3}$

So, $\sqrt{3}\cos x - 3\sin x = 2\sqrt{3}\cos\left(x + \frac{\pi}{3}\right)$.

8. a) Since $\cos 2\theta = 2\cos^2\theta - 1$,

$2\cos^2\frac{2\pi}{3} - 1 = \cos\left((2)\left(\frac{2\pi}{3}\right)\right)$

$= \cos\frac{4\pi}{3} = -\frac{1}{2}$

b) Since $\sin 2\theta = 2\sin\theta\cos\theta$,

$2\sin\frac{11\pi}{12}\cos\frac{11\pi}{12} = \sin\left((2)\left(\frac{11\pi}{12}\right)\right)$

$= \sin\frac{22\pi}{12} = \sin\frac{11\pi}{6} = -\frac{1}{2}$

c) Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$\cos^2\frac{7\pi}{8} - \sin^2\frac{7\pi}{8} = \cos\left((2)\left(\frac{7\pi}{8}\right)\right)$

$= \cos\frac{14\pi}{8} = \cos\frac{7\pi}{8} = \frac{\sqrt{2}}{2}$

d) Since $\cos 2\theta = 1 - 2\sin^2\theta$,

$1 - 2\sin^2\frac{\pi}{2} = \cos\left((2)\left(\frac{\pi}{2}\right)\right) = \cos\pi = -1$

9. a) Since $\cos^2 x = \frac{10}{11}$, $\cos x = \pm\sqrt{\frac{10}{11}}$ $= \pm\frac{\sqrt{10}}{\sqrt{11}} = \pm\frac{\sqrt{110}}{11}$, and since angle x is in the third quadrant, $\cos x$ is negative. For this reason, $\cos x = -\frac{\sqrt{110}}{11}$. Since $\cos x = -\frac{\sqrt{110}}{11}$, the leg adjacent to the angle x in a right triangle has a length of $\sqrt{110}$, while the hypotenuse of the right triangle has a length of 11. For this reason, the other leg of the right triangle can be calculated as follows:

$(\sqrt{110})^2 + y^2 = 11^2$

$110 + y^2 = 121$

$110 + y^2 - 110 = 121 - 110$

$y^2 = 11$

$y = -\sqrt{11}$, in quadrant II

Since $\sin x = \frac{\text{opposite leg}}{\text{hypotenuse}}$, and since angle x is in the third quadrant, $\sin x = -\frac{\sqrt{11}}{11}$.

b) Since $\cos^2 x = \frac{10}{11}$, $\cos x = \pm\sqrt{\frac{10}{11}} = \pm\frac{\sqrt{10}}{\sqrt{11}}$ $= \pm\frac{\sqrt{110}}{11}$, and since angle x is in the third quadrant, $\cos x$ is negative. For this reason, $\cos x = -\frac{\sqrt{110}}{11}$.

c) Since $\cos^2 x = \frac{10}{11}$, $\cos x = \pm\sqrt{\frac{10}{11}} = \pm\frac{\sqrt{10}}{\sqrt{11}}$ $= \pm\frac{\sqrt{110}}{11}$, and since angle x is in the third quadrant, $\cos x$ is negative. For this reason, $\cos x = -\frac{\sqrt{110}}{11}$. Since $\cos x = -\frac{\sqrt{110}}{11}$, the leg adjacent to the angle x in a right triangle has a length of $\sqrt{110}$, while the hypotenuse of the right triangle has a length of 11. For this reason, the other leg of the right triangle can be calculated as follows:

$(\sqrt{110})^2 + y^2 = 11^2$

$110 + y^2 = 121$

$110 + y^2 - 110 = 121 - 110$

$y^2 = 11$

$y = -\sqrt{11}$, in quadrant II

Since $\sin x = \frac{\text{opposite leg}}{\text{hypotenuse}}$, and since angle x is in the third quadrant, $\sin x = -\frac{\sqrt{11}}{11}$. Since $\sin 2\theta = 2\sin\theta\cos\theta$, $\sin 2x = 2\sin x\cos x$

$= (2)\left(-\dfrac{\sqrt{11}}{11}\right)\left(-\dfrac{\sqrt{110}}{11}\right) = \dfrac{2\sqrt{10}}{11}$.

d) Since $\cos^2 x = \frac{10}{11}$, $\cos x = \pm\sqrt{\frac{10}{11}} = \pm\frac{\sqrt{10}}{\sqrt{11}}$ $= \pm\frac{\sqrt{110}}{11}$, and since angle x is in the third quadrant, $\cos x$ is negative. For this reason, $\cos x = -\frac{\sqrt{110}}{11}$. Since $\cos x = -\frac{\sqrt{110}}{11}$, the leg adjacent to the angle x in a right triangle has a length of $\sqrt{110}$, while the hypotenuse of the right triangle has a length of 11. For this reason, the other leg of the right triangle can be calculated as follows:

$(\sqrt{110})^2 + y^2 = 11^2$

$110 + y^2 = 121$

$110 + y^2 - 110 = 121 - 110$

$$\begin{aligned} y^2 &= 11 \\ y &= -\sqrt{11}\text{, in quadrant II} \end{aligned}$$

Since $\sin x = \frac{\text{opposite leg}}{\text{hypotenuse}}$, and since angle x is in the third quadrant, $\sin x = -\frac{\sqrt{11}}{11}$. Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$, $\cos 2x = \cos^2 x - \sin^2 x = \frac{10}{11} - \left(-\frac{\sqrt{11}}{11}\right)^2 = \frac{10}{11} - \frac{1}{11} = \frac{9}{11}$. (The formulas $\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 2\theta = 1 - 2\sin^2\theta$ could also have been used.)

10. Since $\sin x = \frac{3}{5}$, the leg opposite the angle x in a right triangle has a length of 3, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$\begin{aligned} x^2 + 3^2 &= 5^2 \\ x^2 + 9 &= 25 \\ x^2 + 9 - 9 &= 25 - 9 \\ x^2 &= 16 \\ x &= 4\text{, in quadrant I} \end{aligned}$$

Since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos x = \frac{4}{5}$. Therefore, since $\sin 2\theta = 2\sin\theta\cos\theta$,

$$\sin 2x = 2\sin x\cos x = (2)\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}.$$

Also, since $\cos 2\theta = \cos^2\theta - \sin^2\theta$, $\cos 2x = \cos^2 x - \sin^2 x = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$. (The formulas $\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 2\theta = 1 - 2\sin^2\theta$ could also have been used.)

11. Since $\sin x = \frac{5}{13}$, the leg opposite the angle x in a right triangle has a length of 5, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$\begin{aligned} x^2 + 5^2 &= 13^2 \\ x^2 + 25 &= 169 \\ x^2 + 25 - 25 &= 169 - 25 \\ x^2 &= 144 \\ x &= 12\text{, in quadrant I} \end{aligned}$$

Since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos x = \frac{12}{13}$.

Therefore, since $\sin 2\theta = 2\sin\theta\cos\theta$,

$$\sin 2x = 2\sin x\cos x = (2)\left(\frac{5}{13}\right)\left(\frac{12}{13}\right) = \frac{120}{169}.$$

12. Since $\cos x = -\frac{4}{5}$, the leg adjacent to the angle x in a right triangle has a length of 4, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$\begin{aligned} 4^2 + y^2 &= 5^2 \\ 16 + y^2 &= 25 \\ 16 + y^2 - 16 &= 25 - 16 \\ y^2 &= 9 \\ y &= 3\text{, in quadrant III} \end{aligned}$$

Since $\tan x = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan x = \frac{3}{4}$. (The reason the sign is positive is because angle x is in the third quadrant.) Since $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$,

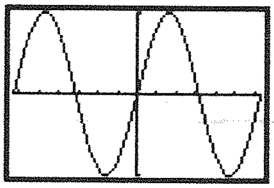
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{(2)\left(\frac{3}{4}\right)}{1 - \left(\frac{3}{4}\right)^2} = \frac{\frac{3}{2}}{1 - \frac{9}{16}}$$

$$= \frac{\frac{3}{2}}{\frac{16}{16} - \frac{9}{16}} = \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{3}{2} \times \frac{16}{7} = \frac{24}{7}$$

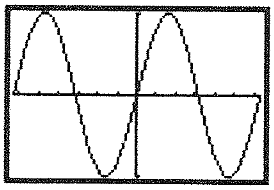
7.4 Proving Trigonometric Identities, pp. 417–418

1. Although $\sin x = \cos x$ is true for $x = \frac{\pi}{4}$, it is not true for all values of x, and therefore, it is not an identity. A counterexample is a value of x for which $\sin x = \cos x$ is not true. Many counterexamples exist, so answers may vary. One counterexample is $x = \frac{\pi}{6}$, since $\sin\frac{\pi}{6} = \frac{1}{2}$, and since $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

2. a) The graphs of $f(x) = \sin x$ and $g(x) = \tan x\cos x$ are as follows: $f(x) = \sin x$:



$g(x) = \tan x\cos x$:



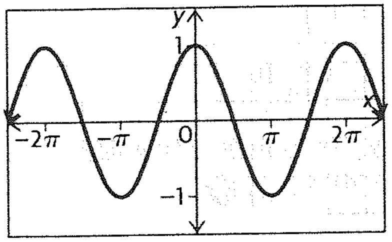
b) Since the graphs of $f(x) = \sin x$ and $g(x) = \tan x\cos x$ are the same, $\sin x = \tan x\cos x$.

c) To prove that the identity $\sin x = \tan x\cos x$ is true, $\tan x\cos x$ can be simplified as follows:

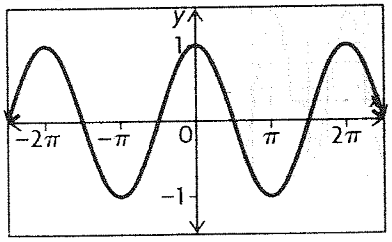
$$\tan x\cos x = \left(\frac{\sin x}{\cos x}\right)(\cos x) = \frac{\sin x\cos x}{\cos x} = \sin x$$

d) The identity is not true when $\cos x = 0$ because when $\cos x = 0$, $\tan x$, or $\frac{\sin x}{\cos x}$, is undefined. This is because 0 cannot be in the denominator of a fraction.

3. a) The graph of $y = \sin x \cot x$ is as follows:

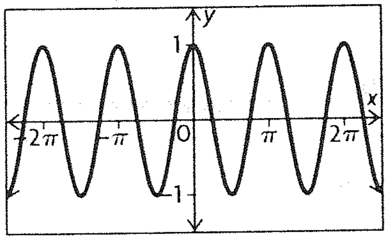


The graph of $y = \cos x$ is as follows:

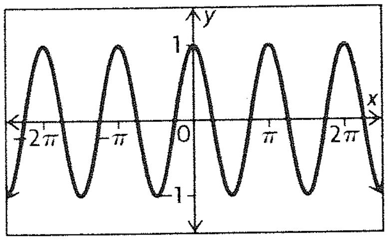


Since the graphs are the same, the answer is C.

b) The graph of $y = 1 - 2\sin^2 x$ is as follows:

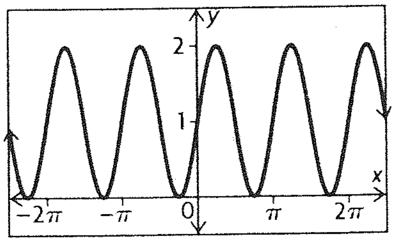


The graph of $y = 2\cos^2 x - 1$ is as follows:

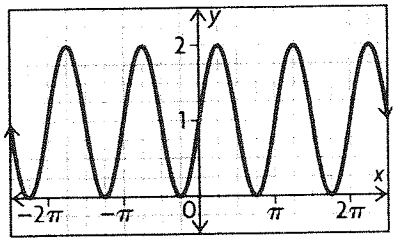


Since the graphs are the same, the answer is D.

c) The graph of $y = (\sin x + \cos x)^2$ is as follows:

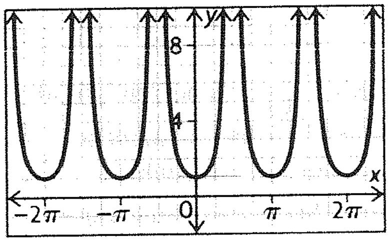


The graph of $y = 1 + 2\sin x \cos x$ is as follows:

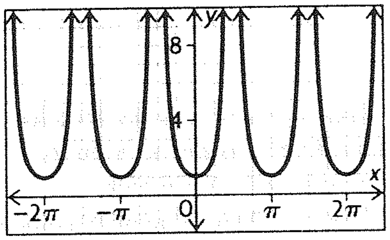


Since the graphs are the same, the answer is B.

d) The graph of $y = \sec^2 x$ is as follows:



The graph of $y = \sin^2 x + \cos^2 x + \tan^2 x$ is as follows:



Since the graphs are the same, the answer is A.

4. a) The identity $\sin x \cot x = \cos x$ can be proven as follows:

$$\sin x \cot x = (\sin x)\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x \cos x}{\sin x} = \cos x$$

b) The identity $1 - 2\sin^2 x = 2\cos^2 x - 1$ can be proven as follows:

$$\begin{aligned} 1 - 2\sin^2 x &= 2\cos^2 x - 1 \\ 1 - 2\sin^2 x - 2\cos^2 x + 1 &= 0 \\ 2 - 2\sin^2 x - 2\cos^2 x &= 0 \\ 2 - 2(\sin^2 x + \cos^2 x) &= 0 \\ 2 - 2(1) &= 0 \\ 2 - 2 &= 0 \\ 0 &= 0 \end{aligned}$$

c) The identity $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$ can be proven as follows:

$$\begin{aligned} (\sin x + \cos x)^2 &= \sin^2 x + 2\sin x \cos x + \cos^2 x \\ &= (\sin^2 x + \cos^2 x) + 2\sin x \cos x \\ &= 1 + 2\sin x \cos x \end{aligned}$$

d) The identity $\sec^2 x = \sin^2 x + \cos^2 x + \tan^2 x$ can be proven as follows:

$$\begin{aligned}&\sin^2 x + \cos^2 x + \tan^2 x\\ &= (\sin^2 x + \cos^2 x) + \tan^2 x\\ &= 1 + \tan^2 x\\ &= \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}\\ &= \frac{\sin^2 x + \cos^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\\ &= \sec^2 x\end{aligned}$$

5. a) The equation $\cos x = \frac{1}{\cos x}$ is not true for all values of x, and therefore, it is not an identity. A counterexample is a value of x for which $\cos x = \frac{1}{\cos x}$ is not true. Many counterexamples exist, so answers may vary. One counterexample is $x = \frac{\pi}{6}$, since $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, and since $\frac{1}{\cos\frac{\pi}{6}}$ $= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

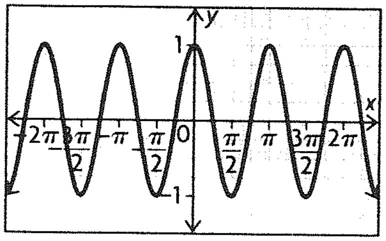
b) The equation $1 - \tan^2 x = \sec^2 x$ is not true for all values of x, and therefore, it is not an identity. A counterexample is a value of x for which $1 - \tan^2 x = \sec^2 x$ is not true. Many counterexamples exist, so answers may vary. One counterexample is $x = \frac{\pi}{4}$, since $1 - \tan^2\left(\frac{\pi}{4}\right) = 1 - (1)^2$ $= 1 - 1 = 0$, and since $\sec^2\left(\frac{\pi}{4}\right) = (\sqrt{2})^2 = 2$.

c) The equation $\sin(x + y) = \cos x \cos y + \sin x \sin y$ is not true for all values of x and y, and therefore, it is not an identity. A counterexample is values for x and y for which $\sin(x + y) = \cos x \cos y + \sin x \sin y$ is not true. Many counterexamples exist, so answers may vary. One counterexample is $x = \frac{\pi}{2}$ and $y = \pi$, since $\sin\left(\frac{\pi}{2} + \pi\right) = \sin\left(\frac{3\pi}{2}\right) = -1$, and since $\cos\left(\frac{\pi}{2}\right)\cos\pi + \sin\left(\frac{\pi}{2}\right)\sin\pi$ $= (0)(-1) + (1)(0) = 0 + 0 = 0$.

d) The equation $\cos 2x = 1 + 2\sin^2 x$ is not true for all values of x, and therefore, it is not an identity. A counterexample is a value of x for which $\cos 2x = 1 + 2\sin^2 x$ is not true. Many counterexamples exist, so answers may vary. One counterexample is $x = \frac{\pi}{3}$, since $\cos\left(2\left(\frac{\pi}{3}\right)\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$, and since

$$\begin{aligned}1 + 2\sin^2\left(\frac{\pi}{3}\right) &= 1 + (2)\left(\frac{\sqrt{3}}{2}\right)^2\\ &= 1 + (2)\left(\frac{3}{4}\right) = \frac{4}{4} + \frac{6}{4} = \frac{10}{4} = \frac{5}{2}\end{aligned}$$

6. Answers may vary. For example, the graph of the function $y = \frac{1 - \tan^2 x}{1 + \tan^2 x}$ is as follows:



The graph is the same as that of the function $y = \cos 2x$, so an appropriate conjecture is that $\cos 2x$ is another expression that is equivalent to $\frac{1 - \tan^2 x}{1 + \tan^2 x}$.

7. The identity $\frac{1 - \tan^2 x}{1 + \tan^2 x} = \cos 2x$ can be proven as follows:

$$\begin{aligned}\frac{1 - \tan^2 x}{1 + \tan^2 x} &= \frac{\frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}}{\sec^2 x} = \frac{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}}{\sec^2 x}\\ &= \frac{\cos^2 x - \sin^2 x}{\cos^2 x} \times \frac{1}{\sec^2 x}\\ &= \frac{\cos^2 x - \sin^2 x}{\cos^2 x} \times \cos^2 x = \cos^2 x - \sin^2 x\\ &= \cos 2x\end{aligned}$$

8. The identity $\frac{1 + \tan x}{1 + \cot x} = \frac{1 - \tan x}{\cot x - 1}$ can be proven as follows:

$$\begin{aligned}\text{LS} &= \frac{1 + \tan x}{1 + \cot x}\\ &= \frac{1 + \tan x}{1 + \frac{1}{\tan x}}\\ &= \frac{1 + \tan x}{\frac{\tan x + 1}{\tan x}}\\ &= \tan x\end{aligned}
\qquad
\begin{aligned}\text{RS} &= \frac{1 - \tan x}{\cot x - 1}\\ &= \frac{1 - \tan x}{\frac{1}{\tan x} - 1}\\ &= \frac{1 - \tan x}{\frac{1 - \tan x}{\tan x}}\\ &= \tan x\end{aligned}$$

Since the right side and the left side are equal, $\frac{1 + \tan x}{1 + \cot x} = \frac{1 - \tan x}{\cot x - 1}$

9. a) The identity $\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin\theta\cos\theta} = 1 - \tan\theta$ can be proven as follows:

$$\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin\theta\cos\theta}$$

$$= \frac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{(\cos\theta)(\cos\theta + \sin\theta)}$$

$$= \frac{\cos\theta - \sin\theta}{\cos\theta} = \frac{\cos\theta}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = 1 - \tan\theta$$

b) The identity $\tan^2 x - \sin^2 x = \sin^2 x \tan^2 x$ can be proven as follows:

$$\text{LS} = \tan^2 x - \sin^2 x$$

$$= \frac{\sin^2 x}{\cos^2 x} - \sin^2 x$$

$$= \sin^2 x\left(\frac{1}{\cos^2 x} - 1\right)$$

$$= \sin^2 x(\sec^2 x - 1)$$

$$= \sin^2 x \tan^2 x$$

$$= \text{RS}$$

So $\tan^2 x - \sin^2 x = \sin^2 x \tan^2 x$.
Since $\csc^2 x = 1 + \cot^2 x$ is a known identity, $\tan^2 x - \sin^2 x$ must equal $\sin^2 x \tan^2 x$.

c) The identity

$\tan^2 x - \cos^2 x = \frac{1}{\cos^2 x} - 1 - \cos^2 x$ can be proven as follows:

$$\tan^2 x - \cos^2 x = \frac{1}{\cos^2 x} - 1 - \cos^2 x;$$

$$\tan^2 x - \cos^2 x + \cos^2 x = \frac{1}{\cos^2 x} - 1 - \cos^2 x + \cos^2 x;$$

$$\tan^2 x = \frac{1}{\cos^2 x} - 1;$$

$$\tan^2 x = \frac{1}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x};$$

$$\tan^2 x = \frac{1 - \cos^2 x}{\cos^2 x};$$

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x};$$

$$\tan^2 x = \tan^2 x$$

d) The identity $\frac{1}{1 + \cos\theta} + \frac{1}{1 - \cos\theta} = \frac{2}{\sin^2\theta}$ can be proven as follows:

$$\frac{1}{1 + \cos\theta} + \frac{1}{1 - \cos\theta} = \frac{1 - \cos\theta}{(1 + \cos\theta)(1 - \cos\theta)} + \frac{1 + \cos\theta}{(1 - \cos\theta)(1 + \cos\theta)}$$

$$= \frac{1 - \cos\theta}{1 - \cos^2\theta} + \frac{1 + \cos\theta}{1 - \cos^2\theta}$$

$$= \frac{1 - \cos\theta + 1 + \cos\theta}{1 - \cos^2\theta} = \frac{2}{1 - \cos^2\theta} = \frac{2}{\sin^2\theta}$$

10. a) The identity $\cos x \tan^3 x = \sin x \tan^2 x$ can be proven as follows:

$$\cos x \tan^3 x = \sin x \tan^2 x$$

$$\frac{\cos x \tan^3 x}{\tan^2 x} = \frac{\sin x \tan^2 x}{\tan^2 x}$$

$$\cos x \tan x = \sin x$$

$$(\cos x)\left(\frac{\sin x}{\cos x}\right) = \sin x$$

$$\sin x = \sin x$$

b) The identity $\sin^2\theta + \cos^4\theta = \cos^2\theta + \sin^4\theta$ can be proven as follows:

$$\sin^2\theta + \cos^4\theta = \cos^2\theta + \sin^4\theta$$

$$\sin^2\theta + \cos^4\theta - \sin^4\theta = \cos^2\theta + \sin^4\theta - \sin^4\theta$$

$$\sin^2\theta + \cos^4\theta - \sin^4\theta = \cos^2\theta$$

$$\sin^2\theta + \cos^4\theta - \sin^4\theta - \sin^2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$$

$$(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 = 1$$

c) The identity

$(\sin x + \cos x)\left(\frac{\tan^2 x + 1}{\tan x}\right) = \frac{1}{\cos x} + \frac{1}{\sin x}$

can be proven as follows:

$$(\sin x + \cos x)\left(\frac{\tan^2 x + 1}{\tan x}\right) = \frac{1}{\cos x} + \frac{1}{\sin x}$$

$$(\sin x + \cos x)\left(\frac{\sec^2 x}{\tan x}\right) = \frac{\sin x}{\cos x \sin x} + \frac{\cos x}{\sin x \cos x}$$

$$(\sin x + \cos x)\left(\frac{1}{\cos^2 x}\right)\left(\frac{1}{\tan x}\right) = \frac{\sin x + \cos x}{\cos x \sin x}$$

$$(\sin x + \cos x)\left(\frac{1}{\cos^2 x}\right)\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x + \cos x}{\cos x \sin x}$$

$$(\sin x + \cos x)\left(\frac{1}{\cos x \sin x}\right) = \frac{\sin x + \cos x}{\cos x \sin x}$$

$$\frac{\sin x + \cos x}{\cos x \sin x} = \frac{\sin x + \cos x}{\cos x \sin x}$$

d) The identity $\tan^2\beta + \cos^2\beta + \sin^2\beta = \frac{1}{\cos^2\beta}$ can be proven as follows:

$$\tan^2\beta + \cos^2\beta + \sin^2\beta = \frac{1}{\cos^2\beta}$$

$$\tan^2\beta + 1 = \frac{1}{\cos^2\beta}$$

$\tan^2 \beta + 1 = \sec^2 \beta$

Since $\tan^2 \beta + 1 = \sec^2 \beta$ is a known identity, $\tan^2 \beta + \cos^2 \beta + \sin^2 \beta$ must equal $\frac{1}{\cos^2 \beta}$.

e) The identity

$$\sin\left(\frac{\pi}{4} + x\right) + \sin\left(\frac{\pi}{4} - x\right) = \sqrt{2} \cos x$$

can be proven as follows:

$$\sin\left(\frac{\pi}{4} + x\right) + \sin\left(\frac{\pi}{4} - x\right) = \sqrt{2} \cos x;$$

$$\sin \frac{\pi}{4} \cos x + \cos \frac{\pi}{4} \sin x + \sin \frac{\pi}{4} \cos x - \cos \frac{\pi}{4} \sin x = \sqrt{2} \cos x;$$

$$2 \sin \frac{\pi}{4} \cos x = \sqrt{2} \cos x;$$

$$(2)\left(\frac{\sqrt{2}}{2}\right)(\cos x) = \sqrt{2} \cos x;$$

$$\sqrt{2} \cos x = \sqrt{2} \cos x$$

f) The identity $\sin\left(\frac{\pi}{2} - x\right) \cot\left(\frac{\pi}{2} + x\right) = -\sin x$ can be proven as follows:

$$\sin\left(\frac{\pi}{2} - x\right) \cot\left(\frac{\pi}{2} + x\right) = -\sin x;$$

$$\sin\left(\frac{\pi}{2} - x\right)\left(\frac{\cos\left(\frac{\pi}{2} + x\right)}{\sin\left(\frac{\pi}{2} + x\right)}\right) = -\sin x;$$

$$\left(\sin \frac{\pi}{2} \cos x - \cos \frac{\pi}{2} \sin x\right) \times \left(\frac{\cos \frac{\pi}{2} \cos x - \sin \frac{\pi}{2} \sin x}{\sin \frac{\pi}{2} \cos x + \cos \frac{\pi}{2} \sin x}\right) = -\sin x;$$

$$((1)(\cos x) - (0)(\sin x)) \left(\frac{(0)(\cos x) - (1)(\sin x)}{(1)(\cos x) + (0)(\sin x)}\right) = -\sin x;$$

$$(\cos x - 0)\left(\frac{0 - \sin x}{\cos x + 0}\right) = -\sin x;$$

$$(\cos x)\left(-\frac{\sin x}{\cos x}\right) = -\sin x;$$

$$-\sin x = -\sin x$$

11. a) The identity $\frac{\cos 2x + 1}{\sin 2x} = \cot x$ can be proven as follows:

$$\frac{\cos 2x + 1}{\sin 2x} = \cot x$$

$$\frac{2 \cos^2 x - 1 + 1}{2 \sin x \cos x} = \cot x$$

$$\frac{2 \cos^2 x}{2 \sin x \cos x} = \cot x$$

$$\frac{\cos x}{\sin x} = \cot x$$

$$\cot x = \cot x$$

b) The identity $\frac{\sin 2x}{1 - \cos 2x} = \cot x$ can be proven as follows:

$$\frac{\sin 2x}{1 - \cos 2x} = \cot x$$

$$\frac{2 \sin x \cos x}{1 - (1 - 2 \sin^2 x)} = \cot x$$

$$\frac{2 \sin x \cos x}{1 - 1 + 2 \sin^2 x} = \cot x$$

$$\frac{2 \sin x \cos x}{2 \sin^2 x} = \cot x$$

$$\frac{\cos x}{\sin x} = \cot x$$

$$\cot x = \cot x$$

c) The identity $(\sin x + \cos x)^2 = 1 + \sin 2x$ can be proven as follows:

$$(\sin x + \cos x)^2 = 1 + \sin 2x;$$

$$\sin^2 x + \sin x \cos x + \sin x \cos x + \cos^2 x = 1 + 2 \sin x \cos x;$$

$$\sin^2 x + 2 \sin x \cos x + \cos^2 x = 1 + 2 \sin x \cos x;$$

$$(\cos^2 x + \sin^2 x) + 2 \sin x \cos x = 1 + 2 \sin x \cos x;$$

$$1 + 2 \sin x \cos x = 1 + 2 \sin x \cos x$$

d) The identity $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$ can be proven as follows:

$$\cos^4 \theta - \sin^4 \theta = \cos 2\theta$$

$$(\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta) = \cos^2 \theta - \sin^2 \theta$$

$$(1)(\cos^2 \theta - \sin^2 \theta) = \cos^2 \theta - \sin^2 \theta$$

$$\cos^2 \theta - \sin^2 \theta = \cos^2 \theta - \sin^2 \theta$$

e) The identity $\cot \theta - \tan \theta = 2 \cot 2\theta$ can be proven as follows:

$$\cot \theta - \tan \theta = 2 \cot 2\theta$$

$$\frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} = 2\frac{\cos 2\theta}{\sin 2\theta}$$

$$\frac{\cos^2 \theta}{\sin \theta \cos \theta} - \frac{\sin^2 \theta}{\cos \theta \sin \theta} = (2)\left(\frac{\cos 2\theta}{2 \cos \theta \sin \theta}\right)$$

$$\frac{\cos^2 \theta - \sin^2 \theta}{\sin \theta \cos \theta} = \frac{\cos 2\theta}{\cos \theta \sin \theta}$$

$$\frac{\cos 2\theta}{\cos \theta \sin \theta} = \frac{\cos 2\theta}{\cos \theta \sin \theta}$$

f) The identity $\cot \theta + \tan \theta = 2 \csc 2\theta$ can be proven as follows:

$$\cot \theta + \tan \theta = 2 \csc 2\theta$$

$$\frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} = 2\frac{1}{\sin 2\theta}$$

$$\frac{\cos^2 \theta}{\sin \theta \cos \theta} + \frac{\sin^2 \theta}{\cos \theta \sin \theta} = (2)\left(\frac{1}{2 \cos \theta \sin \theta}\right)$$

$$\frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta \cos \theta} = \frac{1}{\cos \theta \sin \theta}$$

$$\frac{1}{\cos \theta \sin \theta} = \frac{1}{\cos \theta \sin \theta}$$

g) The identity $\frac{1 + \tan x}{1 - \tan x} = \tan\left(x + \frac{\pi}{4}\right)$ can be proven as follows:

$$\frac{1 + \tan x}{1 - \tan x} = \tan\left(x + \frac{\pi}{4}\right)$$

$$\frac{1 + \tan x}{1 - \tan x} = \frac{\tan x + \tan \frac{\pi}{4}}{1 - \tan x \tan \frac{\pi}{4}}$$

$$\frac{1 + \tan x}{1 - \tan x} = \frac{\tan x + 1}{1 - (\tan x)(1)}$$

$$\frac{1 + \tan x}{1 - \tan x} = \frac{1 + \tan x}{1 - \tan x}$$

h) The identity $\csc 2x + \cot 2x = \cot x$ can be proven as follows:

$$\csc 2x + \cot 2x = \cot x;$$

$$\frac{1}{\sin 2x} + \frac{1}{\tan 2x} = \cot x;$$

$$\frac{1}{2 \sin x \cos x} + \frac{1}{\frac{2 \tan x}{1 - \tan^2 x}} = \cot x;$$

$$\frac{1}{2 \sin x \cos x} + \frac{1 - \tan^2 x}{2 \tan x} = \cot x;$$

$$\frac{1}{2 \sin x \cos x} + \frac{1 - \tan^2 x}{2\frac{\sin x}{\cos x}} = \cot x;$$

$$\frac{1}{2 \sin x \cos x} + \frac{(\cos x)(1 - \tan^2 x)}{2 \sin x} = \frac{\cos x}{\sin x};$$

$$\frac{1}{2 \sin x \cos x} + \frac{(\cos x)(1 - \tan^2 x)(\cos x)}{2 \sin x \cos x}$$

$$= \frac{(\cos x)(2 \cos x)}{(\sin x)(2 \cos x)};$$

$$\frac{1}{2 \sin x \cos x} + \frac{(\cos^2 x)(1 - \tan^2 x)}{2 \sin x \cos x}$$

$$= \frac{2 \cos^2 x}{2 \sin x \cos x};$$

$$\frac{1}{2 \sin x \cos x} + \frac{\cos^2 x - (\tan^2 x)(\cos^2 x)}{2 \sin x \cos x}$$

$$= \frac{2 \cos^2 x}{2 \sin x \cos x};$$

$$\frac{1}{2 \sin x \cos x} + \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x} = \frac{2 \cos^2 x}{2 \sin x \cos x};$$

$$\frac{1 + \cos^2 x - \sin^2 x}{2 \sin x \cos x} = \frac{2 \cos^2 x}{2 \sin x \cos x};$$

$$\frac{1 + \cos^2 x - \sin^2 x}{2 \sin x \cos x} - \frac{2 \cos^2 x}{2 \sin x \cos x}$$

$$= \frac{2 \cos^2 x}{2 \sin x \cos x} - \frac{2 \cos^2 x}{2 \sin x \cos x};$$

$$\frac{1 + \cos^2 x - \sin^2 x - 2 \cos^2 x}{2 \sin x \cos x} = 0;$$

$$\frac{1 - \sin^2 x - \cos^2 x}{2 \sin x \cos x} = 0;$$

$$\frac{1 - (\sin^2 x + \cos^2 x)}{2 \sin x \cos x} = 0;$$

$$\frac{1 - 1}{2 \sin x \cos x} = 0;$$

$$\frac{0}{2 \sin x \cos x} = 0;$$

$$0 = 0$$

i) The identity $\frac{2 \tan x}{1 + \tan^2 x} = \sin 2x$ can be proven as follows:

$$\frac{2 \tan x}{1 + \tan^2 x} = \sin 2x$$

$$\frac{2 \tan x}{\sec^2 x} = \sin 2x$$

$$\frac{2 \tan x}{\frac{1}{\cos^2 x}} = \sin 2x$$

$$(2 \tan x)(\cos^2 x) = \sin 2x$$

$$\left(\frac{2 \sin x}{\cos x}\right)(\cos^2 x) = \sin 2x$$

$$\sin 2x = 2 \sin x \cos x$$

Since $\sin 2x = 2 \sin x \cos x$ is a known identity, $\frac{2 \tan x}{1 - \tan^2 x}$ must equal $\sin 2x$.

j) The identity $\sec 2t = \frac{\csc t}{\csc t - 2 \sin t}$ can be proven as follows:

$$\sec 2t = \frac{\csc t}{\csc t - 2 \sin t}$$

$$\frac{1}{\cos 2t} = \frac{\frac{1}{\sin t}}{\frac{1}{\sin t} - 2 \sin t}$$

$$\frac{1}{\cos 2t} = \frac{\frac{1}{\sin t}}{\frac{1}{\sin t} - \frac{2 \sin^2 t}{\sin t}}$$

$$\frac{1}{\cos 2t} = \frac{\frac{1}{\sin t}}{\frac{1 - 2\sin^2 t}{\sin t}}$$

$$\frac{1}{\cos 2t} = \frac{1}{\sin t} \times \frac{\sin t}{1 - 2\sin^2 t}$$

$$\frac{1}{\cos 2t} = \frac{1}{1 - 2\sin^2 t}$$

$$\frac{1}{\cos 2t} = \frac{1}{\cos 2t}$$

k) The identity $\csc 2\theta = \frac{1}{2}\sec\theta\csc\theta$ can be proven as follows:

$$\csc 2\theta = \frac{1}{2}\sec\theta\csc\theta$$

$$\frac{1}{\sin 2\theta} = \left(\frac{1}{2}\right)\left(\frac{1}{\cos\theta}\right)\left(\frac{1}{\sin\theta}\right)$$

$$\frac{1}{\sin 2\theta} = \frac{1}{2\cos\theta\sin\theta}$$

$$\frac{1}{2\sin\theta\cos\theta} = \frac{1}{2\sin\theta\cos\theta}$$

l) The identity $\sec t = \frac{\sin 2t}{\sin t} - \frac{\cos 2t}{\cos t}$ can be proven as follows:

$$\frac{1}{\cos t} = \frac{2\sin t\cos t}{\sin t} - \frac{2\cos^2 t - 1}{\cos t}$$

$$\frac{\sin t}{\cos t\sin t} = \frac{2\sin t\cos^2 t}{\sin t\cos t} - \frac{(\sin t)(2\cos^2 t - 1)}{\cos t\sin t}$$

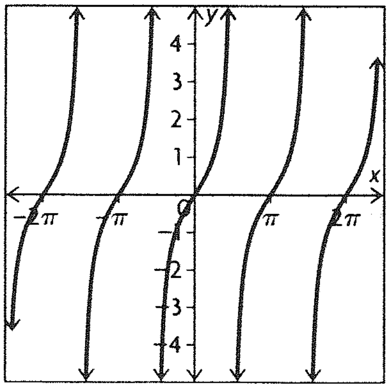
$$\frac{\sin t}{\cos t\sin t} = \frac{2\sin t\cos^2 t}{\sin t\cos t} - \frac{2\cos^2 t\sin t - \sin t}{\cos t\sin t}$$

$$\frac{\sin t}{\cos t\sin t} = \frac{2\sin t\cos^2 t - (2\cos^2 t\sin t - \sin t)}{\sin t\cos t}$$

$$\frac{\sin t}{\cos t\sin t} = \frac{2\sin t\cos^2 t - 2\sin t\cos^2 t + \sin t}{\sin t\cos t}$$

$$\frac{\sin t}{\cos t\sin t} = \frac{\sin t}{\cos t\sin t}$$

12. Answers may vary. For example, the graph of the function $y = \frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x}$ is as follows:



The graph is the same as that of the function $y = \tan x$, so an expression equivalent to $\frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x}$ is $\tan x$.

13. The identity $\frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x} = \tan x$ can be proven as follows:

$$\frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x} = \tan x$$

$$\frac{\sin x + 2\sin x\cos x}{1 + \cos x + \cos 2x} = \tan x$$

$$\frac{(\sin x)(1 + 2\cos x)}{1 + \cos x + \cos 2x} = \tan x$$

$$\frac{(\sin x)(1 + 2\cos x)}{1 + \cos x + 2\cos^2 x - 1} = \tan x$$

$$\frac{(\sin x)(1 + 2\cos x)}{\cos x + 2\cos^2 x} = \tan x$$

$$\frac{(\sin x)(1 + 2\cos x)}{(\cos x)(1 + 2\cos x)} = \tan x$$

$$\frac{\sin x}{\cos x} = \tan x$$

$$\tan x = \tan x$$

14. A trigonometric identity is a statement of the equivalence of two trigonometric expressions. To prove it, both sides of the equation must be shown to be equivalent through graphing or simplifying/rewriting. Therefore, the chart can be completed as follows:

Definition A statement of the equivalence of two trigonometric expressions	**Methods of Proof** Both sides of the equation must be shown to be equivalent through graphing or simplifying/rewriting.
Trigonometric Identities	
Examples $\cos 2x + \sin^2 x = \cos^2 x$ $\cos 2x + 1 = 2\cos^2 x$	**Non-Examples** $\cos 2x - 2\sin^2 x = 1$ $\cot^2 x + \csc^2 x = 1$

15. She can determine whether the equation $2\sin x\cos x = \cos 2x$ is an identity by trying to simplify and/or rewrite the left side of the equation so that it is equivalent to the right side of the equation. Alternatively, she can graph the functions $y = 2 \sin x \cos x$ and $y = \cos 2x$ and see if the graphs are the same. If they're the same, it's an identity, but if they're not the same, it's not an identity. By doing this she can determine it's not an identity, but she can make it an identity by changing the equation to $2 \sin x \cos x = \sin 2x$.

16. a) To write the expression $2 \cos^2 x + 4 \sin x \cos x$ in the form $a \sin 2x + b \cos 2x + c$, rewrite the expression as follows:

$2 \cos^2 x + 4 \sin x \cos x$

$= 2 \cos^2 x + (2)(2 \sin x \cos x)$

$= (2 \cos^2 x - 1) + (2)(2 \sin x \cos x) + 1$

$= \cos 2x + 2 \sin 2x + 1$

$= 2 \sin 2x + \cos 2x + 1$

Since the expression can be rewritten as $2 \sin 2x + \cos 2x + 1$, the values of a, b, and c are $a = 2$, $b = 1$, and $c = 1$.

b) To write the expression $-2 \sin x \cos x - 4 \sin^2 x$ in the form $a \sin 2x + b \cos 2x + c$, rewrite the expression as follows:

$-2 \sin x \cos x - 4 \sin^2 x$

$= -2 \sin x \cos x + (2 - 4 \sin^2 x) - 2$

$= -2 \sin x \cos x + (2)(1 - 2 \sin^2 x) - 2$

$= -\sin 2x + 2 \cos 2x - 2$

Since the expression can be rewritten as $-\sin 2x + 2 \cos 2x - 2$, the values of a, b, and c are $a = -1$, $b = 2$, and $c = -2$.

17. To write the expression $8 \cos^4 x$ in the form $a \cos 4x + b \cos 2x + c$, it's first necessary to develop a formula for $\cos 4x$. Since $\cos 2x = 2 \cos^2 x - 1$, $\cos 4x = 2 \cos^2 2x - 1$. The formula $\cos 2x = 2 \cos^2 x - 1$ can now be used again to further develop the formula for $\cos 4x$ as follows:

$\cos 4x = 2 \cos^2 2x - 1$

$\cos 4x = (2)(2 \cos^2 x - 1)^2 - 1$

$\cos 4x = (2)(4 \cos^4 x - 2 \cos^2 x$
$\quad - 2 \cos^2 x + 1) - 1$

$\cos 4x = (2)(4 \cos^4 x - 4 \cos^2 x + 1) - 1$

$\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 2 - 1$

$\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1$

Since $\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1$, the value of a must be 1. Now, to turn the expression $8 \cos^4 x - 8 \cos^2 x + 1$ into the expression $8 \cos^4 x$, the terms $-8 \cos^2 x$ and 1 must be eliminated. First, to eliminate $-8 \cos^2 x$, $4 \cos 2x$, or $(4)(2 \cos^2 x - 1)$, can be added to the expression $8 \cos^4 x - 8 \cos^2 x + 1$ as follows:

$8 \cos^4 x - 8 \cos^2 x + 1 + (4)(2 \cos^2 x - 1)$

$8 \cos^4 x - 8 \cos^2 x + 1 + 8 \cos^2 x - 4$

$8 \cos^4 x - 3$

Since adding $4 \cos 2x$ to the expression eliminated the term $-8 \cos^2 x$, the value of b must be 4. Now, to turn the expression $8 \cos^4 x - 3$ into the expression $8 \cos^4 x$, 3 must be added to it as follows:

$8 \cos^4 x - 3 + 3$

$8 \cos^4 x$

Therefore, the value of c must be 3, and the expression must be $\cos 4x + 4 \cos 2x + 3$.

$a = 1, b = 4, c = 3$

7.5 Solving Linear Trigonometric Equations, pp. 426–428

1. a) From the graph of $y = \sin \theta$, the equation $\sin \theta = 1$ is true when $\theta = \frac{\pi}{2}$.

b) From the graph of $y = \sin \theta$, the equation $\sin \theta = -1$ is true when $\theta = \frac{3\pi}{2}$.

c) From the graph of $y = \sin \theta$, the equation $\sin \theta = 0.5$ is true when $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

d) From the graph of $y = \sin \theta$, the equation $\sin \theta = -0.5$ is true when $\theta = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

e) From the graph of $y = \sin \theta$, the equation $\sin \theta = 0$ is true when $\theta = 0, \pi$, or 2π.

f) From the graph of $y = \sin \theta$, the equation $\sin \theta = \frac{\sqrt{3}}{2}$ is true when $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$.

2. a) From the graph of $y = \cos \theta$, the equation $\cos \theta = 1$ is true when $\theta = 0$ or 2π.

b) From the graph of $y = \cos \theta$, the equation $\cos \theta = -1$ is true when $\theta = \pi$.

c) From the graph of $y = \cos \theta$, the equation $\cos \theta = 0.5$ is true when $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

d) From the graph of $y = \cos \theta$, the equation $\cos \theta = -0.5$ is true when $\theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

e) From the graph of $y = \cos \theta$, the equation $\cos \theta = 0$ is true when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

f) From the graph of $y = \cos \theta$, the equation $\cos \theta = \frac{\sqrt{3}}{2}$ is true when $\theta = \frac{\pi}{6}$ or $\frac{11\pi}{6}$.

3. a) Given $\sin x = \frac{\sqrt{3}}{2}$ and $0 \le x \le 2\pi$, 2 solutions must be possible, since $\sin x = \frac{\sqrt{3}}{2}$ in 2 of the 4 quadrants.

b) Given $\sin x = \frac{\sqrt{3}}{2}$ and $0 \le x \le 2\pi$, the solutions for x must occur in the 1st and 2nd quadrants, since the sine function is positive in these quadrants.

c) Given $\sin x = \frac{\sqrt{3}}{2}$ and $0 \le x \le 2\pi$, the related acute angle for the equation must be $x = \frac{\pi}{3}$, since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

d) Given $\sin x = \frac{\sqrt{3}}{2}$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$, since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$.

4. a) Given $\cos x = -0.8667$ and $0° \le x \le 360°$, two solutions must be possible, since $\cos x = -0.8667$ in two of the four quadrants.

b) Given $\cos x = -0.8667$ and $0° \le x \le 360°$, the solutions for x must occur in the second and third quadrants, since the cosine function is negative in these quadrants.

c) Given $\cos x = -0.8667$ and $0° \le x \le 360°$, the related acute angle for the equation must be $x = 30°$, since $\cos 30° = -0.866$.

d) Given $\cos x = -0.8667$ and $0° \le x \le 360°$, the solutions to the equation must be $x = 150°$ or $210°$, since $\cos 150° = -0.866$ and $\cos 210° = -0.866$.

5. a) Given $\tan \theta = 2.7553$ and $0 \le \theta \le 2\pi$, two solutions must be possible, since $\tan \theta = 2.7553$ in two of the four quadrants.

b) Given $\tan \theta = 2.7553$ and $0 \le \theta \le 2\pi$, the solutions for θ must occur in the first and third quadrants, since the tangent function is positive in these quadrants.

c) Given $\tan \theta = 2.7553$ and $0 \le \theta \le 2\pi$, the related acute angle for the equation must be $\theta = 1.22$, since $\tan 1.22 = 2.7553$.

d) Given $\tan \theta = 2.7553$ and $0 \le \theta \le 2\pi$, the solutions to the equation must be $\theta = 1.22$ or 4.36, since $\tan 1.22 = 2.7553$ and $\tan 4.36 = 2.7553$.

6. a) Since $\tan \frac{\pi}{4} = 1$ and $\tan \frac{5\pi}{4} = 1$, the solutions to the equation $\tan \theta = 1$ are $\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

b) Since $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$, the solutions solutions to the equation $\sin \theta = \frac{1}{\sqrt{2}}$ are $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.

c) Since $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\cos \frac{11\pi}{6} = \frac{\sqrt{3}}{2}$, the solutions to the equation $\cos \theta = \frac{\sqrt{3}}{2}$ are $\theta = \frac{\pi}{6}$ or $\frac{11\pi}{6}$.

d) Since $\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$ and $\sin \frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$, the solutions to the equation $\sin \theta = -\frac{\sqrt{3}}{2}$ are $\theta = \frac{4\pi}{3}$ or $\frac{5\pi}{3}$.

e) Since $\cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$ and $\cos \frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$, the solutions to the equation $\cos \theta = -\frac{1}{\sqrt{2}}$ are $\theta = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$.

f) Since $\tan \frac{\pi}{3} = \sqrt{3}$ and $\tan \frac{4\pi}{3} = \sqrt{3}$, the solutions to the equation $\tan \theta = \sqrt{3}$ are $\theta = \frac{\pi}{3}$ or $\frac{4\pi}{3}$.

7. a) The equation $2 \sin \theta = -1$ can be rewritten as follows:

$$2 \sin \theta = -1$$
$$\frac{2 \sin \theta}{2} = -\frac{1}{2}$$
$$\sin \theta = -\frac{1}{2}$$

Given $\sin \theta = -\frac{1}{2}$ and $0° \le \theta \le 360°$, the solutions to the equation must be $\theta = 210°$ or $330°$, since $\sin 210° = -\frac{1}{2}$ and $\sin 330° = -\frac{1}{2}$.

b) The equation $3 \cos \theta = -2$ can be rewritten as follows:

$$3 \cos \theta = -2$$
$$\frac{3 \cos \theta}{3} = -\frac{2}{3}$$
$$\cos \theta = -\frac{2}{3}$$

Given $\cos \theta = -\frac{2}{3}$ and $0° \le \theta \le 360°$, the solutions to the equation must be $\theta = 131.8°$ or $228.2°$.

c) The equation $2 \tan \theta = 3$ can be rewritten as follows:

$$2 \tan \theta = 3$$
$$\frac{2 \tan \theta}{2} = \frac{3}{2}$$
$$\tan \theta = \frac{3}{2}$$

Given $\tan \theta = \frac{3}{2}$ and $0° \le \theta \le 360°$, the solutions to the equation must be $\theta = 56.3°$ or $236.3°$.

d) The equation $-3\sin\theta - 1 = 1$ can be rewritten as follows:

$$\begin{aligned} -3\sin\theta - 1 &= 1 \\ -3\sin\theta - 1 + 1 &= 1 + 1 \\ -3\sin\theta &= 2 \\ \frac{-3\sin\theta}{-3} &= \frac{2}{-3} \\ \sin\theta &= -\frac{2}{3} \end{aligned}$$

Given $\sin\theta = -\frac{2}{3}$ and $0° \leq \theta \leq 360°$, the solutions to the equation must be $\theta = 221.8°$ or $318.2°$.

e) The equation $-5\cos\theta + 3 = 2$ can be rewritten as follows:

$$\begin{aligned} -5\cos\theta + 3 &= 2 \\ -5\cos\theta + 3 - 3 &= 2 - 3 \\ -5\cos\theta &= -1 \\ \frac{-5\cos\theta}{-5} &= \frac{-1}{-5} \\ \cos\theta &= \frac{1}{5} \end{aligned}$$

Given $\cos\theta = \frac{1}{5}$ and $0° \leq \theta \leq 360°$, the solutions to the equation must be $\theta = 78.5°$ or $281.5°$.

f) The equation $8 - \tan\theta = 10$ can be rewritten as follows:

$$\begin{aligned} 8 - \tan\theta &= 10 \\ 8 - \tan\theta + \tan\theta &= 10 + \tan\theta \\ 8 &= 10 + \tan\theta \\ 8 - 10 &= 10 + \tan\theta - 10 \\ \tan\theta &= -2 \end{aligned}$$

Given $\tan\theta = -2$ and $0° \leq \theta \leq 360°$, the solutions to the equation must be $\theta = 116.6°$ or $296.6°$.

8. a) The equation $3\sin x = \sin x + 1$ can be rewritten as follows:

$$\begin{aligned} 3\sin x &= \sin x + 1 \\ 3\sin x - \sin x &= \sin x + 1 - \sin x \\ 2\sin x &= 1 \\ \frac{2\sin x}{2} &= \frac{1}{2} \\ \sin x &= \frac{1}{2} \end{aligned}$$

Given $\sin x = \frac{1}{2}$ and $0 \leq x \leq 2\pi$, the solutions to the equation must be $x = 0.52$ or 2.62.

b) The equation $5\cos x - \sqrt{3} = 3\cos x$ can be rewritten as follows:

$$\begin{aligned} 5\cos x - \sqrt{3} &= 3\cos x \\ 5\cos x - \sqrt{3} - 3\cos x &= 3\cos x - 3\cos x \\ 2\cos x - \sqrt{3} &= 0 \\ 2\cos x - \sqrt{3} + \sqrt{3} &= 0 + \sqrt{3} \\ 2\cos x &= \sqrt{3} \\ \frac{2\cos x}{2} &= \frac{\sqrt{3}}{2} \\ \cos x &= \frac{\sqrt{3}}{2} \end{aligned}$$

Given $\cos x = \frac{\sqrt{3}}{2}$ and $0 \leq x \leq 2\pi$, the solutions to the equation must be $x = 0.52$ or 5.76.

c) The equation $\cos x - 1 = -\cos x$ can be rewritten as follows:

$$\begin{aligned} \cos x - 1 &= -\cos x \\ \cos x - 1 + \cos x &= -\cos x + \cos x \\ 2\cos x - 1 &= 0 \\ 2\cos x - 1 + 1 &= 0 + 1 \\ 2\cos x &= 1 \\ \frac{2\cos x}{2} &= \frac{1}{2} \\ \cos x &= \frac{1}{2} \end{aligned}$$

Given $\cos x = \frac{1}{2}$ and $0 \leq x \leq 2\pi$, the solutions to the equation must be $x = 1.05$ or 5.24.

d) The equation $5\sin x + 1 = 3\sin x$ can be rewritten as follows:

$$\begin{aligned} 5\sin x + 1 &= 3\sin x \\ 5\sin x + 1 - 3\sin x &= 3\sin x - 3\sin x \\ 2\sin x + 1 &= 0 \\ 2\sin x + 1 - 1 &= 0 - 1 \\ 2\sin x &= -1 \\ \frac{2\sin x}{2} &= -\frac{1}{2} \\ \sin x &= -\frac{1}{2} \end{aligned}$$

Given $\sin x = -\frac{1}{2}$ and $0 \leq x \leq 2\pi$, the solutions to the equation must be $x = 3.67$ or 5.76.

9. a) The equation $2 - 2\cot x = 0$ can be rewritten as follows:

$$\begin{aligned} 2 - 2\cot x &= 0 \\ 2 - 2\cot x + 2\cot x &= 0 + 2\cot x \\ 2 &= 2\cot x \\ \frac{2}{2} &= \frac{2\cot x}{2} \\ \cot x &= 1 \end{aligned}$$

Given $\cot x = 1$ and $0 \leq x \leq 2\pi$, the solutions to the equation must be $x = 0.79$ or 3.93.

b) The equation $\csc x - 2 = 0$ can be rewritten as follows:

$$\begin{aligned} \csc x - 2 &= 0 \\ \csc x - 2 + 2 &= 0 + 2 \\ \csc x &= 2 \end{aligned}$$

Given $\csc x = 2$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = 0.52$ or 2.62.

c) The equation $7 \sec x = 7$ can be rewritten as follows:

$$7 \sec x = 7$$
$$\frac{7 \sec x}{7} = \frac{7}{7}$$
$$\sec x = 1$$

Given $\sec x = 1$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = 0$ or 6.28.

d) The equation $2 \csc x + 17 = 15 + \csc x$ can be rewritten as follows:

$$2 \csc x + 17 = 15 + \csc x$$
$$2 \csc x + 17 - \csc x = 15 + \csc x - \csc x$$
$$\csc x + 17 = 15$$
$$\csc x + 17 - 17 = 15 - 17$$
$$\csc x = -2$$

Given $\csc x = -2$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = 3.67$ or 5.76.

e) The equation $2 \sec x + 1 = 6$ can be rewritten as follows:

$$2 \sec x + 1 = 6$$
$$2 \sec x + 1 - 1 = 6 - 1$$
$$2 \sec x = 5$$
$$\frac{2 \sec x}{2} = \frac{5}{2}$$
$$\sec x = \frac{5}{2}$$

Given $\sec x = \frac{5}{2}$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = 1.16$ or 5.12.

f) The equation $8 + 4 \cot x = 10$ can be rewritten as follows:

$$8 + 4 \cot x = 10$$
$$8 + 4 \cot x - 8 = 10 - 8$$
$$4 \cot x = 2$$
$$\frac{4 \cot x}{4} = \frac{2}{4}$$
$$\cot x = \frac{1}{2}$$

Given $\cot x = \frac{1}{2}$ and $0 \le x \le 2\pi$, the solutions to the equation must be $x = 1.11$ or 4.25.

10. a) Given $\sin 2x = \frac{1}{\sqrt{2}}$ and $0 \le x \le 2\pi$, $2x$ must equal $0.79 + 2k\pi$ or $2.36 + 2k\pi$. Therefore, the solutions to the equation must be $x = \frac{0.79}{2} = 0.39$, $\frac{2.36}{2} = 1.18, 3.53$, or 4.32.

b) Given $\sin 4x = \frac{1}{2}$ and $0 \le x \le 2\pi$, $4x$ must equal $0.52 + 2k\pi$ or $2.62 + 2k\pi$. Therefore, the solutions to the equation must be $x = \frac{0.52}{4} = 0.13$, $\frac{2.62}{4} = 0.65, 1.70, 2.23, 3.27, 3.80, 4.84$, and 5.37.

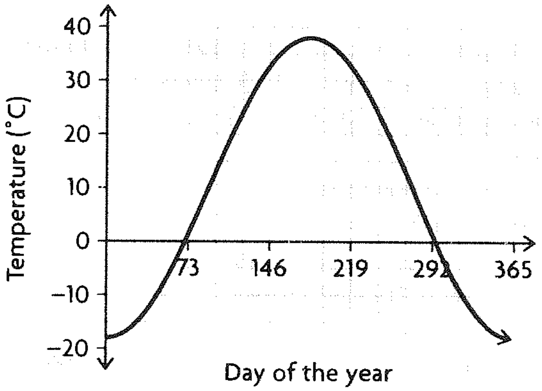
c) Given $\sin 3x = -\frac{\sqrt{3}}{2}$ and $0 \le x \le 2\pi$, $3x$ must equal $4.19 + 2k\pi$ or $5.24 + 2k\pi$. Therefore, the solutions to the equation must be $x = \frac{4.19}{3} = 1.40$, $\frac{5.24}{3} = 1.75, 3.49, 3.84, 5.59$, or 5.93.

d) Given $\cos 4x = -\frac{1}{\sqrt{2}}$ and $0 \le x \le 2\pi$, $4x$ must equal $2.36 + 2k\pi$ or $3.93 + 2k\pi$. Therefore, the solutions to the equation must be $x = \frac{2.36}{4} = 0.59$, $\frac{3.93}{4} = 0.985, 2.16, 2.55, 3.73, 4.12, 5.30$, or 5.697.

e) Given $\cos 2x = -\frac{1}{2}$ and $0 \le x \le 2\pi$, $2x$ must equal $2.09 + 2k\pi$ or $4.19 + 2k\pi$. Therefore, the solutions to the equation must be $x = \frac{2.09}{2} = 1.05$, $\frac{4.19}{2} = 2.09, 4.19$, or 5.24.

f) Given $\cos \frac{x}{2} = \frac{\sqrt{3}}{2}$ and $0 \le x \le 2\pi$, $\frac{x}{2}$ must equal 0.52 or 5.76. Therefore, the solution to the equation must be $x = (2)(0.52) = 1.05$. 5.76 is not a solution to the equation since $(2)(5.76) = 11.52$ is greater than 2π.

11. When graphed, the function modelling the city's daily high temperature is as follows:



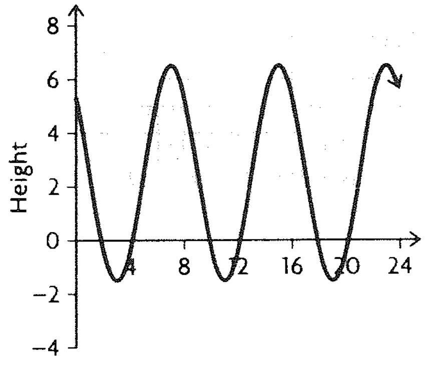
First it's necessary to find the first day when the temperature is approximately 32°C. This can be done as follows:

$$32 = -28 \cos \frac{2\pi}{365}d + 10$$
$$32 - 10 = -28 \cos \frac{2\pi}{365}d + 10 - 10$$
$$22 = -28 \cos \frac{2\pi}{365}d$$
$$\frac{22}{-28} = \frac{-28}{-28} \cos \frac{2\pi}{365}d$$
$$-0.7857 = \cos \frac{2\pi}{365}d$$
$$\cos^{-1}(-0.7857) = \cos^{-1}\left(\cos \frac{2\pi}{365}d\right)$$
$$2.4746 = \frac{2\pi}{365}d$$

$$2.4746 \times \frac{365}{2\pi} = \frac{2\pi}{365}d \times \frac{365}{2\pi}$$
$$d = 143.76$$

Since the first day when the temperature is approximately 32 °C is day 144, and since the period of the function is 365 days, the other day when the temperature is approximately 32 °C is $365 - 144 = 221$. Therefore, the temperature is approximately 32 °C or above from about day 144 to about day 221, and those are the days of the year when the air conditioners are running at the City Hall.

12. When graphed, the function modelling the height of the nail above the surface of the water is as follows:

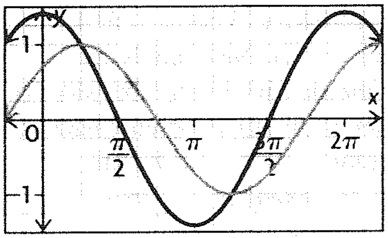


First it's necessary to find the first time when the nail is at the surface of the water. This can be done as follows:

$$0 = -4\sin\frac{\pi}{4}(t - 1) + 2.5$$
$$0 - 2.5 = -4\sin\frac{\pi}{4}(t - 1) + 2.5 - 2.5$$
$$-2.5 = -4\sin\frac{\pi}{4}(t - 1)$$
$$\frac{-2.5}{-4} = \frac{-4}{-4}\sin\frac{\pi}{4}(t - 1)$$
$$0.625 = \sin\frac{\pi}{4}(t - 1)$$
$$\sin^{-1}(0.625) = \sin^{-1}\left(\sin\frac{\pi}{4}(t - 1)\right)$$
$$0.6751 = \frac{\pi}{4}(t - 1)$$
$$0.6751 \times \frac{4}{\pi} = \frac{\pi}{4}(t - 1) \times \frac{4}{\pi}$$
$$t - 1 = 0.8596$$
$$t - 1 + 1 = 0.8596 + 1$$
$$t = 1.86$$

Since the first time when the nail is at the surface of the water is 1.86 s, and since the period of the function is 8 s, the next time the nail is at the surface of the water is $6 - 1.86 = 4.14$ s. Therefore, the nail is below the water when $1.86\text{ s} < t < 4.14$ s. Since the cycle repeats itself two more times in the first 24 s that the wheel is rotating, the nail is also below the water when $9.86\text{ s} < t < 12.14$ and when $17.86\text{ s} < t < 20.14$ s.

13. To solve $\sin\left(x + \frac{\pi}{4}\right) = \sqrt{2}\cos x$ for $0 \le x \le 2\pi$, graph the functions $y = \sin\left(x + \frac{\pi}{4}\right)$ and $y = \sqrt{2}\cos x$ on the same coordinate grid as follows:

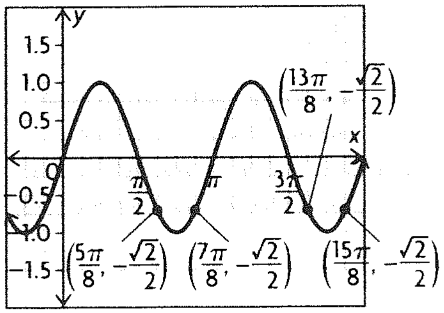


Since the graphs intersect when $x = \frac{\pi}{4}$ and when $x = \frac{5\pi}{4}$, the solution to the equation is $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

14. Given $\sin 2\theta = -\frac{1}{\sqrt{2}}$ and $0 \le x \le 2\pi$, 2θ must equal $\frac{5\pi}{4} + 2\pi k$ or $\frac{7\pi}{4} + 2\pi k$. Therefore, the solutions to the equation must be

$$\theta = \frac{\frac{5\pi}{4} + 2\pi k}{2} = \frac{5\pi}{8} + \pi k \text{ or}$$
$$\frac{\frac{7\pi}{4} + 2\pi k}{2} = \frac{7\pi}{8} + \pi k$$

So the solution are $\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}$, and $\frac{15\pi}{8}$. When the solutions are plotted on the graph of the function $y = \sin 2\theta$ for $0 \le \theta \le 2\pi$, the graph appears as follows:



15. The value of $f(x) = \sin x$ is the same at x and $\pi - x$. In other words, it is the same at x and half the period minus x. Since the period of $f(x) = 25 \sin \frac{\pi}{50}(x + 20) - 55$ is 100, if the function were not horizontally translated, its value at x would be the same as at $50 - x$. The function is horizontally translated 20 units to the left, however, so it goes through half its period from $x = -20$ to $x = 30$. At $x = 3$, the function is 23 units away from the left end of the range, so it will have the same value at $x = 30 - 23$ or $x = 7$, which is 23 units away from the right end of the range.

16. To solve a trigonometric equation **algebraically**, first isolate the trigonometric function on one side of the equation. For example, the trigonometric equation $5 \cos x - 3 = 2$ would become $5 \cos x = 5$, which would then become $\cos x = 1$. Next, apply the inverse of the trigonometric function to both sides of the equation. For example, the trigonometric equation $\cos x = 1$ would become $x = \cos^{-1} 1$. Finally, simplify the equation. For example, $x = \cos^{-1} 1$ would become $x = 0 + 2n\pi$, where $n \in \mathbf{I}$.

To solve a trigonometric equation **graphically**, first isolate the trigonometric function on one side of the equation. For example, the trigonometric equation $5 \cos x - 3 = 2$ would become $5 \cos x = 5$, which would then become $\cos x = 1$. Next, graph both sides of the equation. For example, the functions $f(x) = \cos x$ and $f(x) = 1$ would both be graphed. Finally, find the points where the two graphs intersect. For example, $f(x) = \cos x$ and $f(x) = 1$ would intersect at $x = 0 + 2n\pi$, where $n \in \mathbf{I}$.

Similarity: Both trigonometric functions are first isolated on one side of the equation.

Differences: The inverse of a trigonometric function is not applied in the graphical method, and the points of intersection are not obtained in the algebraic method.

17. To solve the trigonometric equation $2 \sin x \cos x + \sin x = 0$, first factor $\sin x$ from the left side of the equation as follows:

$2 \sin x \cos x + \sin x = 0$

$(\sin x)(2 \cos x + 1) = 0$

Since $(\sin x)(2 \cos x + 1) = 0$, either $\sin x = 0$ or $2 \cos x + 1 = 0$ (or both). If $\sin x = 0$, one solution to the equation is $x = 0 + n\pi$, where $n \in \mathbf{I}$. If $2 \cos x + 1 = 0$, x can be found by first rewriting the equation as follows:

$$2 \cos x + 1 = 0$$
$$2 \cos x + 1 - 1 = 0 - 1$$
$$2 \cos x = -1$$
$$\frac{2 \cos x}{2} = -\frac{1}{2}$$
$$\cos x = -\frac{1}{2}$$

The solutions to the equation $\cos x = -\frac{1}{2}$ occur at $x = \frac{2\pi}{3} + 2n\pi$ or $\frac{4\pi}{3} + 2n\pi$, where $n \in \mathbf{I}$, since the period of the cosine function is 2π. Therefore, the solutions to the equation $2 \sin x \cos x + \sin x = 0$ are $x = 0 + n\pi$, $\frac{2\pi}{3} + 2n\pi$, or $\frac{4\pi}{3} + 2n\pi$, where $n \in \mathbf{I}$.

18. a) Since $\sin 2\theta = 2 \sin \theta \cos \theta$, the equation $\sin 2x - 2 \cos^2 x = 0$ can be rewritten as follows:

$$\sin 2x - 2 \cos^2 x = 0$$
$$2 \sin x \cos x - 2 \cos^2 x = 0$$
$$(2 \cos x)(\sin x - \cos x) = 0$$

Since $(2 \cos x)(\sin x - \cos x) = 0$, either $2 \cos x = 0$ or $\sin x - \cos x = 0$ (or both). If $2 \cos x = 0$, x can be found by first rewriting the equation as follows:

$$2 \cos x = 0$$
$$\frac{2 \cos x}{2} = \frac{0}{2}$$
$$\cos x = 0$$

The solutions to the equation $\cos x = 0$ occur at $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. If $\sin x - \cos x = 0$, x can be found by first rewriting the equation as follows:

$$\sin x - \cos x = 0$$
$$\sin x - \cos x + \cos x = 0 + \cos x$$
$$\sin x = \cos x$$

The solutions to the equation $\sin x = \cos x$ occur at $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. Therefore, the solutions to the equation $\sin 2x - 2 \cos^2 x = 0$ are $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}$, or $\frac{3\pi}{2}$.

b) Since $\cos 2\theta = 1 - 2 \sin^2 \theta$, the equation $3 \sin x + \cos 2x = 2$ can be rewritten as follows:

$$3 \sin x + \cos 2x = 2$$
$$3 \sin x + 1 - 2 \sin^2 x = 2$$
$$3 \sin x + 1 - 2 \sin^2 x - 2 = 2 - 2$$
$$3 \sin x - 1 - 2 \sin^2 x = 0$$
$$-2 \sin^2 x + 3 \sin x - 1 = 0$$
$$(-1)(-2 \sin^2 x + 3 \sin x - 1) = (-1)(0)$$
$$2 \sin^2 x - 3 \sin x + 1 = 0$$
$$(2 \sin x - 1)(\sin x - 1) = 0$$

Since $(2 \sin x - 1)(\sin x - 1) = 0$, either $2 \sin x - 1 = 0$ or $\sin x - 1 = 0$ (or both).

If $\sin x - 1 = 0$, x can be found by first rewriting the equation as follows:

$$2 \sin x - 1 = 0$$
$$2 \sin x - 1 + 1 = 0 + 1$$
$$2 \sin x = 1$$
$$\frac{2 \sin x}{2} = \frac{1}{2}$$
$$\sin x = \frac{1}{2}$$

The solutions to the equation $\sin x = \frac{1}{2}$ occur at $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. If $\sin x - 1 = 0$, x can be found by first rewriting the equation as follows:

$$\sin x - 1 = 0$$
$$\sin x - 1 + 1 = 0 + 1$$
$$\sin x = 1$$

The solution to the equation $\sin x = 1$ occurs at $x = \frac{\pi}{2}$. Therefore, the solutions to the equation $3 \sin x + \cos 2x = 2$ are $x = \frac{\pi}{6}, \frac{\pi}{2}$, or $\frac{5\pi}{6}$.

7.6 Solving Quadratic Trigonometric Equations, pp. 435–437

1. a) The expression $\sin^2 \theta - \sin \theta$ can be factored as follows:

$\sin^2 \theta - \sin \theta$
$(\sin \theta)(\sin \theta - 1)$

b) The expression $\cos^2 \theta - 2 \cos \theta + 1$ can be factored as follows:

$\cos^2 \theta - 2 \cos \theta + 1$
$(\cos \theta - 1)(\cos \theta - 1)$

c) The expression $3 \sin^2 \theta - \sin \theta - 2$ can be factored as follows:

$(3 \sin \theta + 2)(\sin \theta - 1)$

d) The expression $4 \cos^2 \theta - 1$ can be factored as follows:

$4 \cos^2 \theta - 1$
$(2 \cos \theta - 1)(2 \cos \theta + 1)$

e) The expression $24 \sin^2 x - 2 \sin x - 2$ can be factored as follows:

$24 \sin^2 x - 2 \sin x - 2$
$(6 \sin x - 2)(4 \sin x + 1)$

f) The expression $49 \tan^2 x - 64$ can be factored as follows:

$49 \tan^2 x - 64$
$(7 \tan x + 8)(7 \tan x - 8)$

2. a) The equation $y^2 = \frac{1}{3}$ can be solved as follows:

$$y^2 = \frac{1}{3}$$
$$\sqrt{y^2} = \pm\sqrt{\frac{1}{3}}$$
$$y = \pm\sqrt{\frac{1}{3}}$$
$$y = \pm\frac{\sqrt{3}}{3}$$

Likewise, the equation $\tan^2 x = \frac{1}{3}$ can be solved as follows:

$$\tan^2 x = \frac{1}{3}$$
$$\sqrt{\tan^2 x} = \pm\sqrt{\frac{1}{3}}$$
$$\tan x = \pm\sqrt{\frac{1}{3}}$$
$$\tan x = \pm\frac{\sqrt{3}}{3}$$

The solutions to the equation $\tan x = \pm\frac{\sqrt{3}}{3}$ occur at $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$.

b) The equation $y^2 + y = 0$ can be solved as follows:

$$y^2 + y = 0$$
$$(y)(y + 1) = 0$$

Since $(y)(y + 1) = 0$, either $y = 0$ or $y + 1 = 0$ (or both). If $y + 1 = 0$, y can be solved for as follows:

$$y + 1 = 0$$
$$y + 1 - 1 = 0 - 1$$
$$y = -1$$

Therefore, the solutions to the equation $y^2 + y = 0$ are $y = 0$ or $y = -1$. Likewise, the equation $\sin^2 x + \sin x = 0$ can be solved as follows:

$$\sin^2 x + \sin x = 0$$
$$(\sin x)(\sin x + 1) = 0$$

Since $(\sin x)(\sin x + 1) = 0$, either $\sin x = 0$ or $\sin x + 1 = 0$ (or both). The solutions to the equation $\sin x = 0$ occur at $x = 0, \pi$, or 2π. If $\sin x + 1 = 0$, x can be solved for as follows:

$$\sin x + 1 = 0$$
$$\sin x + 1 - 1 = 0 - 1$$
$$\sin x = -1$$

The solution to the equation $\sin x = -1$ occurs at $x = \frac{3\pi}{2}$, so the solutions to the equation $\sin^2 x + \sin x = 0$ are $x = 0, \pi, \frac{3\pi}{2}$, or 2π.

c) The equation $y - 2yz = 0$ can be solved as follows:

$y - 2yz = 0$
$(y)(1 - 2z) = 0$

Since $(y)(1 - 2z) = 0$, either $y = 0$ or $1 - 2z = 0$ (or both). If $1 - 2z = 0$, z can be solved for as follows:

$1 - 2z = 0$
$1 - 2z + 2z = 0 + 2z$
$1 = 2z$
$\frac{1}{2} = \frac{2z}{2}$
$z = \frac{1}{2}$

Likewise, the equation $\cos x - 2 \cos x \sin x = 0$ can be solved as follows:

$\cos x - 2 \cos x \sin x = 0$
$(\cos x)(1 - 2 \sin x) = 0$

Since $(\cos x)(1 - 2 \sin x) = 0$, either $\cos x = 0$ or $1 - 2 \sin x = 0$ (or both). The solutions to the equation $\cos x = 0$ occur at $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. If $1 - 2 \sin x = 0$, x can be solved for as follows:

$1 - 2 \sin x = 0$
$1 - 2 \sin x + 2 \sin x = 0 + 2 \sin x$
$1 = 2 \sin x$
$\frac{1}{2} = \frac{2 \sin x}{2}$
$\sin x = \frac{1}{2}$

The solutions to the equation $\sin x = \frac{1}{2}$ occur at $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, so the solutions to the equation $\cos x - 2 \cos x \sin x = 0$ are $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, or $\frac{3\pi}{2}$.

d) The equation $yz = y$ can be solved as follows:

$yz = y$
$yz - y = y - y$
$yz - y = 0$
$(y)(z - 1) = 0$

Since $(y)(z - 1) = 0$, either $y = 0$ or $z - 1 = 0$ (or both). If $z - 1 = 0$, z can be solved for as follows:

$z - 1 = 0$
$z - 1 + 1 = 0 + 1$
$z = 1$

Therefore, the solutions to the equation $yz = y$ are $y = 0$ or $z = 1$. Likewise, the equation $\tan x \sec x = \tan x$ can be solved as follows:

$\tan x \sec x = \tan x$
$\tan x \sec x - \tan x = \tan x - \tan x$
$\tan x \sec x - \tan x = 0$
$(\tan x)(\sec x - 1) = 0$

Since $(\tan x)(\sec x - 1) = 0$, either $\tan x = 0$ or $\sec x - 1 = 0$ (or both). The solutions to the equation $\tan x = 0$ occur at $x = 0$, π, or 2π. If $\sec x - 1 = 0$, x can be solved for as follows:

$\sec x - 1 = 0$
$\sec x - 1 + 1 = 0 + 1$
$\sec x = 1$

The solutions to the equation $\sec x = 1$ occur at $x = 0$ or 2π, so the solutions to the equation $(\tan x)(\sec x - 1) = 0$ are $x = 0$, π, or 2π.

3. a) The equation $6y^2 - y - 1 = 0$ can be solved as follows:

$6y^2 - y - 1 = 0$
$(2y - 1)(3y + 1) = 0$

Since $(2y - 1)(3y + 1) = 0$, either $2y - 1 = 0$ or $3y + 1 = 0$ (or both). If $2y - 1 = 0$, y can be solved for as follows:

$2y - 1 = 0$
$2y - 1 + 1 = 0 + 1$
$2y = 1$
$\frac{2y}{2} = \frac{1}{2}$
$y = \frac{1}{2}$

If $3y + 1 = 0$, y can be solved for as follows:

$3y + 1 = 0$
$3y + 1 - 1 = 0 - 1$
$3y = -1$
$\frac{3y}{3} = -\frac{1}{3}$
$y = -\frac{1}{3}$

Therefore, the solutions to the equation $6y^2 - y - 1 = 0$ are $y = \frac{1}{2}$ or $-\frac{1}{3}$.

b) The equation $6 \cos^2 x - \cos x - 1 = 0$ for $0 \leq x \leq 2\pi$ can be solved as follows:

$6 \cos^2 x - \cos x - 1 = 0$
$(2 \cos x - 1)(3 \cos x + 1) = 0$

Since $(2 \cos x - 1)(3 \cos x + 1) = 0$, either $2 \cos x - 1 = 0$ or $3 \cos x + 1 = 0$ (or both). If $2 \cos x - 1 = 0$, x can be solved for as follows:

$2 \cos x - 1 = 0$
$2 \cos x - 1 + 1 = 0 + 1$

$$2\cos x = 1$$
$$\frac{2\cos x}{2} = \frac{1}{2}$$
$$\cos x = \frac{1}{2}$$

The solutions to the equation $\cos x = \frac{1}{2}$ occur at $x = 1.05$ or 5.24. If $3\cos x + 1 = 0$, x can be solved for as follows:

$$3\cos x + 1 = 0$$
$$3\cos x + 1 - 1 = 0 - 1$$
$$3\cos x = -1$$
$$\frac{3\cos x}{3} = -\frac{1}{3}$$
$$\cos x = -\frac{1}{3}$$

The solutions to the equation $\cos x = -\frac{1}{3}$ occur at $x = 1.91$ or 4.37. Therefore, the solutions to the equation $6\cos^2 x - \cos x - 1 = 0$ are $x = 1.05$, 1.91, 4.37, or 5.24.

4. a) The equation $\sin^2\theta = 1$ can be solved as follows:

$$\sin^2\theta = 1$$
$$\sqrt{\sin^2\theta} = \pm\sqrt{1}$$
$$\sin\theta = \pm 1$$

The solutions to the equation $\sin\theta = \pm 1$ occur at $\theta = 90°$ or $270°$.

b) The equation $\cos^2\theta = 1$ can be solved as follows:

$$\cos^2\theta = 1$$
$$\sqrt{\cos^2\theta} = \pm\sqrt{1}$$
$$\cos\theta = \pm 1$$

The solutions to the equation $\cos\theta = \pm 1$ occur at $\theta = 0°$, $180°$, or $360°$.

c) The equation $\tan^2\theta = 1$ can be solved as follows:

$$\tan^2\theta = 1$$
$$\sqrt{\tan^2\theta} = \pm\sqrt{1}$$
$$\tan\theta = \pm 1$$

The solutions to the equation $\tan\theta = \pm 1$ occur at $\theta = 45°$, $135°$, $225°$, or $315°$.

d) The equation $4\cos^2\theta = 1$ can be solved as follows:

$$4\cos^2\theta = 1$$
$$\frac{4\cos^2\theta}{4} = \frac{1}{4}$$
$$\cos^2\theta = \frac{1}{4}$$
$$\sqrt{\cos^2\theta} = \pm\sqrt{\frac{1}{4}}$$
$$\cos\theta = \pm\frac{1}{2}$$

The solutions to the equation $\cos\theta = \pm\frac{1}{2}$ occur at $\theta = 60°$, $120°$, $240°$, or $300°$.

e) The equation $3\tan^2\theta = 1$ can be solved as follows:

$$3\tan^2\theta = 1$$
$$\frac{3\tan^2\theta}{3} = \frac{1}{3}$$
$$\tan^2\theta = \frac{1}{3}$$
$$\sqrt{\tan^2\theta} = \pm\sqrt{\frac{1}{3}}$$
$$\tan\theta = \pm\frac{\sqrt{3}}{3}$$

The solutions to the equation $\tan\theta = \pm\frac{\sqrt{3}}{3}$ occur at $\theta = 30°$, $150°$, $210°$, or $330°$.

f) The equation $2\sin^2\theta = 1$ can be solved as follows:

$$2\sin^2\theta = 1$$
$$\frac{2\sin^2\theta}{2} = \frac{1}{2}$$
$$\sin^2\theta = \frac{1}{2}$$
$$\sqrt{\sin^2\theta} = \pm\sqrt{\frac{1}{2}}$$
$$\sin\theta = \pm\frac{\sqrt{2}}{2}$$

The solutions to the equation $\sin\theta = \pm\frac{\sqrt{2}}{2}$ occur at $\theta = 45°$, $135°$, $225°$, or $315°$.

5. a) Since $\sin x\cos x = 0$, either $\sin x = 0$ or $\cos x = 0$ (or both). If $\sin x = 0$, the solutions for x occur at $x = 0°$, $180°$, or $360°$. If $\cos x = 0$, the solutions for x occur at $x = 90°$ or $270°$. Therefore, the solutions to the equation occur at $x = 0°$, $90°$, $180°$, $270°$, or $360°$.

b) Since $\sin x(\cos x - 1) = 0$, either $\sin x = 0$ or $\cos x - 1 = 0$ (or both). If $\sin x = 0$, the solutions for x occur at $x = 0°$, $180°$, or $360°$. If $\cos x - 1 = 0$, x can be solved for as follows:

$$\cos x - 1 = 0$$
$$\cos x - 1 + 1 = 0 + 1$$
$$\cos x = 1$$

The solutions to the equation $\cos x = 1$ occur at $x = 0°$ or $360°$. Therefore, the solutions to the equation $\sin x(\cos x - 1) = 0$ occur at $x = 0°$, $180°$, or $360°$.

c) Since $(\sin x + 1)\cos x = 0$, either $\sin x + 1 = 0$ or $\cos x = 0$ (or both). If $\sin x + 1 = 0$, x can be solved for as follows:

$$\sin x + 1 = 0$$
$$\sin x + 1 - 1 = 0 - 1$$
$$\sin x = -1$$

The solution to the equation $\sin x = -1$ occurs at $x = 270°$. If $\cos x = 0$, the solutions for x occur at $x = 90°$ or $270°$. Therefore, the solutions to the equation $(\sin x + 1)\cos x = 0$ occur at $x = 90°$ or $270°$.

d) Since $\cos x\,(2\sin x - \sqrt{3}) = 0$, either $\cos x = 0$ or $2\sin x - \sqrt{3} = 0$ (or both). If $\cos x = 0$, the solutions for x occur at $x = 90°$ or $270°$. If $2\sin x - \sqrt{3} = 0$, x can be solved for as follows:

$$2\sin x - \sqrt{3} = 0$$
$$2\sin x - \sqrt{3} + \sqrt{3} = 0 + \sqrt{3}$$
$$2\sin x = \sqrt{3}$$
$$\frac{2\sin x}{2} = \frac{\sqrt{3}}{2}$$
$$\sin x = \frac{\sqrt{3}}{2}$$

The solutions to the equation $\sin x = \frac{\sqrt{3}}{2}$ occur at $x = 60°$ and $120°$. Therefore, the solutions to the equation $\cos x(2\sin x - \sqrt{3}) = 0$ occur at $x = 60°$, $90°$, $120°$, or $270°$.

e) Since $(\sqrt{2}\sin x - 1)(\sqrt{2}\sin x + 1) = 0$, either $\sqrt{2}\sin x - 1 = 0$ or $\sqrt{2}\sin x + 1 = 0$ (or both). If $\sqrt{2}\sin x - 1 = 0$, x can be solved for as follows:

$$\sqrt{2}\sin x - 1 = 0$$
$$\sqrt{2}\sin x - 1 + 1 = 0 + 1$$
$$\sqrt{2}\sin x = 1$$
$$\frac{\sqrt{2}\sin x}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$\sin x = \frac{1}{\sqrt{2}}$$
$$\sin x = \frac{\sqrt{2}}{2}$$

The solutions to the equation $\sin x = \frac{\sqrt{2}}{2}$ occur at $x = 45°$ or $135°$. If $\sqrt{2}\sin x + 1 = 0$, x can be solved for as follows:

$$\sqrt{2}\sin x + 1 = 0$$
$$\sqrt{2}\sin x + 1 - 1 = 0 - 1$$
$$\sqrt{2}\sin x = -1$$
$$\frac{\sqrt{2}\sin x}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$
$$\sin x = -\frac{1}{\sqrt{2}}$$
$$\sin x = -\frac{\sqrt{2}}{2}$$

The solutions to the equation $\sin x = -\frac{\sqrt{2}}{2}$ occur at $x = 225°$ or $315°$. Therefore, the solutions to the equation $(\sqrt{2}\sin x - 1)(\sqrt{2}\sin x + 1) = 0$ occur at $x = 45°$, $135°$, $225°$, or $315°$.

f) Since $(\sin x - 1)(\cos x + 1) = 0$, either $\sin x - 1 = 0$ or $\cos x + 1 = 0$ (or both). If $\sin x - 1 = 0$, x can be solved for as follows:

$$\sin x - 1 = 0$$
$$\sin x - 1 + 1 = 0 + 1$$
$$\sin x = 1$$

The solution to the equation $\sin x = 1$ occurs at $x = 90°$. If $\cos x + 1 = 0$, x can be solved for as follows:

$$\cos x + 1 = 0$$
$$\cos x + 1 - 1 = 0 - 1$$
$$\cos x = -1$$

The solution to the equation $\cos x = -1$ occurs at $x = 180°$. Therefore, the solutions to the equation $(\sin x - 1)(\cos x + 1) = 0$ occur at $x = 90°$ or $180°$.

6. a) Since $(2\sin x - 1)\cos x = 0$, either $2\sin x - 1 = 0$ or $\cos x = 0$. If $2\sin x - 1 = 0$, x can be solved for as follows:

$$2\sin x - 1 = 0$$
$$2\sin x - 1 + 1 = 0 + 1$$
$$2\sin x = 1$$
$$\frac{2\sin x}{2} = \frac{1}{2}$$
$$\sin x = \frac{1}{2}$$

The solutions to the equation $\sin x = \frac{1}{2}$ occur at $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. Also, the solutions to the equation $\cos x = 0$ occur at $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Therefore, the solutions to the equation $(2\sin x - 1)\cos x = 0$ occur at $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, or $\frac{3\pi}{2}$.

b) Since $(\sin x + 1)^2 = 0$, $\sin x + 1 = 0$, so x can be solved for as follows:

$$\sin x + 1 = 0$$
$$\sin x + 1 - 1 = 0 - 1$$
$$\sin x = -1$$

The solution to the equation $\sin x = -1$ occurs at $x = \frac{3\pi}{2}$, so the solution to the equation $(\sin x + 1)^2 = 0$ occurs at $x = \frac{3\pi}{2}$.

c) Since $(2\cos x + \sqrt{3})\sin x = 0$, either $2\cos x + \sqrt{3} = 0$ or $\sin x = 0$ (or both). If $2\cos x + \sqrt{3} = 0$, x can be solved for as follows:

$$\begin{aligned} 2\cos x + \sqrt{3} &= 0 \\ 2\cos x + \sqrt{3} - \sqrt{3} &= 0 - \sqrt{3} \\ 2\cos x &= -\sqrt{3} \\ \frac{2\cos x}{2} &= -\frac{\sqrt{3}}{2} \\ \cos x &= -\frac{\sqrt{3}}{2} \end{aligned}$$

The solutions to the equation $\cos x = -\frac{\sqrt{3}}{2}$ occur at $x = \frac{5\pi}{6}$ or $\frac{7\pi}{6}$. Also, the solutions to the equation $\sin x = 0$ occur at $x = 0$, π, or 2π. Therefore, the solutions to the equation $(2\cos x + \sqrt{3})\sin x = 0$ occur at $x = 0, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}$, or 2π.

d) Since $(2\cos x - 1)(2\sin x + \sqrt{3}) = 0$, either $2\cos x - 1 = 0$ or $2\sin x + \sqrt{3} = 0$. If $2\cos x - 1 = 0$, x can be solved for as follows:

$$\begin{aligned} 2\cos x - 1 &= 0 \\ 2\cos x - 1 + 1 &= 0 + 1 \\ 2\cos x &= 1 \\ \frac{2\cos x}{2} &= \frac{1}{2} \\ \cos x &= \frac{1}{2} \end{aligned}$$

The solutions to the equation $\cos x = \frac{1}{2}$ occur at $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. If $2\sin x + \sqrt{3} = 0$, x can be solved for as follows:

$$\begin{aligned} 2\sin x + \sqrt{3} &= 0 \\ 2\sin x + \sqrt{3} - \sqrt{3} &= 0 - \sqrt{3} \\ 2\sin x &= -\sqrt{3} \\ \frac{2\sin x}{2} &= -\frac{\sqrt{3}}{2} \\ \sin x &= -\frac{\sqrt{3}}{2} \end{aligned}$$

The solutions to the equation $\sin x = -\frac{\sqrt{3}}{2}$ occur at $x = \frac{4\pi}{3}$ or $\frac{5\pi}{3}$. Therefore, the solutions to the equation $(2\cos x - 1)(2\sin x + \sqrt{3}) = 0$ occur at $x = \frac{\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$.

e) Since $(\sqrt{2}\cos x - 1)(\sqrt{2}\cos x + 1) = 0$, either $\sqrt{2}\cos x - 1 = 0$ or $\sqrt{2}\cos x + 1 = 0$ (or both). If $\sqrt{2}\cos x - 1 = 0$, x can be solved for as follows:

$$\begin{aligned} \sqrt{2}\cos x - 1 &= 0 \\ \sqrt{2}\cos x - 1 + 1 &= 0 + 1 \\ \sqrt{2}\cos x &= 1 \\ \frac{\sqrt{2}\cos x}{\sqrt{2}} &= \frac{1}{\sqrt{2}} \\ \cos x &= \frac{1}{\sqrt{2}} \\ \cos x &= \frac{\sqrt{2}}{2} \end{aligned}$$

The solutions to the equation $\cos x = \frac{\sqrt{2}}{2}$ occur at $x = \frac{\pi}{4}$ or $\frac{7\pi}{4}$. If $\sqrt{2}\cos x + 1 = 0$, x can be solved for as follows:

$$\begin{aligned} \sqrt{2}\cos x + 1 &= 0 \\ \sqrt{2}\cos x + 1 - 1 &= 0 - 1 \\ \sqrt{2}\cos x &= -1 \\ \frac{\sqrt{2}\cos x}{\sqrt{2}} &= -\frac{1}{\sqrt{2}} \\ \cos x &= -\frac{1}{\sqrt{2}} \\ \cos x &= -\frac{\sqrt{2}}{2} \end{aligned}$$

The solutions to the equation $\cos x = -\frac{\sqrt{2}}{2}$ occur at $x = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$. Therefore, the solutions to the equation $(\sqrt{2}\cos x - 1)(\sqrt{2}\cos x + 1) = 0$ occur at $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$.

f) Since $(\sin x + 1)(\cos x - 1) = 0$, either $\sin x + 1 = 0$ or $\cos x - 1 = 0$ (or both). If $\sin x + 1 = 0$, x can be solved for as follows:

$$\begin{aligned} \sin x + 1 &= 0 \\ \sin x + 1 - 1 &= 0 - 1 \\ \sin x &= -1 \end{aligned}$$

The solution to the equation $\sin x = -1$ occurs at $x = \frac{3\pi}{2}$. If $\cos x - 1 = 0$, x can be solved for as follows:

$$\begin{aligned} \cos x - 1 &= 0 \\ \cos x - 1 + 1 &= 0 + 1 \\ \cos x &= 1 \end{aligned}$$

The solutions to the equation $\cos x = 1$ occur at $x = 0$ or 2π. Therefore, the solutions to the equation $(\sin x + 1)(\cos x - 1) = 0$ occur at $x = 0, \frac{3\pi}{2}$, or 2π.

7. a) Since $2\cos^2\theta + \cos\theta - 1 = 0$, $(2\cos\theta - 1)(\cos\theta + 1) = 0$. For this reason, either $2\cos\theta - 1 = 0$ or $\cos\theta + 1 = 0$ (or both). If $2\cos\theta - 1 = 0$, θ can be solved for as follows:

$$2\cos\theta - 1 = 0$$
$$2\cos\theta - 1 + 1 = 0 + 1$$
$$2\cos\theta = 1$$
$$\frac{2\cos\theta}{2} = \frac{1}{2}$$
$$\cos\theta = \frac{1}{2}$$

The solutions to the equation $\cos\theta = \frac{1}{2}$ occur at $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. If $\cos\theta + 1 = 0$, θ can be solved for as follows:

$$\cos\theta + 1 = 0$$
$$\cos\theta + 1 - 1 = 0 - 1$$
$$\cos\theta = -1$$

The solution to the equation $\cos\theta = -1$ occurs at $\theta = \pi$. Therefore, the solutions to the equation $2\cos^2\theta + \cos\theta - 1 = 0$ occur at $\theta = \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3}$.

b) The equation $2\sin^2\theta = 1 - \sin\theta$ can be rewritten and factored as follows:

$$2\sin^2\theta = 1 - \sin\theta$$
$$2\sin^2\theta - 1 + \sin\theta = 1 - \sin\theta - 1 + \sin\theta$$
$$2\sin^2\theta + \sin\theta - 1 = 0$$
$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$

Since $(2\sin\theta - 1)(\sin\theta + 1) = 0$, either $2\sin\theta - 1 = 0$ or $\sin\theta + 1 = 0$ (or both). If $2\sin\theta - 1 = 0$, θ can be solved for as follows:

$$2\sin\theta - 1 = 0$$
$$2\sin\theta - 1 + 1 = 0 + 1$$
$$2\sin\theta = 1$$
$$\frac{2\sin\theta}{2} = \frac{1}{2}$$
$$\sin\theta = \frac{1}{2}$$

The solutions to the equation $\sin\theta = \frac{1}{2}$ occur at $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. If $\sin\theta + 1 = 0$, θ can be solved for as follows:

$$\sin\theta + 1 = 0$$
$$\sin\theta + 1 - 1 = 0 - 1$$
$$\sin\theta = -1$$

The solution to the equation $\sin\theta = -1$ occurs at $x = \frac{3\pi}{2}$. Therefore, the solutions to the equation $2\sin^2\theta = 1 - \sin\theta$ occur at $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$, or $\frac{3\pi}{2}$.

c) The equation $\cos^2\theta = 2 + \cos\theta$ can be rewritten and factored as follows:

$$\cos^2\theta = 2 + \cos\theta$$
$$\cos^2\theta - 2 - \cos\theta = 2 + \cos\theta - 2 - \cos\theta$$
$$\cos^2\theta - \cos\theta - 2 = 0$$
$$(\cos\theta - 2)(\cos\theta + 1) = 0$$

Since $(\cos\theta - 2)(\cos\theta + 1) = 0$, either $\cos\theta - 2 = 0$ or $\cos\theta + 1 = 0$. If $\cos\theta - 2 = 0$, the equation can be rewritten as follows:

$$\cos\theta - 2 = 0$$
$$\cos\theta - 2 + 2 = 0 + 2$$
$$\cos\theta = 2$$

Since $\cos\theta$ can never equal 2, the factor $\cos\theta - 2$ can be ignored. If $\cos\theta + 1 = 0$, θ can be solved for as follows:

$$\cos\theta + 1 = 0$$
$$\cos\theta + 1 - 1 = 0 - 1$$
$$\cos\theta = -1$$

The solution to the equation $\cos\theta = -1$ occurs at $\theta = \pi$. Therefore, the solution to the equation $\cos^2\theta = 2 + \cos\theta$ occurs at $\theta = \pi$.

d) Since $2\sin^2\theta + 5\sin\theta - 3 = 0$, $(2\sin\theta - 1)(\sin\theta + 3) = 0$. For this reason, either $2\sin\theta - 1 = 0$ or $\sin\theta + 3 = 0$. If $2\sin\theta - 1 = 0$, θ can be solved for as follows:

$$2\sin\theta - 1 = 0$$
$$2\sin\theta - 1 + 1 = 0 + 1$$
$$2\sin\theta = 1$$
$$\frac{2\sin\theta}{2} = \frac{1}{2}$$
$$\sin\theta = \frac{1}{2}$$

The solutions to the equation $\sin\theta = \frac{1}{2}$ occur at $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. If $\sin\theta + 3 = 0$, the equation can be rewritten as follows:

$$\sin\theta + 3 = 0$$
$$\sin\theta + 3 - 3 = 0 - 3$$
$$\sin\theta = -3$$

Since $\sin\theta$ can never equal -3, the factor $\sin\theta + 3$ can be ignored. Therefore, the solutions to the equation $2\sin^2\theta + 5\sin\theta - 3 = 0$ occur at $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

e) The equation $3\tan^2\theta - 2\tan\theta = 1$ can be rewritten and factored as follows:

$$3\tan^2\theta - 2\tan\theta = 1$$
$$3\tan^2\theta - 2\tan\theta - 1 = 1 - 1$$
$$3\tan^2\theta - 2\tan\theta - 1 = 0$$
$$(3\tan\theta + 1)(\tan\theta - 1) = 0$$

Since $(3\tan\theta + 1)(\tan\theta - 1) = 0$, either $3\tan\theta + 1 = 0$ or $\tan\theta - 1 = 0$ (or both). If $3\tan\theta + 1 = 0$, θ can be solved for as follows:

$$3\tan\theta + 1 = 0$$
$$3\tan\theta + 1 - 1 = 0 - 1$$
$$3\tan\theta = -1$$
$$\frac{3\tan\theta}{3} = -\frac{1}{3}$$
$$\tan\theta = -\frac{1}{3}$$

The solutions to the equation $\tan\theta = -\frac{1}{3}$ occur at $\theta = 2.81$ or 5.96. If $\tan\theta - 1 = 0$, θ can be solved for as follows:

$$\tan\theta - 1 = 0$$
$$\tan\theta - 1 + 1 = 0 + 1$$
$$\tan\theta = 1$$

The solutions to the equation $\tan\theta = 1$ occur at $\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. Therefore, the solutions to the equation $3\tan^2\theta - 2\tan\theta = 1$ occur at $\theta = \frac{\pi}{4}, 2.82, \frac{5\pi}{4}$, or 5.96.

f) Since $12\sin^2\theta + \sin\theta - 6 = 0$, $(4\sin\theta + 3)(3\sin\theta - 2) = 0$. For this reason, either $4\sin\theta + 3 = 0$ or $3\sin\theta - 2 = 0$ (or both). If $4\sin\theta + 3 = 0$, θ can be solved for as follows:

$$4\sin\theta + 3 = 0$$
$$4\sin\theta + 3 - 3 = 0 - 3$$
$$4\sin\theta = -3$$
$$\frac{4\sin\theta}{4} = -\frac{3}{4}$$
$$\sin\theta = -\frac{3}{4}$$

The solutions to the equation $\sin\theta = -\frac{3}{4}$ occur at $\theta = 3.99$ or 5.44. If $3\sin\theta - 2 = 0$, θ can be solved for as follows:

$$3\sin\theta - 2 = 0$$
$$3\sin\theta - 2 + 2 = 0 + 2$$
$$3\sin\theta = 2$$
$$\frac{3\sin\theta}{2} = \frac{2}{3}$$
$$\sin\theta = \frac{2}{3}$$

The solutions to the equation $\sin\theta = \frac{2}{3}$ occur at $\theta = 0.73$ or 2.41. Therefore, the solutions to the equation $12\sin^2\theta + \sin\theta - 6 = 0$ occur at $\theta = 0.73, 2.41, 3.99$, or 5.44.

8. a) Since $\sec x\csc x - 2\csc x = 0$, $(\csc x)(\sec x - 2) = 0$. For this reason, either $\csc x = 0$ or $\sec x - 2 = 0$. Since $\csc x$ can never equal 0, the factor $\csc x$ can be ignored. If $\sec x - 2 = 0$, x can be solved for as follows:

$$\sec x - 2 = 0$$
$$\sec x - 2 + 2 = 0 + 2$$
$$\sec x = 2$$

The solutions to the equation $\sec x = 2$ occur at $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. Therefore, the solutions to the equation $\sec x\csc x - 2\csc x = 0$ occur at $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

b) Since $3\sec^2 x - 4 = 0$, $(\sqrt{3}\sec x - 2)(\sqrt{3}\sec x + 2) = 0$. For this reason, either $\sqrt{3}\sec x - 2 = 0$ or $\sqrt{3}\sec x + 2 = 0$ (or both). If $\sqrt{3}\sec x - 2 = 0$, x can be solved for as follows:

$$\sqrt{3}\sec x - 2 = 0$$
$$\sqrt{3}\sec x - 2 + 2 = 0 + 2$$
$$\sqrt{3}\sec x = 2$$
$$\frac{\sqrt{3}\sec x}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$
$$\sec x = \frac{2}{\sqrt{3}}$$
$$\sec x = \frac{2\sqrt{3}}{3}$$

The solutions to the equation $\sec x = \frac{2\sqrt{3}}{3}$ occur at $x = \frac{\pi}{6}$ or $\frac{11\pi}{6}$. If $\sqrt{3}\sec x + 2 = 0$, x can be solved for as follows:

$$\sqrt{3}\sec x + 2 = 0$$
$$\sqrt{3}\sec x + 2 - 2 = 0 - 2$$
$$\sqrt{3}\sec x = -2$$
$$\frac{\sqrt{3}\sec x}{\sqrt{3}} = -\frac{2}{\sqrt{3}}$$
$$\sec x = -\frac{2}{\sqrt{3}}$$
$$\sec x = -\frac{2\sqrt{3}}{3}$$

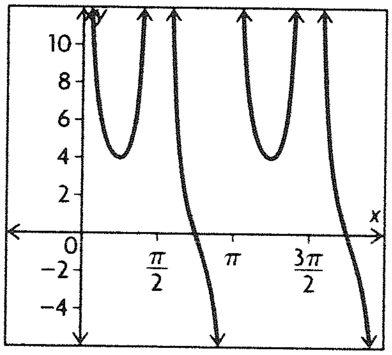
The solutions to the equation $\sec x = -\frac{2\sqrt{3}}{3}$ occur at $x = \frac{5\pi}{6}$ or $\frac{7\pi}{6}$. Therefore, the solutions to the equation $3\sec^2 x - 4 = 0$ occur at $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$.

c) Since $2\sin x \sec x - 2\sqrt{3}\sin x = 0$, $(\sin x)(2\sec x - 2\sqrt{3}) = 0$. For this reason, either $\sin x = 0$ or $2\sec x - 2\sqrt{3} = 0$ (or both). The solutions to the equation $\sin x = 0$ occur at $x = 0, \pi$, or 2π. If $2\sec x - 2\sqrt{3} = 0$, x can be solved for as follows:

$$2\sec x - 2\sqrt{3} = 0$$
$$2\sec x - 2\sqrt{3} + 2\sqrt{3} = 0 + 2\sqrt{3}$$
$$2\sec x = 2\sqrt{3}$$
$$\frac{2\sec x}{2} = \frac{2\sqrt{3}}{2}$$
$$\sec x = \sqrt{3}$$

The solutions to the equation $\sec x = \sqrt{3}$ occur at $x = 0.96$ or 5.33. Therefore, the solutions to the equation $2\sin x \sec x - 2\sqrt{3}\sin x = 0$ occur at $x = 0, 0.96, \pi, 5.33$, or 2π.

d) To solve the equation $2\cot x + \sec^2 x = 0$, graph the function $y = 2\cot x + \sec^2 x$ as follows:



Since the graph intersects the x-axis at $x = \frac{3\pi}{4}$ and $\frac{7\pi}{4}$, the solutions to the equation $2\cot x + \sec^2 x = 0$ occur at $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$.

e) The equation $\cot x \csc^2 x = 2\cot x$ can be rewritten and factored as follows:

$$\cot x \csc^2 x = 2\cot x$$
$$\cot x \csc^2 x - 2\cot x = 2\cot x - 2\cot x$$
$$\cot x \csc^2 x - 2\cot x = 0$$
$$(\cot x)(\csc x - \sqrt{2})(\csc x + \sqrt{2}) = 0$$

Since $(\cot x)(\csc x - \sqrt{2})(\csc x + \sqrt{2}) = 0$, either $\cot x = 0$, $\csc x - \sqrt{2} = 0$, or $\csc x + \sqrt{2} = 0$. The solutions to the equation $\cot x = 0$ occur at $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. If $\csc x - \sqrt{2} = 0$, x can be solved for as follows:

$$\csc x - \sqrt{2} = 0$$
$$\csc x - \sqrt{2} + \sqrt{2} = 0 + \sqrt{2}$$
$$\csc x = \sqrt{2}$$

The solutions to the equation $\csc x = \sqrt{2}$ occur at $x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. If $\csc x + \sqrt{2} = 0$, x can be solved for as follows:

$$\csc x + \sqrt{2} = 0$$
$$\csc x + \sqrt{2} - \sqrt{2} = 0 - \sqrt{2}$$
$$\csc x = -\sqrt{2}$$

The solutions to the equation $\csc x = -\sqrt{2}$ occur at $x = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$. Therefore, the solutions to the equation $\cot x \csc^2 x = 2\cot x$ occur at $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}$ or $\frac{7\pi}{4}$.

f) Since $3\tan^3 x - \tan x = 0$, $(\tan x)(\sqrt{3}\tan x - 1)(\sqrt{3}\tan x + 1) = 0$. For this reason, either $\tan x = 0$, $\sqrt{3}\tan x - 1 = 0$, or $\sqrt{3}\tan x + 1 = 0$. The solutions to the equation $\tan x = 0$ occur at $x = 0, \pi$, and 2π. If $\sqrt{3}\tan x - 1 = 0$, x can be solved for as follows:

$$\sqrt{3}\tan x - 1 = 0$$
$$\sqrt{3}\tan x - 1 + 1 = 0 + 1$$
$$\sqrt{3}\tan x = 1$$
$$\frac{\sqrt{3}\tan x}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$
$$\tan x = \frac{1}{\sqrt{3}}$$
$$\tan x = \frac{\sqrt{3}}{3}$$

The solutions to the equation $\tan x = \frac{\sqrt{3}}{3}$ occur at $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$. If $\sqrt{3}\tan x + 1 = 0$, x can be solved for as follows:

$$\sqrt{3}\tan x + 1 = 0$$
$$\sqrt{3}\tan x + 1 - 1 = 0 - 1$$
$$\sqrt{3}\tan x = -1$$
$$\frac{\sqrt{3}\tan x}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$\tan x = -\frac{1}{\sqrt{3}}$$

$$\tan x = -\frac{\sqrt{3}}{3}$$

The solutions to the equation $\tan x = \frac{\sqrt{3}}{3}$ occur at $x = \frac{5\pi}{6}$ or $\frac{11\pi}{6}$. Therefore, the solutions to the equation $3\tan^3 x - \tan x = 0$ occur at $x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$, or 2π.

9. a) Since $\cos 2\theta = 2\cos^2\theta - 1$, the equation $5\cos 2x - \cos x + 3 = 0$ can be rewritten and factored as follows:

$$\begin{aligned}
5\cos 2x - \cos x + 3 &= 0\\
(5)(2\cos^2 x - 1) - \cos x + 3 &= 0\\
10\cos^2 x - 5 - \cos x + 3 &= 0\\
10\cos^2 x - \cos x - 2 &= 0\\
(5\cos x + 2)(2\cos x - 1) &= 0
\end{aligned}$$

For this reason, either $5\cos x + 2 = 0$ or $2\cos x - 1 = 0$ (or both). If $5\cos x + 2 = 0$, x can be solved for as follows:

$$\begin{aligned}
5\cos x + 2 &= 0\\
5\cos x + 2 - 2 &= 0 - 2\\
5\cos x &= -2\\
\frac{5\cos x}{5} &= -\frac{2}{5}\\
\cos x &= -\frac{2}{5}
\end{aligned}$$

The solutions to the equation $\cos x = -\frac{2}{5}$ occur at $x = 1.98$ or 4.30.

If $2\cos x - 1 = 0$, x can be solved for as follows:

$$\begin{aligned}
2\cos x - 1 &= 0\\
2\cos x - 1 + 1 &= 0 + 1\\
2\cos x &= 1\\
\frac{2\cos x}{2} &= \frac{1}{2}\\
\cos x &= \frac{1}{2}
\end{aligned}$$

The solutions to the equation $\cos x = \frac{1}{2}$ occur at $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. Therefore, the solutions to the equation $5\cos 2x - \cos x + 3 = 0$ occur at $x = \frac{\pi}{3}$, 1.98, 4.30, or $\frac{5\pi}{3}$.

b) Since $\cos 2\theta = 2\cos^2\theta - 1$, the equation $10\cos 2x - 8\cos x + 1 = 0$ can be rewritten and factored as follows:

$$\begin{aligned}
10\cos 2x - 8\cos x + 1 &= 0\\
(10)(2\cos^2 x - 1) - 8\cos x + 1 &= 0\\
20\cos^2 x - 10 - 8\cos x + 1 &= 0\\
20\cos^2 x - 8\cos x - 9 &= 0\\
(10\cos x - 9)(2\cos x + 1) &= 0
\end{aligned}$$

For this reason, either $10\cos x - 9 = 0$ or $2\cos x + 1 = 0$ (or both). If $10\cos x - 9 = 0$, x can be solved for as follows:

$$\begin{aligned}
10\cos x - 9 &= 0\\
10\cos x - 9 + 9 &= 0 + 9\\
10\cos x &= 9\\
\frac{10\cos x}{10} &= \frac{9}{10}\\
\cos x &= \frac{9}{10}
\end{aligned}$$

The solutions to the equation $\cos x = \frac{9}{10}$ occur at $x = 0.45$ or 5.83. If $2\cos x + 1 = 0$, x can be solved for as follows:

$$\begin{aligned}
2\cos x + 1 &= 0\\
2\cos x + 1 - 1 &= 0 - 1\\
2\cos x &= -1\\
\frac{2\cos x}{2} &= -\frac{1}{2}\\
\cos x &= -\frac{1}{2}
\end{aligned}$$

The solutions to the equation $\cos x = -\frac{1}{2}$ occur at $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. Therefore, the solutions to the equation $10\cos 2x - 8\cos x + 1 = 0$ occur at $x = 0.45, \frac{2\pi}{3}, \frac{4\pi}{3}$, or 5.83.

c) Since $\cos 2\theta = 1 - 2\sin^2\theta$, the equation $4\cos 2x + 10\sin x - 7 = 0$ can be rewritten and factored as follows:

$$\begin{aligned}
4\cos 2x + 10\sin x - 7 &= 0\\
(4)(1 - 2\sin^2 x) + 10\sin x - 7 &= 0\\
4 - 8\sin^2 x + 10\sin x - 7 &= 0\\
-8\sin^2 x + 10\sin x - 3 &= 0\\
(-1)(-8\sin^2 x + 10\sin x - 3) &= (-1)(0)\\
8\sin^2 x - 10\sin x + 3 &= 0\\
(4\sin x - 3)(2\sin x - 1) &= 0
\end{aligned}$$

For this reason, either $4\sin x - 3 = 0$ or $2\sin x - 1 = 0$ (or both). If $4\sin x - 3 = 0$, x can be solved for as follows:

$$\begin{aligned}
4\sin x - 3 &= 0\\
4\sin x - 3 + 3 &= 0 + 3\\
4\sin x &= 3\\
\frac{4\sin x}{4} &= \frac{3}{4}\\
\sin x &= \frac{3}{4}
\end{aligned}$$

The solutions to the equation $\sin x = \frac{3}{4}$ occur at $x = 0.85$ or 2.29. If $2 \sin x - 1 = 0$, x can be solved for as follows:

$$
\begin{aligned}
2 \sin x - 1 &= 0 \\
2 \sin x - 1 + 1 &= 0 + 1 \\
2 \sin x &= 1 \\
\frac{2 \sin x}{2} &= \frac{1}{2} \\
\sin x &= \frac{1}{2}
\end{aligned}
$$

The solutions to the equation $\sin x = \frac{1}{2}$ occur at $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. Therefore, the solutions to the equation $4 \cos 2x + 10 \sin x - 7 = 0$ occur at $x = \frac{\pi}{6}$, 0.85, $\frac{5\pi}{6}$, or 2.29.

d) Since $\cos 2\theta = 1 - 2 \sin^2 \theta$, the equation $-2 \cos 2x = 2 \sin x$ can be rewritten and factored as follows:

$$
\begin{aligned}
-2 \cos 2x &= 2 \sin x \\
(-2)(1 - 2 \sin^2 x) &= 2 \sin x \\
-2 + 4 \sin^2 x &= 2 \sin x \\
-2 + 4 \sin^2 x - 2 \sin x &= 2 \sin x - 2 \sin x \\
4 \sin^2 x - 2 \sin x - 2 &= 0 \\
(2)(2 \sin x + 1)(\sin x - 1) &= 0
\end{aligned}
$$

For this reason, either $2 \sin x + 1 = 0$ or $\sin x - 1 = 0$ (or both). If $2 \sin x + 1 = 0$, x can be solved for as follows:

$$
\begin{aligned}
2 \sin x + 1 &= 0 \\
2 \sin x + 1 - 1 &= 0 - 1 \\
2 \sin x &= -1 \\
\frac{2 \sin x}{2} &= -\frac{1}{2} \\
\sin x &= -\frac{1}{2}
\end{aligned}
$$

The solutions to the equation $\sin x = -\frac{1}{2}$ occur at $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$. If $\sin x - 1 = 0$, x can be solved for as follows:

$$
\begin{aligned}
\sin x - 1 &= 0 \\
\sin x - 1 + 1 &= 0 + 1 \\
\sin x &= 1
\end{aligned}
$$

The solution to the equation $\sin x = 1$ occurs at $x = \frac{\pi}{2}$. Therefore, the solutions to the equation $-2 \cos 2x = 2 \sin x$ occur at $x = \frac{\pi}{2}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$.

10. To solve the equation $8 \sin^2 x - 8 \sin x + 1 = 0$, first substitute θ for $\sin x$. The equation then becomes $8\theta^2 - 8\theta + 1$. Next, use the quadratic formula to solve for θ as follows:

$$
\begin{aligned}
\theta &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(8)(1)}}{2(8)} \\
&= \frac{8 \pm \sqrt{64 - 32}}{16} \\
&= \frac{8 \pm \sqrt{32}}{16} \\
&= \frac{8 \pm 4\sqrt{2}}{16} \\
&= \frac{2 \pm \sqrt{2}}{4} \\
&= 0.1464 \text{ or } 0.8536
\end{aligned}
$$

Since $\theta = 0.1464$ or 0.8536, $\sin x = 0.1464$ or 0.8536. If $\sin x = 0.1464$, $x = 0.15$ or 2.99. If $\sin x = 0.8536$, $x = 1.02$ or 2.12. Therefore, the solutions to the equation $8 \sin^2 x - 8 \sin x + 1 = 0$ occur at $x = 0.15, 1.02, 2.12$, or 2.99.

11. Since the solutions of the quadratic trigonometric equation $\cot^2 x - b \cot x + c = 0$ are $\frac{\pi}{6}, \frac{\pi}{4}, \frac{7\pi}{6}$, and $\frac{5\pi}{4}$, the cotangent of these solutions must be found. The cotangent of both $\frac{\pi}{6}$ and $\frac{7\pi}{6}$ is $\sqrt{3}$, while the cotangent of both $\frac{\pi}{4}$ and $\frac{5\pi}{4}$ is 1. For this reason, $\cot x = \sqrt{3}$ and $\cot x = 1$, so $\cot x - \sqrt{3} = 0$ and $\cot x - 1 = 0$. If the factors $\cot x - \sqrt{3}$ and $\cot x - 1$ are multiplied together as follows, the quadratic trigonometric equation $\cot^2 x - b \cot x + c = 0$ is formed:

$$
\begin{aligned}
(\cot x - \sqrt{3})(\cot x - 1) &= 0 \\
\cot^2 x - \sqrt{3} \cot x - \cot x + (-1)(-\sqrt{3}) &= 0 \\
\cot^2 x - (1 + \sqrt{3}) \cot x + \sqrt{3} &= 0
\end{aligned}
$$

Therefore, $b = 1 + \sqrt{3}$ and $c = \sqrt{3}$.

12. It's clear from the graph that the quadratic trigonometric equation $\sin^2 x - c = 0$ has solutions at $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$, so the sine of these solutions must be found. The sine of both $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ is $\frac{\sqrt{2}}{2}$, while the sine of both $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ is $-\frac{\sqrt{2}}{2}$. For this reason, $\sin x = \frac{\sqrt{2}}{2}$ and $\sin x = -\frac{\sqrt{2}}{2}$, so $\sin x - \frac{\sqrt{2}}{2} = 0$ and $\sin x + \frac{\sqrt{2}}{2} = 0$. If the factors $\sin x - \frac{\sqrt{2}}{2}$ and $\sin x + \frac{\sqrt{2}}{2}$ are multiplied together as follows, the quadratic trigonometric equation $\sin^2 x - c = 0$ is formed:

$$\left(\sin x - \frac{\sqrt{2}}{2}\right)\left(\sin x + \frac{\sqrt{2}}{2}\right) = 0$$

$$\sin^2 x - \frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\sin x + \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = 0$$

$$\sin^2 x - \frac{2}{4} = 0$$

$$\sin^2 x - \frac{1}{2} = 0$$

Therefore, $c = \frac{1}{2}$.

13. To solve the problem, first the zeros of the function $h(d) = 4\cos^2 d - 1$ must be found as follows:

$$h(d) = 4\cos^2 d - 1$$

$$0 = 4\cos^2 d - 1$$

$$(2\cos d - 1)(2\cos d + 1) = 0$$

Since $(2\cos d - 1)(2\cos d + 1) = 0$, either $2\cos d - 1 = 0$ or $2\cos d + 1 = 0$ (or both). If $2\cos d - 1 = 0$, d can be solved for as follows:

$$2\cos d - 1 = 0$$

$$2\cos d - 1 + 1 = 0 + 1$$

$$2\cos d = 1$$

$$\frac{2\cos d}{2} = \frac{1}{2}$$

$$\cos d = \frac{1}{2}$$

The solutions to the equation $\cos d = \frac{1}{2}$ occur at $d = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. If $2\cos d + 1 = 0$, d can be solved for as follows:

$$2\cos d + 1 = 0$$

$$2\cos d + 1 - 1 = 0 - 1$$

$$2\cos d = -1$$

$$\frac{2\cos d}{2} = -\frac{1}{2}$$

$$\cos d = -\frac{1}{2}$$

The solutions to the equation $\cos d = -\frac{1}{2}$ occur at $d = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. Therefore, the zeros of the function $h(d) = 4\cos^2 d - 1$ are at $d = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$.

With the zeros known, the 2π stretch of rolling hills can be broken down into the following regions: $d < \frac{\pi}{3}$; $\frac{\pi}{3} < d < \frac{2\pi}{3}$; $\frac{2\pi}{3} < d < \frac{4\pi}{3}$; $\frac{4\pi}{3} < d < \frac{5\pi}{3}$; $d > \frac{5\pi}{3}$. First, the region $d < \frac{\pi}{3}$ can be tested by finding $h\left(\frac{\pi}{6}\right)$ as follows:

$$h\left(\frac{\pi}{6}\right) = 4\cos^2\left(\frac{\pi}{6}\right) - 1$$

$$= 4\left(\frac{\sqrt{3}}{2}\right)^2 - 1$$

$$= 4\left(\frac{3}{4}\right) - 1 = 3 - 1 = 2$$

Since $h\left(\frac{\pi}{6}\right)$ is positive, the height of the rolling hills above sea level relative to Natasha's home is positive in the region $d < \frac{\pi}{3}$. Next, the region $\frac{\pi}{3} < d < \frac{2\pi}{3}$ can be tested by finding $h\left(\frac{\pi}{2}\right)$ as follows:

$$h\left(\frac{\pi}{2}\right) = 4\cos^2\left(\frac{\pi}{2}\right) - 1$$

$$= 4(0)^2 - 1 = 4(0) - 1$$

$$= 0 - 1 = -1$$

Since $h\left(\frac{\pi}{2}\right)$ is negative, the height of the rolling hills above sea level relative to Natasha's home is negative in the region $\frac{\pi}{3} < d < \frac{2\pi}{3}$. Next, the region $\frac{2\pi}{3} < d < \frac{4\pi}{3}$ can be tested by finding $h(\pi)$ as follows:

$$h(1) = 4\cos^2(\pi) - 1$$

$$= 4(-1)^2 - 1 = 4(1) - 1$$

$$= 4 - 1 = 3$$

Since $h(\pi)$ is positive, the height of the rolling hills above sea level relative to Natasha's home is positive in the region $\frac{2\pi}{3} < d < \frac{4\pi}{3}$. Next, the region $\frac{4\pi}{3} < d < \frac{5\pi}{3}$ can be tested by finding $h\left(\frac{3\pi}{2}\right)$ as follows:

$$h\left(\frac{3\pi}{2}\right) = 4\cos^2\left(\frac{3\pi}{2}\right) - 1$$

$$= 4(0)^2 - 1$$

$$= 4(0) - 1 = 0 - 1 = -1$$

Since $h\left(\frac{3\pi}{2}\right)$ is negative, the height of the rolling negative in the region $\frac{4\pi}{3} < d < \frac{5\pi}{3}$. Finally, the region $d > \frac{5\pi}{3}$ can be tested by finding $h\left(\frac{11\pi}{6}\right)$ as follows:

$$\begin{aligned} h\left(\frac{11\pi}{6}\right) &= 4\cos^2\left(\frac{11\pi}{6}\right) - 1 \\ &= 4\left(\frac{\sqrt{3}}{2}\right)^2 - 1 \\ &= 4\left(\frac{3}{4}\right) - 1 = 3 - 1 = 2 \end{aligned}$$

Since $h\left(\frac{11\pi}{6}\right)$ is positive, the height of the rolling hills above sea level relative to Natasha's home is positive in the region $d > \frac{5\pi}{3}$. Therefore, the intervals at which the height of the rolling hills above sea level relative to Natasha's home is negative are $\frac{\pi}{3}\text{km} < d < \frac{2\pi}{3}\text{km}$ and $\frac{4\pi}{3}\text{km} < d < \frac{5\pi}{3}\text{km}$.

14. Since $\sin^2\theta + \cos^2\theta = 1$, or $\sin^2\theta = 1 - \cos^2\theta$, the equation $6\sin^2 x = 17\cos x + 11$ can be rewritten and factored as follows:

$$\begin{aligned} 6\sin^2 x &= 17\cos x + 11; \\ 6\sin^2 x - 17\cos x - 11 &= 17\cos x + 11 \\ &\quad - 17\cos x - 11; \\ 6\sin^2 x - 17\cos x - 11 &= 0; \\ 6(1 - \cos^2 x) - 17\cos x - 11 &= 0; \\ 6 - 6\cos^2 x - 17\cos x - 11 &= 0; \\ -6\cos^2 x - 17\cos x - 5 &= 0; \\ (-1)(-6\cos^2 x - 17\cos x - 5) &= (-1)(0); \\ 6\cos^2 x + 17\cos x + 5 &= 0; \\ (2\cos x + 5)(3\cos x + 1) &= 0 \end{aligned}$$

For this reason, either $2\cos x + 5 = 0$ or $3\cos x + 1 = 0$. If $2\cos x + 5 = 0$, x can be solved for as follows:

$$\begin{aligned} 2\cos x + 5 &= 0 \\ 2\cos x + 5 - 5 &= 0 - 5 \\ 2\cos x &= -5 \\ \frac{2\cos x}{2} &= -\frac{5}{2} \\ \cos x &= -\frac{5}{2} \end{aligned}$$

Since $\cos x$ can never equal $-\frac{5}{2}$, the factor $2\cos x + 5$ can be ignored. If $3\cos x + 1 = 0$, x can be solved for as follows:

$$\begin{aligned} 3\cos x + 1 &= 0 \\ 3\cos x + 1 - 1 &= 0 - 1 \\ 3\cos x &= -1 \\ \frac{3\cos x}{3} &= -\frac{1}{3} \\ \cos x &= -\frac{1}{3} \end{aligned}$$

The solutions to the equation $\cos x = -\frac{1}{3}$ occur at $x = 1.91$ or 4.37. Therefore, the solutions to the equation $6\sin^2 x = 17\cos x + 11$ occur at $x = 1.91$ or 4.37.

15. a) Since $\sin^2\theta + \cos^2\theta = 1$, or $\sin^2\theta = 1 - \cos^2\theta$, the equation $\sin^2 x - \sqrt{2}\cos x = \cos^2 x + \sqrt{2}\cos x + 2$ can be rewritten and factored as follows:

$$\begin{aligned} \sin^2 x - \sqrt{2}\cos x &= \cos^2 x + \sqrt{2}\cos x + 2; \\ 1 - \cos^2 x - \sqrt{2}\cos x &= \cos^2 x + \sqrt{2}\cos x + 2; \\ 1 - \cos^2 x - \sqrt{2}\cos x - \cos^2 x - \sqrt{2}\cos x - 2 & \\ = \cos^2 x + \sqrt{2}\cos x + 2 - \cos^2 x - \sqrt{2}\cos x - 2; & \\ 1 - \cos^2 x - \sqrt{2}\cos x - \cos^2 x & \\ - \sqrt{2}\cos x - 2 &= 0; \\ -2\cos^2 x - 2\sqrt{2}\cos x - 1 &= 0; \\ \left(-\frac{1}{2}\right)(-2\cos^2 x - 2\sqrt{2}\cos x - 1) &= 0; \\ \cos^2 x + \sqrt{2}\cos x + \frac{1}{2} &= 0; \\ \left(\cos x + \frac{\sqrt{2}}{2}\right)^2 &= 0 \end{aligned}$$

Since $\left(\cos x + \frac{\sqrt{2}}{2}\right)^2 = 0$, $\cos x + \frac{\sqrt{2}}{2} = 0$. For this reason, x can be solved for as follows:

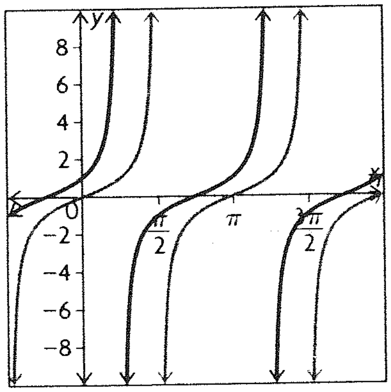
$$\begin{aligned} \cos x + \frac{\sqrt{2}}{2} &= 0 \\ \cos x + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} &= 0 - \frac{\sqrt{2}}{2} \\ \cos x &= -\frac{\sqrt{2}}{2} \end{aligned}$$

The solutions to the equation $\cos x = -\frac{\sqrt{2}}{2}$ occur at $x = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$, so the solutions to the equation $\sin^2 x - \sqrt{2}\cos x = \cos^2 x + \sqrt{2}\cos x + 2$ occur at $x = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$.

b) Since the period of the cosine function is 2π, a general solution for the equation in part (a) is $x = \frac{3\pi}{4} + 2n\pi$ or $\frac{5\pi}{4} + 2n\pi$, where $n \in \mathbf{I}$.

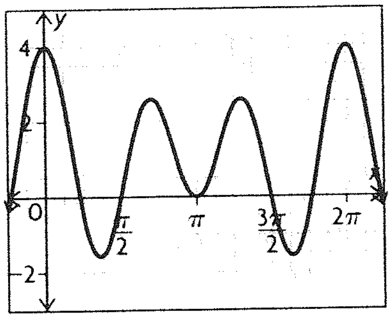
16. It is possible to have different numbers of solutions for quadratic trigonometric equations because, when factored, a quadratic trigonometric equation can be one expression multiplied by another expression or it can be a single expression squared. For example, the equation $\cos^2 x + \frac{3}{2}\cos x + \frac{1}{2}$ becomes $(\cos x + 1)(\cos x + \frac{1}{2})$ when factored, and it has the solutions $\frac{2\pi}{3}$, π, and $\frac{4\pi}{3}$ in the interval $0 \le x \le 2\pi$. In comparison, the equation $\cos^2 x + 2\cos x + 1 = 0$ becomes $(\cos x + 1)^2$ when factored, and it has only one solution, π, in the interval $0 \le x \le 2\pi$. Also, different expressions produce different numbers of solutions. For example, the expression $\cos x + \frac{1}{2}$ produces two solutions in the interval $0 \le x \le 2\pi$ $\left(\frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}\right)$ because $\cos x = -\frac{1}{2}$ for two different values of x. The expression $\cos x + 1$, however, produces only one solution in the interval $0 \le x \le 2\pi(\pi)$, because $\cos x = -1$ for only one value of x.

17. To determine all the values of a such that $f(x) = \tan(x + a)$, first graph the functions $f(x) = \frac{\tan x}{1 - \tan x} - \frac{\cot x}{1 - \cot x}$ and $g(x) = \tan x$ on the same coordinate grid as follows:



It's apparent from the graphs that if the graph of the function $g(x) = \tan x$ were translated $\frac{\pi}{4}$ units to the left, it would be the same as the graph of the function $f(x) = \frac{\tan x}{1 - \tan x} - \frac{\cot x}{1 - \cot x}$. For this reason, the graph of the function $g(x) = \tan(x + \frac{\pi}{4})$ would be the same as the graph of the function $f(x) = \frac{\tan x}{1 - \tan x} - \frac{\cot x}{1 - \cot x}$. The same is true for $\frac{5\pi}{4}$. Therefore, the values of a such that $f(x) = \tan(x + a)$ are $a = \frac{\pi}{4}$ and $\frac{5\pi}{4}$.

18. To solve the equation $2\cos 3x + \cos 2x + 1 = 0$, graph the function $f(x) = 2\cos 3x + \cos 2x + 1$ on a coordinate grid as follows:



It's apparent from the graph that the x-intercepts are at $x = 0.72, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 5.56. Therefore, the solutions to the equation $2\cos 3x + \cos 2x + 1 = 0$ occur at $x = 0.72, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, or 5.56.

19. Since $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$, the equation $3\tan^2 2x = 1$ can be rewritten as follows:

$$3\tan^2 2x = 1;$$

$$3\left(\frac{2\tan x}{1 - \tan^2 x}\right)^2 = 1;$$

$$3\left(\frac{4\tan^2 x}{1 - \tan^2 x - \tan^2 x + \tan^4 x}\right) = 1;$$

$$\frac{12\tan^2 x}{1 - 2\tan^2 x + \tan^4 x} = 1;$$

$$1 - 2\tan^2 x + \tan^4 x = 12\tan^2 x;$$

$$1 - 2\tan^2 x + \tan^4 x - 12\tan^2 x = 12\tan^2 x - 12\tan^2 x;$$

$$\tan^4 x - 14\tan^2 x + 1 = 0$$

At this point the quadratic formula can be used to solve for $\tan^2 x$ as follows:

$$\begin{aligned}\tan^2 x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\ &= \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(1)}}{2(1)}\\ &= \frac{14 \pm \sqrt{196 - 4}}{2}\\ &= \frac{14 \pm \sqrt{192}}{2}\\ &= \frac{14 \pm 8\sqrt{3}}{2}\\ &= 7 \pm 4\sqrt{3}\\ &= 0.0718 \text{ or } 13.9282\end{aligned}$$

Since $\tan^2 x = 0.0718$ or 13.9282, $\tan x = \pm 0.2679$ or ± 3.7321. Therefore, four solutions for x are $x = 15°, 75°, 285°$, or $345°$. Also, since the value of $\tan x$ repeats itself every 180°, four more solutions for x are $x = 105°, 165°, 195°$, or $255°$.

20. To solve the equation $\sqrt{2}\sin\theta = \sqrt{3} - \cos\theta$, first square both sides of the equation as follows:

$$\sqrt{2}\sin\theta = \sqrt{3} - \cos\theta$$
$$(\sqrt{2}\sin\theta)^2 = (\sqrt{3} - \cos\theta)^2$$
$$2\sin^2\theta = 3 - \sqrt{3}\cos\theta - \sqrt{3}\cos\theta + \cos^2\theta$$
$$2\sin^2\theta = 3 - 2\sqrt{3}\cos\theta + \cos^2\theta$$

Since $\sin^2\theta + \cos^2\theta = 1$, or $\sin^2\theta = 1 - \cos^2\theta$, the equation $2\sin^2\theta = 3 - 2\sqrt{3}\cos\theta + \cos^2\theta$ can be rewritten as follows:

$$2\sin^2\theta = 3 - 2\sqrt{3}\cos\theta + \cos^2\theta;$$
$$2(1 - \cos^2\theta) = 3 - 2\sqrt{3}\cos\theta + \cos^2\theta;$$
$$2 - 2\cos^2\theta = 3 - 2\sqrt{3}\cos\theta + \cos^2\theta;$$
$$2 - 2\cos^2\theta - 2 + 2\cos^2\theta$$
$$= 3 - 2\sqrt{3}\cos\theta + \cos^2\theta - 2 + 2\cos^2\theta;$$
$$3\cos^2\theta - 2\sqrt{3}\cos\theta + 1 = 0$$

At this point the quadratic formula can be used to solve for $\cos\theta$ as follows:

$$\cos\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-2\sqrt{3}) \pm \sqrt{(-2\sqrt{3})^2 - 4(3)(1)}}{2(3)}$$
$$= \frac{2\sqrt{3} \pm \sqrt{12 - 12}}{6}$$
$$= \frac{2\sqrt{3} \pm \sqrt{0}}{6}$$
$$= \frac{2\sqrt{3} \pm 0}{6} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$$

Since $\cos\theta = \frac{\sqrt{3}}{3}$, $\theta = 0.96$

Chapter Review, p. 440

1. a) Answers may vary. For example: Since $\sin(\pi - \theta) = \sin\theta$, $\sin\theta = \sin(\pi - \theta)$.

Therefore, $\sin\frac{3\pi}{10} = \sin\left(\pi - \frac{3\pi}{10}\right)$

$$= \sin\left(\frac{10\pi}{10} - \frac{3\pi}{10}\right) = \sin\frac{7\pi}{10}$$

b) Answers may vary. For example: Since $\cos(2\pi - \theta) = \cos\theta$, $\cos\theta = \cos(2\pi - \theta)$. Therefore,

$$\cos\frac{6\pi}{7} = \cos\left(2\pi - \frac{6\pi}{7}\right)$$
$$= \cos\left(\frac{14\pi}{7} - \frac{6\pi}{7}\right) = \cos\frac{8\pi}{7}$$

c) Answers may vary. For example: Since $-\sin\theta = \sin(\pi + \theta)$,

$$-\sin\frac{13\pi}{7} = \sin\left(\pi + \frac{13\pi}{7}\right)$$
$$= \sin\left(\frac{7\pi}{7} + \frac{13\pi}{7}\right) = \sin\frac{20\pi}{7}$$

Since $\sin\theta = \sin(\theta - 2\pi)$,

$$\sin\frac{20\pi}{7} = \sin\left(\frac{20\pi}{7} - 2\pi\right)$$
$$= \sin\left(\frac{20\pi}{7} - \frac{14\pi}{7}\right) = \sin\frac{6\pi}{7}$$

Therefore, $-\sin\frac{13\pi}{7} = \sin\frac{6\pi}{7}$.

d) Answers may vary. For example: Since $-\cos\theta = \cos(\pi + \theta)$,

$$-\cos\frac{8\pi}{7} = \cos\left(\pi + \frac{8\pi}{7}\right)$$
$$= \cos\left(\frac{7\pi}{7} + \frac{8\pi}{7}\right) = \cos\frac{15\pi}{7}$$

Since $\cos\theta = \cos(\theta - 2\pi)$,

$$\cos\frac{15\pi}{7} = \cos\left(\frac{15\pi}{7} - 2\pi\right)$$
$$= \cos\left(\frac{15\pi}{7} - \frac{14\pi}{7}\right) = \cos\frac{\pi}{7}$$

Therefore, $-\cos\frac{8\pi}{7} = \cos\frac{\pi}{7}$.

2. Since $\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$, the equation $y = -5\sin\left(x - \frac{\pi}{2}\right) - 8$ can be rewritten

$$y = -5\cos\left(x - \frac{\pi}{2} - \frac{\pi}{2}\right) - 8$$

$= -5\cos(x - \pi) - 8$. Since a horizontal translation of π to the left or right is equivalent to a reflection in the x-axis, the equation $y = -5\cos(x - \pi) - 8$ can be rewritten $y = 5\cos x - 8$. Therefore, the equation $y = -5\sin x\left(-\frac{\pi}{2}\right) - 8$ can be rewritten $y = 5\cos x - 8$.

3. a) Since $\sin(a - b) = \sin a\cos b - \cos a\sin b$,

$$\sin\left(x - \frac{4\pi}{3}\right) = \sin x\cos\frac{4\pi}{3} - \cos x\sin\frac{4\pi}{3}$$
$$= (\sin x)\left(-\frac{1}{2}\right) - (\cos x)\left(-\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$$

b) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\cos\left(x + \frac{3\pi}{4}\right) = \cos x \cos\frac{3\pi}{4} - \sin x \sin\frac{3\pi}{4}$$
$$= (\cos x)\left(-\frac{\sqrt{2}}{2}\right) - (\sin x)\left(\frac{\sqrt{2}}{2}\right)$$
$$= -\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x$$

c) Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\tan\left(x + \frac{\pi}{3}\right) = \frac{\tan x + \tan\frac{\pi}{3}}{1 - \tan x \tan\frac{\pi}{3}}$$
$$= \frac{\tan x + \sqrt{3}}{1 - (\tan x)(\sqrt{3})}$$
$$= \frac{\tan x + \sqrt{3}}{1 - \sqrt{3}\tan x}$$

d) Since $\cos(a - b) = \cos a \cos b + \sin a \sin b$,

$$\cos\left(x - \frac{5\pi}{4}\right)$$
$$= (\cos x)\left(\cos\frac{5\pi}{4}\right) + (\sin x)\left(\sin\frac{5\pi}{4}\right)$$
$$= (\cos x)\left(-\frac{\sqrt{2}}{2}\right) + (\sin x)\left(-\frac{\sqrt{2}}{2}\right)$$
$$= -\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x$$

4. a) Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$,

$$\frac{\tan\frac{\pi}{12} + \tan\frac{7\pi}{4}}{1 - \tan\frac{\pi}{12}\tan\frac{7\pi}{4}} = \tan\left(\frac{\pi}{12} + \frac{7\pi}{4}\right)$$
$$= \tan\left(\frac{\pi}{12} + \frac{21\pi}{12}\right) = \tan\frac{22\pi}{12} = \tan\frac{11\pi}{6} = -\frac{\sqrt{3}}{3}$$

b) Since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,

$$\cos\frac{\pi}{9}\cos\frac{19\pi}{18} - \sin\frac{\pi}{9}\sin\frac{19\pi}{18}$$
$$= \cos\left(\frac{\pi}{9} + \frac{19\pi}{18}\right) = \cos\left(\frac{2\pi}{18} + \frac{19\pi}{18}\right)$$
$$= \cos\frac{21\pi}{18} = \cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$$

5. a) Since $\sin 2\theta = 2\sin\theta\cos\theta$,

$$2\sin\frac{\pi}{12}\cos\frac{\pi}{12} = \sin\left((2)\left(\frac{\pi}{12}\right)\right)\sin\frac{2\pi}{12}$$
$$= \sin\frac{\pi}{6} = \frac{1}{2}$$

b) Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$$\cos^2\frac{\pi}{12} - \sin^2\frac{\pi}{12} = \cos\left((2)\left(\frac{\pi}{12}\right)\right)$$
$$= \cos\frac{2\pi}{12} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

c) Since $\cos 2\theta = 1 - 2\sin^2\theta$,

$$1 - 2\sin^2\frac{3\pi}{8} = \cos\left((2)\left(\frac{3\pi}{8}\right)\right)$$
$$= \cos\frac{6\pi}{8} = \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$$

d) Since $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$,

$$\frac{2\tan\frac{\pi}{6}}{1 - \tan^2\frac{\pi}{6}} = \tan 2\left(\frac{\pi}{6}\right) = \tan\frac{\pi}{3} = \sqrt{3}$$

6. a) Since $\sin x = \frac{3}{5}$, the leg opposite to the angle x in a right triangle has a length of 3, while the hypotenuse of the right triangle has a length of 5. For this reason, the other leg of the right triangle can be calculated as follows:

$$3^2 + y^2 = 5^2$$
$$9 + y^2 = 25$$
$$9 + y^2 - 9 = 25 - 9$$
$$y^2 = 16$$
$$y = 4, \text{ in quadrant I}$$

Since $\cos x = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos x = \dfrac{4}{5}$.

Therefore, since $\sin 2x = 2\sin x\cos x$,

$$\sin 2x = (2)\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}.$$

Also, since $\cos 2x = \cos^2 x - \sin^2 x$,

$$\cos 2x = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}.$$

Since $\tan x = \dfrac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan x = \frac{3}{4}$.

Since $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$, $\tan 2x = \dfrac{(2)(\frac{3}{4})}{1 - (\frac{3}{4})^2}$

$$= \frac{\frac{3}{2}}{1 - \frac{9}{16}} = \frac{\frac{3}{2}}{\frac{16}{16} - \frac{9}{16}} = \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{3}{2} \times \frac{16}{7} = \frac{24}{7}$$

b) Since $\cot x = -\frac{7}{24}$, the leg opposite the angle x in a right triangle has a length of 24, while the leg adjacent to the angle x has a length of 7. For this reason, the hypotenuse of the right triangle can be calculated as follows:

$$7^2 + 24^2 = c^2$$
$$49 + 576 = c^2$$

$$625 = c^2$$
$$c = 25$$

Since $\sin x = \frac{\text{opposite leg}}{\text{hypotenuse}}$, $\sin x = \frac{24}{25}$, and since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos x = -\frac{7}{25}$. (The reason the sign is negative is because angle x is in the second quadrant.) Therefore, since $\sin 2x = 2 \sin x \cos x$, $\sin 2x = (2)\left(\frac{24}{25}\right)\left(-\frac{7}{25}\right)$

$= -\frac{336}{625}$. Also, since $\cos 2x = \cos^2 x - \sin^2 x$,

$$\cos 2x = \left(-\frac{7}{25}\right)^2 - \left(\frac{24}{25}\right)^2 = \frac{49}{625} - \frac{576}{625} = -\frac{527}{625}.$$

Since $\cot x = -\frac{7}{24}$, $\tan x = -\frac{24}{7.}$

Since $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$,

$$\tan 2x = \frac{(2)\left(-\frac{24}{7}\right)}{1 - \left(-\frac{24}{7}\right)^2} = \frac{-\frac{48}{7}}{1 - \frac{576}{49}} = \frac{-\frac{48}{7}}{\frac{49}{49} - \frac{576}{49}}$$

$$= \frac{-\frac{48}{7}}{-\frac{527}{49}} = -\frac{48}{7} \times \frac{49}{527} = \frac{336}{527}$$

c) Since $\cos x = \frac{12}{13}$, the leg adjacent to the angle x in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 + 144 - 144 = 169 - 144$$
$$x^2 = 25$$
$$x = 5$$

Since $\sin x = \frac{\text{opposite leg}}{\text{hypotenuse}}$, $\sin x = -\frac{5}{13}$. (Sine is negative because x is in the fourth quadrant.) Therefore, since $\sin 2x = 2 \sin x \cos x$,

$\sin 2x = (2)\left(-\frac{5}{13}\right)\left(\frac{12}{13}\right) = -\frac{120}{169}$. Also, since $\cos 2x = \cos^2 x - \sin^2 x$,

$$\cos 2x = \left(\frac{12}{13}\right)^2 - \left(-\frac{5}{13}\right)^2$$
$$= \frac{144}{169} - \frac{25}{169} = \frac{119}{169}$$

Finally, since $\tan x = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan x = -\frac{5}{12}$. (The reason the sign is negative is because angle x is in the fourth quadrant.)

Since $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$,

$$\tan 2x = \frac{(2)\left(-\frac{5}{12}\right)}{1 - \left(-\frac{5}{12}\right)^2}$$

$$= \frac{-\frac{5}{6}}{1 - \frac{25}{144}} = \frac{-\frac{5}{6}}{\frac{144}{144} - \frac{25}{144}} = \frac{-\frac{5}{6}}{\frac{119}{144}}$$

$$= -\frac{5}{6} \times \frac{144}{119} = -\frac{120}{119}$$

7. a) Since $\sin 2\theta = 2 \sin \theta \cos \theta$, and since $\cos 2\theta = 1 - 2 \sin^2 \theta$, $\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{2 \sin x \cos x}{1 - 2 \sin^2 x}$. Therefore, $\tan 2x = \frac{2 \sin x \cos x}{1 - 2 \sin^2 x}$ is a trigonometric identity.

b) Since $1 + \tan^2 x = \sec^2 x$, $\sec^2 x - \tan^2 x = 1$. Therefore, since $\sec^2 x - \tan^2 x = 1$ is a trigonometric identity, $\sec^2 x - \tan^2 x = \cos x$ must be a trigonometric equation, because $\cos x$ does not always equal 1.

c) Since $1 + \cot^2 x = \csc^2 x$, $\csc^2 x - \cot^2 x = 1$. Therefore, since $\sin^2 x + \cos^2 x = 1$, $\csc^2 x - \cot^2 x = \sin^2 x + \cos^2 x$ is a trigonometric identity.

d) Since $\tan^2 x = 1$, $\tan x = \pm 1$. Therefore, since $\tan x$ does not always equal -1 or 1, $\tan^2 x = 1$ must be a trigonometric equation.

8. The trigonometric identity $\frac{1 - \sin^2 x}{\cot^2 x} = 1 - \cos^2 x$ can be proven as follows:

$$\frac{1 - \sin^2 x}{\cot^2 x} = 1 - \cos^2 x$$

$$\frac{\cos^2 x}{\cot^2 x} = 1 - \cos^2 x$$

$$\frac{\cos^2 x}{\frac{\cos^2 x}{\sin^2 x}} = 1 - \cos^2 x$$

$$\frac{(\cos^2 x)(\sin^2 x)}{\cos^2 x} = 1 - \cos^2 x$$

$$\sin^2 x = 1 - \cos^2 x$$

$$1 - \cos^2 x = 1 - \cos^2 x$$

9. The trigonometric identity $\frac{2 \sec^2 x - 2 \tan^2 x}{\csc x} = \sin 2x \sec x$ can be proven as follows:

$$\frac{2 \sec^2 x - 2 \tan^2 x}{\csc x} = \sin 2x \sec x$$

$$\frac{2(\sec^2 x - \tan^2 x)}{\csc x} = \sin 2x \sec x$$

$$\frac{2(1)}{\csc x} = \sin 2x \sec x$$
$$\frac{2}{\csc x} = \sin 2x \sec x$$
$$2 \sin x = \sin 2x \sec x$$
$$\frac{2 \sin x \cos x}{\cos x} = \sin 2x \sec x$$
$$\frac{\sin 2x}{\cos x} = \sin 2x \sec x$$
$$\sin 2x \sec x = \sin 2x \sec x$$

10. a) The trigonometric equation $\frac{2}{\sin x} + 10 = 6$ can be solved as follows:

$$\frac{2}{\sin x} + 10 = 6$$
$$\frac{2}{\sin x} + 10 - 10 = 6 - 10$$
$$\frac{2}{\sin x} = -4$$
$$-4 \sin x = 2$$
$$\frac{-4 \sin x}{-4} = \frac{2}{-4}$$
$$\sin x = -\frac{1}{2}$$

The solutions to the equation $\sin x = -\frac{1}{2}$ occur at $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

b) The trigonometric equation $-\frac{5 \cot x}{2} + \frac{7}{3} = -\frac{1}{6}$ can be solved as follows:

$$-\frac{5 \cot x}{2} + \frac{7}{3} = -\frac{1}{6}$$
$$-\frac{15 \cot x}{6} + \frac{14}{6} = -\frac{1}{6}$$
$$-15 \cot x + 14 = -1$$
$$-15 \cot x + 14 - 14 = -1 - 14$$
$$-15 \cot x = -15$$
$$\frac{-15 \cot x}{-15} = \frac{-15}{-15}$$
$$\cot x = 1$$

The solutions to the equation $\cot x = 1$ occur at $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

c) The trigonometric equation $3 + 10 \sec x - 1 = -18$ can be solved as follows:

$$3 + 10 \sec x - 1 = -18$$
$$2 + 10 \sec x = -18$$
$$2 + 10 \sec x - 2 = -18 - 2$$
$$10 \sec x = -20$$
$$\frac{10 \sec x}{10} = -\frac{20}{10}$$
$$\sec x = -2$$

The solutions to the equation $\sec x = -2$ occur at $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

11. a) The equation $y^2 - 4 = 0$ can be solved as follows:

$$y^2 - 4 = 0$$
$$(y - 2)(y + 2) = 0$$

Since $(y - 2)(y + 2) = 0$, either $y - 2 = 0$ or $y + 2 = 0$ (or both). If $y - 2 = 0$, y can be solved for as follows:

$$y - 2 = 0$$
$$y - 2 + 2 = 0 + 2$$
$$y = 2$$

If $y + 2 = 0$, y can be solved for as follows:

$$y + 2 = 0$$
$$y + 2 - 2 = 0 - 2$$
$$y = -2$$

Therefore, the solutions to the equation $y^2 - 4 = 0$ are $y = 2$ or $y = -2$.

b) The equation $\csc^2 x - 4 = 0$ can be solved as follows:

$$\csc^2 x - 4 = 0$$
$$(\csc x - 2)(\csc x + 2) = 0$$

Since $(\csc x - 2)(\csc x + 2) = 0$, either $\csc x - 2 = 0$ or $\csc x + 2 = 0$ (or both). If $\csc x - 2 = 0$, x can be solved for as follows:

$$\csc x - 2 = 0$$
$$\csc x - 2 + 2 = 0 + 2$$
$$\csc x = 2$$

The solutions to the equation $\csc x = 2$ occur at $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. If $\csc x + 2 = 0$, x can be solved for as follows:

$$\csc x + 2 = 0$$
$$\csc x + 2 - 2 = 0 - 2$$
$$\csc x = -2$$

The solutions to the equation $\csc x = -2$ occur at $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$. Therefore, the solutions to the equation $\csc^2 x - 4 = 0$ occur at $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$.

12. a) The equation $2 \sin^2 x - \sin x - 1 = 0$ can be solved as follows:

$$2 \sin^2 x - \sin x - 1 = 0$$
$$(2 \sin x + 1)(\sin x - 1) = 0$$

Since $(2\sin x + 1)(\sin x - 1) = 0$, either $2\sin x + 1 = 0$ or $\sin x - 1 = 0$ (or both). If $2\sin x + 1 = 0$, x can be solved for as follows:

$$2\sin x + 1 = 0$$
$$2\sin x + 1 - 1 = 0 - 1$$
$$2\sin x = -1$$
$$\frac{2\sin x}{2} = -\frac{1}{2}$$
$$\sin x = -\frac{1}{2}$$

The solutions to the equation $\sin x = -\frac{1}{2}$ occur at $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$. If $\sin x - 1 = 0$, x can be solved for as follows:

$$\sin x - 1 = 0$$
$$\sin x - 1 + 1 = 0 + 1$$
$$\sin x = 1$$

The solution to the equation $\sin x = 1$ occurs at $x = \frac{\pi}{2}$. Therefore, the solutions to the equation $2\sin^2 x - \sin x - 1 = 0$ occur at $x = \frac{\pi}{2}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$.

b) The equation $\tan^2 x \sin x - \frac{\sin x}{3} = 0$ can be solved as follows:

$$\tan^2 x \sin x - \frac{\sin x}{3} = 0$$
$$(\sin x)\left(\tan^2 x - \frac{1}{3}\right) = 0$$
$$(\sin x)\left(\tan x - \frac{\sqrt{3}}{3}\right)\left(\tan x + \frac{\sqrt{3}}{3}\right) = 0$$

Since $(\sin x)\left(\tan x - \frac{\sqrt{3}}{3}\right)\left(\tan x + \frac{\sqrt{3}}{3}\right) = 0$, either $\sin x = 0$, $\tan x - \frac{\sqrt{3}}{3} = 0$, or $\tan x + \frac{\sqrt{3}}{3} = 0$. The solutions to the equation $\sin x = 0$ occur at $x = 0$, π, or 2π. If $\tan x - \frac{\sqrt{3}}{3} = 0$, x can be solved for as follows:

$$\tan x - \frac{\sqrt{3}}{3} = 0$$
$$\tan x - \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} = 0 + \frac{\sqrt{3}}{3}$$
$$\tan x = \frac{\sqrt{3}}{3}$$

The solutions to the equation $\tan x = \frac{\sqrt{3}}{3}$ occur at $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$. If $\tan x + \frac{\sqrt{3}}{3} = 0$, x can be solved for as follows:

$$\tan x + \frac{\sqrt{3}}{3} = 0$$
$$\tan x + \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3} = 0 - \frac{\sqrt{3}}{3}$$
$$\tan x = -\frac{\sqrt{3}}{3}$$

The solutions to the equation $\tan x = -\frac{\sqrt{3}}{3}$ occur at $x = \frac{5\pi}{6}$ or $\frac{11\pi}{6}$. Therefore, the solutions to the equation $\tan^2 x \sin x - \frac{\sin x}{3} = 0$ occur at $x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$, or 2π.

c) The equation

$$\cos^2 x + \left(\frac{1 - \sqrt{2}}{2}\right)\cos x - \frac{\sqrt{2}}{4} = 0$$

can be solved as follows:

$$\cos^2 x + \left(\frac{1 - \sqrt{2}}{2}\right)\cos x - \frac{\sqrt{2}}{4} = 0$$
$$\left(\cos x - \frac{\sqrt{2}}{2}\right)\left(\cos x + \frac{1}{2}\right) = 0$$

Since $\left(\cos x - \frac{\sqrt{2}}{2}\right)\left(\cos x + \frac{1}{2}\right) = 0$, either $\cos x - \frac{\sqrt{2}}{2} = 0$ or $\cos x + \frac{1}{2} = 0$ (or both). If $\cos x - \frac{\sqrt{2}}{2} = 0$, x can be solved for as follows:

$$\cos x - \frac{\sqrt{2}}{2} = 0$$
$$\cos x - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0 + \frac{\sqrt{2}}{2}$$
$$\cos x = \frac{\sqrt{2}}{2}$$

The solutions to the equation $\cos x = \frac{\sqrt{2}}{2}$ occur at $x = \frac{\pi}{4}$ or $\frac{7\pi}{4}$. If $\cos x + \frac{1}{2} = 0$, x can be solved for as follows:

$$\cos x + \frac{1}{2} = 0$$
$$\cos x + \frac{1}{2} - \frac{1}{2} = 0 - \frac{1}{2}$$
$$\cos x = -\frac{1}{2}$$

The solutions to the equation $\cos x = -\frac{1}{2}$ occur at $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. Therefore, the solutions to the equation $\cos^2 x + \left(\frac{1 - \sqrt{2}}{2}\right)\cos x - \frac{\sqrt{2}}{4} = 0$ occur at $x = \frac{\pi}{4}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{7\pi}{4}$.

d) The equation $25\tan^2 x - 70\tan x = -49$ can be solved as follows:

$$25\tan^2 x - 70\tan x = -49$$
$$25\tan^2 x - 70\tan x + 49 = -49 + 49$$
$$25\tan^2 x - 70\tan x + 49 = 0$$
$$(5\tan x - 7)^2 = 0$$

Since $(5\tan x - 7)^2 = 0$, $5\tan x - 7 = 0$. For this reason, x can be solved for as follows:

$$5\tan x - 7 = 0$$
$$5\tan x - 7 + 7 = 0 + 7$$
$$5\tan x = 7$$
$$\frac{5\tan x}{5} = \frac{7}{5}$$
$$\tan x = \frac{7}{5}$$

The solutions to the equation $\tan x = \frac{7}{5}$ occur at $x = 0.95$ and 4.09. Therefore, the solutions to the equation $25\tan^2 x - 70\tan x = -49$ occur at $x = 0.95$ or 4.09.

13. Since $1 + \tan^2 x = \sec^2 x$, the equation $\frac{1}{1 + \tan^2 x} = -\cos x$ can be rewritten and factored as follows:

$$\frac{1}{1 + \tan^2 x} = -\cos x$$
$$\frac{1}{\sec^2 x} = -\cos x$$
$$\cos^2 x = -\cos x$$
$$\cos^2 x + \cos x = -\cos x + \cos x$$
$$\cos^2 x + \cos x = 0$$
$$(\cos x)(\cos x + 1) = 0$$

Since $(\cos x)(\cos x + 1) = 0$, either $\cos x = 0$ or $\cos x + 1 = 0$ (or both). The solutions to the equation $\cos x = 0$ occur at $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. If $\cos x + 1 = 0$, x can be solved for as follows:

$$\cos x + 1 = 0$$
$$\cos x + 1 - 1 = 0 - 1$$
$$\cos x = -1$$

The solution to the equation $\cos x = -1$ occurs at $x = \pi$. Therefore, the solutions to the equation $\frac{1}{1 + \tan^2 x} = -\cos x$ occur at $x = \frac{\pi}{2}$, π, or $\frac{3\pi}{2}$.

Chapter Self-Test, p. 441

1. The identity

$$\frac{1 - 2\sin^2 x}{\cos x + \sin x} + 2\sin\frac{x}{2}\cos\frac{x}{2} = \cos x$$

can be proven as follows:

$$\frac{1 - 2\sin^2 x}{\cos x + \sin x} + 2\sin\frac{x}{2}\cos\frac{x}{2} = \cos x$$
$$\frac{1 - 2\sin^2 x}{\cos x + \sin x} + \sin x = \cos x$$
$$\frac{1 - 2\sin^2 x}{\cos x + \sin x} + \sin x - \sin x = \cos x - \sin x$$
$$\frac{1 - 2\sin^2 x}{\cos x + \sin x} = \cos x - \sin x$$
$$1 - 2\sin^2 x = (\cos x - \sin x) \times (\cos x + \sin x)$$
$$\cos 2x = (\cos x - \sin x) \times (\cos x + \sin x)$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\cos 2x = \cos 2x$$

2. Since $\cos 2\theta = 1 - 2\sin^2\theta$, the equation $\cos 2x + 2\sin^2 x - 3 = -2$ can be rewritten and solved as follows:

$$\cos 2x + 2\sin^2 x - 3 = -2$$
$$1 - 2\sin^2 x + 2\sin^2 x - 3 = -2$$
$$-2 = -2$$

Since -2 always equals -2, the equation $\cos 2x + 2\sin^2 x - 3 = -2$ is an identity and is true for all real numbers x, where $0 \le x \le 2\pi$.

3. a) The solutions to the equation $\cos x = \frac{\sqrt{3}}{2}$ occur at $x = \frac{\pi}{6}$ or $\frac{11\pi}{6}$.

b) The solutions to the equation $\tan x = -\sqrt{3}$ occur at $x = \frac{2\pi}{3}$ or $\frac{5\pi}{3}$.

c) The solutions to the equation $\sin x = -\frac{\sqrt{2}}{2}$ occur at $x = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$.

4. Since the quadratic trigonometric equation $a\cos^2 x + b\cos x - 1 = 0$ has the solutions $\frac{\pi}{3}$, π, and $\frac{5\pi}{3}$, the left side of the equation must have factors of $\cos x - \frac{1}{2}$ and $\cos x + 1$. This is because the cosine of $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ is $\frac{1}{2}$, and the cosine of π is -1. For this reason, the quadratic trigonometric equation can be found as follows:

$$\left(\cos x - \frac{1}{2}\right)(\cos x + 1) = 0$$
$$\cos^2 x - \frac{1}{2}\cos x + \cos x - \frac{1}{2} = 0$$
$$\cos^2 x + \frac{1}{2}\cos x - \frac{1}{2} = 0$$

$$(2)\left(\cos^2 x + \frac{1}{2}\cos x - \frac{1}{2}\right) = (2)(0)$$

$$2\cos^2 x + \cos x - 1 = 0$$

Therefore, $a = 2$ and $b = 1$.

5. Since the depth of the ocean in metres can be modelled by the function $d(t) = 4 + 2\sin\left(\frac{\pi}{6}t\right)$, when the depth is 3 metres, $3 = 4 + 2\sin\left(\frac{\pi}{6}t\right)$.

The equation $3 = 4 + 2\sin\left(\frac{\pi}{6}t\right)$ can be solved as follows:

$$3 = 4 + 2\sin\left(\frac{\pi}{6}t\right)$$

$$3 - 4 = 4 + 2\sin\left(\frac{\pi}{6}t\right) - 4$$

$$-1 = 2\sin\left(\frac{\pi}{6}t\right)$$

$$-\frac{1}{2} = \frac{2\sin\left(\frac{\pi}{6}t\right)}{2}$$

$$\sin\left(\frac{\pi}{6}t\right) = -\frac{1}{2}$$

$$\sin^{-1}\left(\sin\left(\frac{\pi}{6}t\right)\right) = \sin^{-1}\left(-\frac{1}{2}\right)$$

$$\frac{\pi}{6}t = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$$

If $\frac{\pi}{6}t = \frac{7\pi}{6}$, t can be solved for as follows:

$$\frac{\pi}{6}t = \frac{7\pi}{6}$$

$$\left(\frac{6}{\pi}\right)\left(\frac{\pi}{6}t\right) = \left(\frac{6}{\pi}\right)\left(\frac{7\pi}{6}\right)$$

$$t = 7$$

If $\frac{\pi}{6}t = \frac{11\pi}{6}$, t can be solved for as follows:

$$\frac{\pi}{6}t = \frac{11\pi}{6}$$

$$\left(\frac{6}{\pi}\right)\left(\frac{\pi}{6}t\right) = \left(\frac{6}{\pi}\right)\left(\frac{11\pi}{6}\right)$$

$$t = 11$$

Therefore, two times when the depth of the water is 3 metres are $t = 7$ h or 11 h. Also, since the period of the function $d(t) = 4 + 2\sin\left(\frac{\pi}{6}t\right)$ is $\frac{2\pi}{\frac{\pi}{6}} = (2\pi)\left(\frac{6}{\pi}\right) = 12$ h, the depth of the water is also at 3 metres at $t = 7 + 12 = 19$ h or at $t = 11 + 12 = 23$ h.

6. Nina can find the cosine of $\frac{11\pi}{4}$ by using the formula $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

The cosine of π is -1, and the cosine of $\frac{7\pi}{4}$ is $\frac{\sqrt{2}}{2}$.

Also, the sine of π is 0, and the sine of $\frac{7\pi}{4}$ is $-\frac{\sqrt{2}}{2}$. Therefore, $\cos\frac{11\pi}{4} = \cos\left(\pi + \frac{7\pi}{4}\right)$

$$= \left(-1 \times \frac{\sqrt{2}}{2}\right) - \left(0 \times -\frac{\sqrt{2}}{2}\right)$$

$$= -\frac{\sqrt{2}}{2} - 0 = -\frac{\sqrt{2}}{2}$$

7. The equation $3\sin x + 2 = 1.5$ can be solved as follows:

$$3\sin x + 2 = 1.5$$

$$3\sin x + 2 - 2 = 1.5 - 2$$

$$3\sin x = -0.5$$

$$\frac{3\sin x}{3} = -\frac{0.5}{3}$$

$$\sin x = -0.1667$$

The solutions to the equation $\sin x = -0.1667$ occur at $x = 3.31$ or 6.12.

8. Since $\tan\alpha = 0.75$, the leg opposite the angle α in a right triangle has a length of 3, while the leg adjacent to angle α has a length of 4. For this reason, the hypotenuse of the right triangle can be calculated as follows:

$$3^2 + 4^2 = z^2$$

$$9 + 16 = z^2$$

$$25 = z^2$$

$$z = 5$$

Since $\cos\alpha = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos\alpha = \frac{4}{5}$. Also, since $\sin\alpha = \frac{\text{opposite leg}}{\text{hypotenuse}}$, $\sin\alpha = \frac{3}{5}$. In addition, since $\tan\beta = 2.4$, the leg opposite the angle β in a right triangle has a length of 12, while the leg adjacent to angle β has a length of 5. For this reason, the hypotenuse of the right triangle can be calculated as follows:

$$12^2 + 5^2 = z^2$$

$$144 + 25 = z^2$$

$$169 = z^2$$

$$z = 13$$

Since $\cos\beta = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, $\cos\beta = \frac{5}{13}$. Also, since $\sin\beta = \frac{\text{opposite leg}}{\text{hypotenuse}}$, $\sin\beta = \frac{12}{13}$. Therefore, since

$\sin(a - b) = \sin a \cos b - \cos a \sin b,$
$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

$$= \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) = \frac{15}{65} - \frac{48}{65} = -\frac{33}{65}$$

Also, since $\cos(a + b) = \cos a \cos b - \sin a \sin b$,
$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

$$= \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) = \frac{20}{65} - \frac{36}{65} = -\frac{16}{65}$$

9. a) Since $\sin^2 x = \frac{4}{9}$, and since the angle x is in the second quadrant, $\sin x = \sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$.
Since $\sin x = \frac{2}{3}$, the leg opposite the angle x in a right triangle has a length of 2, while the hypotenuse of the right triangle has a length of 3. For this reason, the other leg of the right triangle can be calculated as follows:

$$\begin{aligned} x^2 + (2)^2 &= 3^2 \\ x^2 + 4 &= 9 \\ x^2 + 4 - 4 &= 9 - 4 \\ x^2 &= 5 \\ x &= \sqrt{5} \end{aligned}$$

Since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, and since the angle x is in the second quadrant, $\cos x = -\frac{\sqrt{5}}{3}$. Since
$\sin 2\theta = 2 \sin \theta \cos \theta,$
$\sin 2x = 2 \sin x \cos x$

$$\begin{aligned} &= (2)\left(\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right) \\ &= -\frac{4\sqrt{5}}{9}. \end{aligned}$$

b) Since $\sin^2 x = \frac{4}{9}$, and since the angle x is in the second quadrant, $\sin x = \sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$. Since $\sin x = \frac{2}{3}$, the leg opposite the angle x in a right triangle has a length of 2, while the hypotenuse of the right triangle has a length of 3. For this reason, the other leg of the right triangle can be calculated as follows:

$$\begin{aligned} x^2 + (2)^2 &= 3^2 \\ x^2 + 4 &= 9 \\ x^2 + 4 - 4 &= 9 - 4 \\ x^2 &= 5 \\ x &= \sqrt{5} \end{aligned}$$

Since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, and since the angle x is in the second quadrant, $\cos x = -\frac{\sqrt{5}}{3}$. Since
$\cos 2\theta = \cos^2 \theta - \sin^2 \theta,$
$\cos 2x = \cos^2 x - \sin^2 x$

$$\begin{aligned} &= \left(-\frac{\sqrt{5}}{3}\right)^2 - \left(\frac{2}{3}\right)^2 \\ &= \frac{5}{9} - \frac{4}{9} = \frac{1}{9} \end{aligned}$$

(The formulas $\cos 2\theta = 2\cos^2 \theta - 1$ and $\cos 2\theta = 1 - 2\sin^2 \theta$ could also have been used.)

c) First note that because x is in the second quadrant, $\frac{x}{2}$ is in the first quadrant, where the cosine is positive. Since $\sin^2 x = \frac{4}{9}$, and since the angle x is in the second quadrant, $\sin x = \sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$.
Since $\sin x = \frac{2}{3}$, the leg opposite the angle x in a right triangle has a length of 2, while the hypotenuse of the right triangle has a length of 3. For this reason, the other leg of the right triangle can be calculated as follows:

$$\begin{aligned} x^2 + (2)^2 &= 3^2 \\ x^2 + 4 &= 9 \\ x^2 + 4 - 4 &= 9 - 4 \\ x^2 &= 5 \\ x &= \sqrt{5} \end{aligned}$$

Since $\cos x = \frac{\text{adjacent leg}}{\text{hypotenuse}}$, and since the angle x is in the second quadrant, $\cos x = -\frac{\sqrt{5}}{3}$. Since $\cos 2\theta = 2\cos^2 \theta - 1$, $\cos \theta = 2\cos^2 \frac{\theta}{2} - 1$, and since $\cos x = -\frac{\sqrt{5}}{3}$, $-\frac{\sqrt{5}}{3} = 2\cos^2 \frac{x}{2} - 1$. The value of $\cos \frac{x}{2}$ can now be determined as follows:

$$\begin{aligned} -\frac{\sqrt{5}}{3} &= 2\cos^2 \frac{x}{2} - 1 \\ -\frac{\sqrt{5}}{3} + 1 &= 2\cos^2 \frac{x}{2} - 1 + 1 \\ \frac{3 - \sqrt{5}}{3} &= 2\cos^2 \frac{x}{2} \\ \frac{\frac{(3 - \sqrt{5})}{3}}{2} &= \frac{2\cos^2 \frac{x}{2}}{2} \\ \cos^2 \frac{x}{2} &= \frac{(3 - \sqrt{5})}{6} \\ \cos \frac{x}{2} &= \sqrt{\left[\frac{(3 - \sqrt{5})}{6}\right]} \end{aligned}$$

d) Since $\sin^2 x = \frac{4}{9}$, and since the angle x is in the second quadrant, $\sin x = \sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$.
Since $\sin x = \frac{2}{3}$, the leg opposite the angle x in a right triangle has a length of 2, while the hypotenuse of the right triangle has a length of 3.

For this reason, the other leg of the right triangle can be calculated as follows:

$$\begin{aligned} x^2 + (2)^2 &= 3^2 \\ x^2 + 4 &= 9 \\ x^2 + 4 - 4 &= 9 - 4 \\ x^2 &= 5 \\ x &= \sqrt{5} \end{aligned}$$

Since $\cos x = \dfrac{\text{adjacent leg}}{\text{hypotenuse}}$, and since the angle x is in the second quadrant, $\cos x = -\frac{\sqrt{5}}{3}$. Since $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$,

$$\begin{aligned} \sin 3x &= 3\cos^2 x\sin x - \sin^3 x \\ &= (3)\left(-\frac{\sqrt{5}}{3}\right)^2\left(\frac{2}{3}\right) - \left(\frac{2}{3}\right)^3 \\ &= (3)\left(\frac{5}{9}\right)\left(\frac{2}{3}\right) - \frac{8}{27} \\ &= \frac{30}{27} - \frac{8}{27} = \frac{22}{27} \end{aligned}$$

10. a) The equation $2 - 14\cos x = -5$ can be solved as follows:

$$\begin{aligned} 2 - 14\cos x &= -5 \\ 2 - 14\cos x - 2 &= -5 - 2 \\ -14\cos x &= -7 \\ \frac{-14\cos x}{-14} &= \frac{-7}{-14} \\ \cos x &= \frac{1}{2} \end{aligned}$$

From the graph, the solutions to the equation $\cos x = \frac{1}{2}$ occur at $x = -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}$, or $\frac{5\pi}{3}$.

b) The equation $9 - 22\cos x - 1 = 19$ can be solved as follows:

$$\begin{aligned} 9 - 22\cos x - 1 &= 19 \\ 8 - 22\cos x &= 19 \\ 8 - 22\cos x - 8 &= 19 - 8 \\ -22\cos x &= 11 \\ \frac{-22\cos x}{-22} &= \frac{11}{-22} \\ \cos x &= -\frac{1}{2} \end{aligned}$$

From the graph, the solutions to the equation $\cos x = -\frac{1}{2}$ occur at $x = -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}$, or $\frac{4\pi}{3}$.

c) The equation $2 + 7.5\cos x = -5.5$ can be solved as follows:

$$\begin{aligned} 2 + 7.5\cos x &= -5.5 \\ 2 + 7.5\cos x - 2 &= -5.5 - 2 \\ 7.5\cos x &= -7.5 \\ \frac{7.5\cos x}{7.5} &= \frac{-7.5}{7.5} \\ \cos x &= -1 \end{aligned}$$

From the graph, the solutions to the equation $\cos x = -1$ occur at $x = -\pi$ and π.

CHAPTER 8
Exponential and Logarithmic Functions

Getting Started, p. 446

1. a) $\frac{1}{5^2} = \frac{1}{25}$

b) 1

c) $\sqrt{36} = 6$

d) $\sqrt[3]{125} = 5$

e) $-\sqrt{121} = -11$

f) $\left(\sqrt[3]{\frac{27}{8}}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

2. a) $3^{(2+5)} = 3^7 = 2187$

b) $(-2)^{(12+(-10))} = (-2)^2 = 4$

c) $10^{(9-6)} = 10^3 = 1000$

d) $7^{(6+(-3)-(-1))} = 7^4 = 2401$

e) $8^{(2)\left(\frac{1}{3}\right)} = 8^{\frac{2}{3}} = 4$

f) $4^{\left(\frac{3}{4}+\frac{1}{4}-\frac{1}{2}\right)} = 4^{\frac{1}{2}} = \sqrt{4} = 2$

3. a) $(2m)(2m)(2m) = 8m^3$

b) $a^{-8}b^{-10} = \frac{1}{a^8b^{10}}$

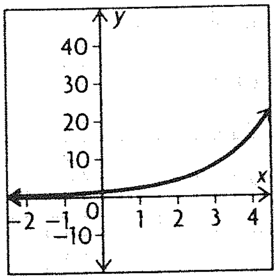
c) $\sqrt{16x^6} = 4|x|^3$

d) $x^{(5-2)}y^{(2-1)} = x^3y$

e) $(-d^4)\left(\frac{c^2}{d^2}\right) = -d^{(4-2)}c^2 = -d^2c^2$

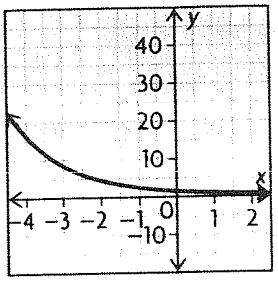
f) $\left(\frac{1}{\sqrt[3]{x^3}}\right)^{-1} = \sqrt[3]{x^3} = x$

4. a)



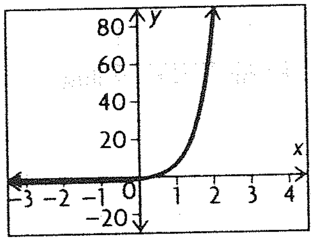
D = $\{x \in \mathbf{R}\}$, R = $\{y \in \mathbf{R} | y > 0\}$, y-intercept 1; horizontal asymptote $y = 0$

b)



D = $\{x \in \mathbf{R}\}$, R = $\{y \in \mathbf{R} | y > 0\}$, y-intercept 1, horizontal asymptote $y = 0$

c)



D = $\{x \in \mathbf{R}\}$, R = $\{y \in \mathbf{R} | y > -2\}$, y-intercept -1, horizontal asymptote $y = -2$

5. a) i) add 6, divide by 3; $y = \frac{x+6}{3}$

ii) add five, take the square root;
$y = \pm\sqrt{x+5}$

iii) divide by 6, take the cube root; $y = \sqrt[3]{\frac{x}{6}}$

iv) subtract 3, take the square root, add 4;
$y = \pm\sqrt{x-3} + 4$

b) The inverses of **(i)** and **(iii)** are functions.

6. a) 12 h ÷ 4 h per doubling = 3
$(100)(2^3) = 800$ bacteria

b) 24 h ÷ 4 h per doubling = 6
$(100)(2^6) = 6400$ bacteria

c) 3.5 days × 24 h/day ÷ 4 h per doubling = 21
$(100)(2^{21}) = 209\,715\,200$

d) 7 days × 24 h/day ÷ 4 h per doubling = 42
$100(2^{42}) = 4.4 \times 10^{15}$

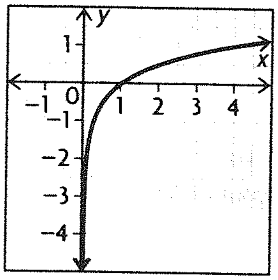
7. x = time in years y = ending population
$y = 15\,000(1 - 0.012)^x$
$y = 15\,000(0.988)^{15}$
$y = 12\,515$ people

8.

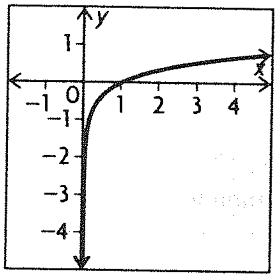
Similarities	Differences
• same y-intercept • same shape • same horizontal asymptote • both are always positive	• one is always increasing, the other is always decreasing • different end behaviour • reflections of each other across the y-axis

8.1 Exploring the Logarithmic Function, p. 451

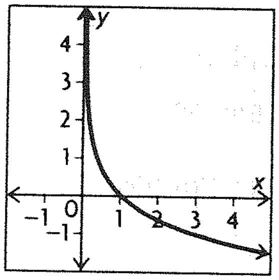
1. a) Inverse function: $x = 4^y$ or $f^{-1}(x) = \log_4 x$



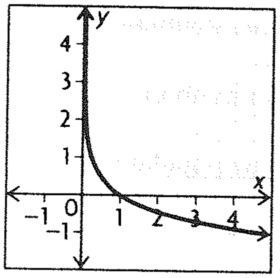
b) Inverse function: $x = 8^y$ or $f^{-1}(x) = \log_8 x$



c) Inverse function: $x = \left(\frac{1}{3}\right)^y$ or $f^{-1}(x) = \log_{\left(\frac{1}{3}\right)} x$



d) Inverse function: $x = \left(\frac{1}{5}\right)^y$ or $f^{-1}(x) = \log_{\left(\frac{1}{5}\right)} x$



2. To write the inverse in exponential form, replace $f(x)$ by y and then switch x and y. Then rewrite in logarithmic form.

a) i) $y = 4^x$

$x = 4^y$

ii) $\log_4 x = y$

b) i) $y = 8^x$

$x = 8^y$

ii) $\log_8 x = y$

c) i) $y = \left(\frac{1}{3}\right)^x$

$x = \left(\frac{1}{3}\right)^y$

ii) $\log_{\frac{1}{3}} x = y$

d) i) $y = \left(\frac{1}{5}\right)^x$

$x = \left(\frac{1}{5}\right)^y$

ii) $\log_{\frac{1}{5}} x = y$

3. All the graphs have the same basic shape, but the last two are reflected over the x-axis, compared with the first two. All the graphs have the same x-intercept, 1. All have the same vertical asymptote, $x = 0$.

4. Locate the point on the graph that has 8 as its x-coordinate. This point is (8, 3). The y-coordinate of this point is the solution to $2^y = 8$, $y = 3$.

5. a) $x = 3^y$
b) $x = 10^y$
c) $x = \left(\frac{1}{4}\right)^y$
d) $x = m^y$

6. a) $\log_3 x = y$
b) $\log_{10} x = y$
c) $\log_{\frac{1}{4}} x = y$
d) $\log_m x = y$

7. a) $x = 5^y$
b) $x = 10^y$
c) $x = 3^y$
d) $x = \left(\frac{1}{4}\right)^y$

8. a) $y = 5^x$
b) $y = 10^x$
c) $y = 3^x$
d) $y = \left(\frac{1}{4}\right)^x$

9. a) $2^x = 4; x = 2$
b) $3^x = 27; x = 3$
c) $4^x = 64; x = 4$
d) $5^x = 1; x = 0$
e) $2^x = \frac{1}{2}; x = -1$
f) $3^x = \sqrt{3}; x = \frac{1}{2}$

10. If a positive base is raised to any power, the resulting value is positive. There is therefore no way to raise positive 3 to a power and wind up with negative 9.

11. a) $-2 = \log_2 x$
$2^{-2} = x$
$x = \frac{1}{4}$
$\left(\frac{1}{4}, -2\right)$
$-1 = \log_2 x$
$2^{-1} = x$
$x = \frac{1}{2}$
$\left(\frac{1}{2}, -1\right)$
$0 = \log_2 x$
$2^0 = x$
$x = 1$
$(1, 0)$
$1 = \log_2 x$
$2^1 = x$
$x = 2$
$(2, 1)$
$2 = \log_2 x$
$2^2 = x$
$x = 4$
$(4, 2)$

b) $-2 = \log_{10} x$
$10^{-2} = x$
$x = \frac{1}{100}$
$\left(\frac{1}{100}, -2\right)$
$-1 = \log_{10} x$
$10^{-1} = x$
$x = \frac{1}{10}$
$\left(\frac{1}{10}, -1\right)$
$0 = \log_{10} x$
$10^0 = x$
$x = 1$
$(1, 0)$
$1 = \log_{10} x$
$10^1 = x$
$x = 10$
$(10, 1)$
$2 = \log_{10} x$
$10^2 = x$
$x = 100$
$(100, 2)$

8.2 Transformations of Logarithmic Functions, pp. 457–458

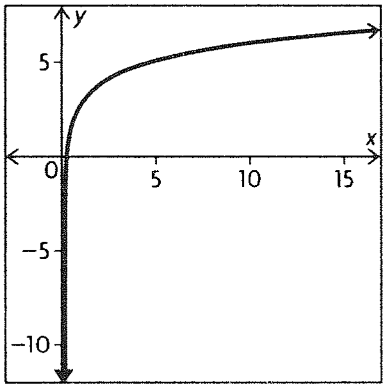
1. The general logarithmic function is $f(x) = a \log_{10}(k(x - d)) + c$
a) $a = 3$; produces a vertical stretch by a factor of 3
b) $k = 2$; produces horizontal compression by a factor of $\frac{1}{2}$
c) $d = -5$; produces a vertical translation 5 units down
d) $c = -4$; produces a horizontal translation 4 units left.

2. a) (a) A vertical stretch by a factor of 3 takes $\left(\frac{1}{10}, -1\right)$ to $\left(\frac{1}{10}, -3\right)$, $(1, 0)$ to $(1, 0)$, and $(10, 1)$ to $(10, 3)$.
(b) A horizontal compression by a factor of $\frac{1}{2}$ takes $\left(\frac{1}{10}, -1\right)$ to $\left(\frac{1}{20}, -1\right)$, $(1, 0)$ to $\left(\frac{1}{2}, 0\right)$, and $(10, 1)$ to $(5, 1)$.
(c) A vertical translation 5 units down takes $\left(\frac{1}{10}, -1\right)$ to $\left(\frac{1}{10}, -6\right)$, $(1, 0)$ to $(1, -5)$, and $(10, 1)$ to $(10, -4)$.
(d) A horizontal translation 4 units to the left takes $\left(\frac{1}{10}, -1\right)$ to $(-3\frac{9}{10}, -1)$, $(1, 0)$ to $(-3, 0)$, and $(10, 1)$ to $(6, 1)$.
b) (a) $D = \{x \in \mathbf{R} | x > 0\}$, $R = \{y \in \mathbf{R}\}$
(b) $D = \{x \in \mathbf{R} | x > 0\}$, $R = \{y \in \mathbf{R}\}$
(c) $D = \{x \in \mathbf{R} | x > 0\}$, $R = \{y \in \mathbf{R}\}$
(d) $D = \{x \in \mathbf{R} | x > -4\}$, $R = \{y \in \mathbf{R}\}$

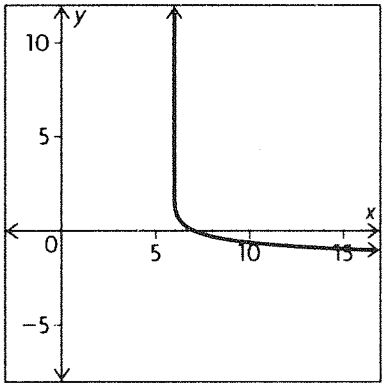
3. a) $a = 5$; $c = 3$; $f(x) = 5 \log_{10} x + 3$
b) $a = -1$ (reflection in the x-axis); $k = 3$ (compression with a factor of 3); $f(x) = -\log_{10}(3x)$
c) $c = -3$; $d = -4$; $f(x) = \log_{10}(x + 4) - 3$
d) $a = -1$; $d = 4$; $f(x) = -\log_{10}(x - 4)$

4. i) a) $a = -4$ resulting in a reflection in the x-axis and a vertical stretch by a factor of 4; $c = 5$ resulting in a translation 5 units up.
b) A vertical stretch by a factor of 4 followed by a reflection in the x-axis and a translation 5 units up takes $(1, 0)$ to $(1, 5)$.
A vertical stretch by a factor of 4 followed by a reflection in the x-axis and a translation 5 units up takes $(10, 1)$ to $(10, 1)$.
c) vertical asymptote is $x = 0$
d) $D = \{x \in \mathbf{R} | x > 0\}$, $R = \{y \in \mathbf{R}\}$
ii) a) $a = \frac{1}{2}$ resulting in a vertical compression by a factor of $\frac{1}{2}$; $d = 6$ resulting in a horizontal translation 6 units to the right; $c = 3$ resulting in a vertical translation 3 units up.

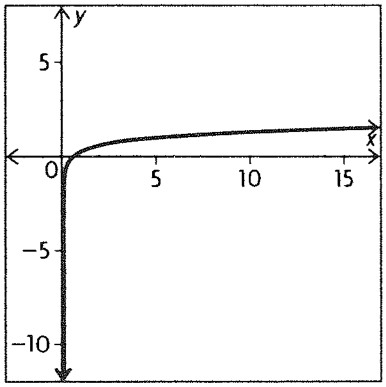
b) A vertical compression by a factor of $\frac{1}{2}$ followed by a translation 6 units to the right and a translation 3 units up takes (1, 0) to (7, 3).
A vertical compression by a factor of $\frac{1}{2}$ followed by a translation 6 units to the right and a translation 3 units up takes (10, 1) to $\left(16, 3\frac{1}{2}\right)$.
c) vertical asymptote is $x = 6$
d) $D = \{x \in \mathbf{R} | x > 6\}$, $R = \{y \in \mathbf{R}\}$
iii) a) $k = 3$ resulting in a horizontal compression by a factor of $\frac{1}{3}$; $c = -4$ resulting in a vertical shift 4 units down.
b) A horizontal compression by a factor of $\frac{1}{3}$ followed by a translation 4 units down takes (1, 0) to $\left(\frac{1}{3}, -4\right)$.
A horizontal compression by a factor of $\frac{1}{3}$ followed by a translation 4 units down takes (10, 1) to $\left(3\frac{1}{3}, -3\right)$.
c) vertical asymptote is $x = 0$
d) $D = \{x \in \mathbf{R} | x > 0\}$, $R = \{y \in \mathbf{R}\}$
iv) a) $a = 2$ resulting in a vertical stretch by a factor of 2; $k = -2$ resulting in a horizontal compression by a factor of $\frac{1}{2}$ and a reflection in the y-axis; $d = -2$ resulting in a horizontal translation 2 units to the left.
b) A horizontal compression by a factor of $\frac{1}{2}$, a reflection in the y-axis, and a vertical stretch by a factor of 2, followed by a translation 2 units to the left takes (1, 0) to $\left(-1\frac{1}{2}, 0\right)$.
A horizontal compression by a factor of $\frac{1}{2}$, a reflection in the y-axis, and a vertical stretch by a factor of 2, followed by a translation 2 units to the left takes (10, 1) to (−7, 2).
c) vertical asymptote is $x = -2$
d) $D = \{x \in \mathbf{R} | x < -2\}$, $R = \{y \in \mathbf{R}\}$
v) a) $\log_{10}(2x + 4) = \log_{10}[2(x + 2)]k = 2$ resulting in a horizontal compression by a factor of $\frac{1}{2}$; $d = -2$ resulting in a horizontal translation 2 units to the left.
b) A horizontal compression by a factor of $\frac{1}{2}$ followed by a translation 2 units to the left takes (1, 0) to $(-1\frac{1}{2}, 0)$.
A horizontal compression by a factor of $\frac{1}{2}$ followed by a translation 2 units to the left takes (10, 1) to (3, 1).
c) vertical asymptote is $x = -2$
d) $D = \{x \in \mathbf{R} | x > -2\}$, $R = \{y \in \mathbf{R}\}$
vi) a) $\log_{10}(-x - 2) = \log_{10}[-1(x + 2)]k = -1$ resulting in a reflection in the x-axis; $d = -2$, resulting in a horizontal translation 2 units to the right.
b) A reflection in the y-axis followed by a translation 2 units to the left takes (1, 0) to (−3, 0)
A reflection in the y-axis followed by a translation 2 units to the left takes (10, 1) to (−12, 1).
c) vertical asymptote is $x = -2$
d) $D = \{x \in \mathbf{R} | x < -2\}$, $R = \{y \in \mathbf{R}\}$
5. a) vertical stretch by a factor of 3;
vertical translation 3 units up
$D = \{x \in \mathbf{R} | x > 0\}$, $R = \{y \in \mathbf{R}\}$



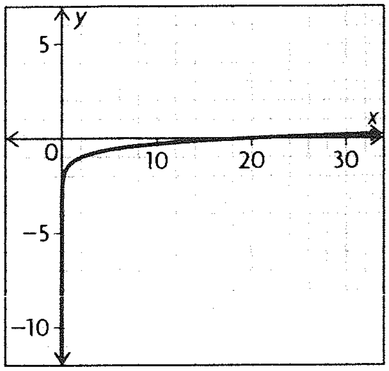
b) reflection in the x-axis;
horizontal translation 6 units to the right
$D = \{x \in \mathbf{R} | x > -6\}$, $R = \{y \in \mathbf{R}\}$



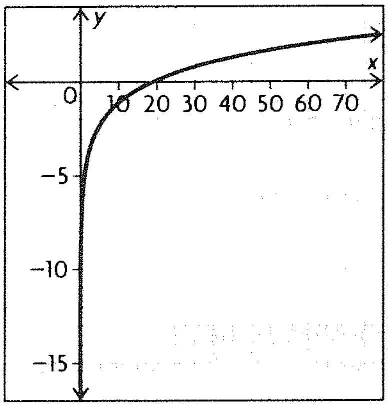
c) horizontal compression by a factor of $\frac{1}{2}$
$D = \{x \in \mathbf{R} | x > 0\}$, $R = \{y \in \mathbf{R}\}$



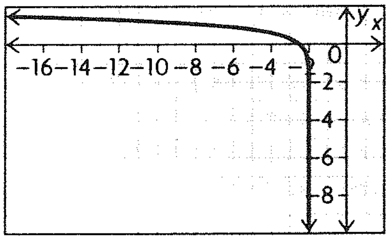
d) horizontal stretch by a factor of 2;
vertical translation 1 unit down
$D = \{x \in \mathbf{R} | x > 0\}$, $R = \{y \in \mathbf{R}\}$



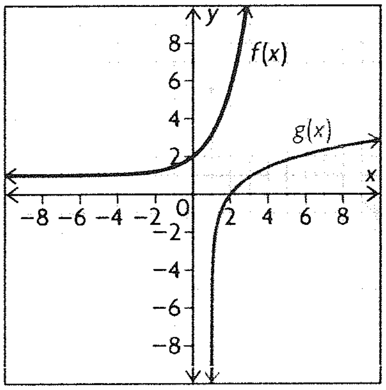
e) vertical stretch by a factor of 4;
horizontal stretch by a factor of 6;
vertical translation 2 units down
$D = \{x \in \mathbf{R} | x > 0\}$, $R = \{y \in \mathbf{R}\}$



f) reflection in the y-axis;
vertical translation 2 units up;
horizontal compression by a factor of $\frac{1}{2}$
$D = \{x \in \mathbf{R} | x < -2\}$, $R = \{y \in \mathbf{R}\}$



6. Graph the two functions.



It appears from the graph that the functions are inverses of each other. Attempt to verify this.

$$y = 10^{\frac{x}{3}} + 1$$
$$x = 10^{\frac{y}{3}} + 1$$
$$x - 1 = 10^{\frac{y}{3}}$$
$$\log (x - 1) = \frac{y}{3}$$
$$3 \log (x - 1) = y$$

This verifies that the functions are inverses of each other.

7. a) The graph of $g(x) = \log_3(x + 4)$ is the same as the graph of $f(x) = \log_3 x$, but horizontally translated 4 units to the left. The graph of $h(x) = \log_3 x + 4$ is the same as the graph of $f(x) = \log_3 x$, but vertically translated 4 units up.

b) The graph of $m(x) = 4 \log_3 x$ is the same as the graph of $f(x) = \log_3 x$, but vertically stretched by a factor of 4. The graph of $n(x) = \log_3 4x$ is the same as the graph of $f(x) = \log_3 x$, but horizontally compressed by a factor of $\frac{1}{4}$.

8. a) $a = -3$ (vertical stretch and reflection in x-axis); $k = \frac{1}{2}$ (horizontal stretch of 2); $d = 5$ (horizontal translation); $c = 2$ (vertical translation)
$f(x) = -3 \log_{10}\left(\frac{1}{2}x - 5\right) + 2$

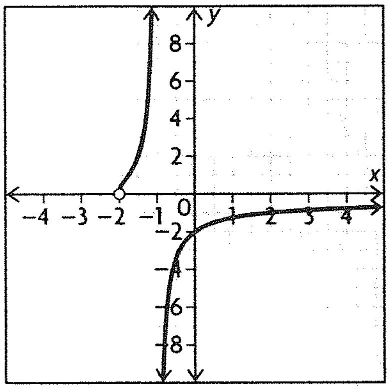
b) multiply the y coordinate by -3 (for the vertical stretch) and add 2 (for the vertical translation), add 5 to the x-coordinate (for the horizontal translation) and multiply by 2 (for the horizontal stretch).
(10, 1) shifts to (30, −1)

c) $D = \{x \in \mathbf{R} | x > 5\}$, $R = \{y \in \mathbf{R}\}$

9. Vertical compression by a factor of $\frac{1}{3}$, reflection in the x-axis, horizontal translation 5 units to the left.

10. domain, range, and vertical asymptote

11.



8.3 Evaluating Logarithms, pp. 466–468

1. a) $\log_4 16 = 2$

b) $\log_3 81 = 4$

c) $\log_8 1 = 0$

d) $\log_6 \frac{1}{36} = -2$

e) $\log_{\frac{1}{3}} \frac{1}{27} = 3$

f) $\log_8 2 = \frac{1}{3}$

2. a) $2^3 = 8$

b) $5^{-2} = \frac{1}{25}$

c) $3^4 = 81$

d) $\left(\frac{1}{6}\right)^{-3} = 216$

e) $6^{\frac{1}{2}} = \sqrt{6}$

f) $10^0 = 1$

3. a) $5^x = 5; x = 1$
b) $7^x = 1; x = 0$
c) $2^x = \frac{1}{4}; x = -2$
d) $7^x = \sqrt{7}; 7^x = 7^{\frac{1}{2}}; x = \frac{1}{2}$
e) $\left(\frac{2}{3}\right)^x = \frac{8}{27}; x = 3$
f) $2^x = \sqrt[3]{2}$
$= 2^{\frac{1}{3}}$
$x = \frac{1}{3}$

4. a) $10^x = \frac{1}{10}; 10^x = 10^{-1}; x = -1$
b) $10^x = 1; 10^x = 10^0; x = 0$
c) $10^x = 1\,000\,000; 10^x = 10^6; x = 6$
d) $10^x = 25; x = 1.40$ (typing log (25) into the calculator); $10^{1.40} \doteq 25$
e) $10^{0.25} = 1.78; x = 1.78$
f) $10^{-2} = 0.01; x = 0.01$
5. a) $6^x = \sqrt{6}$
$6^x = 6^{\frac{1}{2}}$
$x = \frac{1}{2}$
b) $\log_5 125; 5^x = 125; x = 3$
$\log_5 25; 5^x = 25; x = 2$
$\log_5 125 - \log_5 25 = 3 - 2 = 1$
c) $\log_3 81; 3^x = 81; x = 4$
$\log_4 64; 4^x = 64; x = 3$
$\log_3 81 + \log_4 64 = 4 + 3 = 7$
d) $\log_2 \frac{1}{4}; 2^x = \frac{1}{4}; x = -2$
$\log_3 1; 3^x = 1; x = 0$
$\log_2 \frac{1}{4} - \log_3 1; -2 - 0 = -2$
e) $5^x = \sqrt[3]{5}; x = \frac{1}{3}$
f) $3^x = \sqrt{27};$
$3^x = \sqrt{3^3}$
$3^x = 3^{\frac{3}{2}}$
$x = \frac{3}{2}$
6. a) $5^3 = x; x = 125$
b) $x^3 = 27; x = 3$
c) $4^x = \frac{1}{64}; 4^x = \frac{1}{4^3}; 4^x = 4^{-3}; x = -3$
d) $\left(\frac{1}{4}\right)^{-2} = x; 4^2 = x; x = 16$
e) $5^{\frac{1}{2}} = x; x = \sqrt{5}$
f) $4^{1.5} = 8$
7. $f(x) = 3^x$ can be written as $\log_3 f(x)$.
a) $f(x) = 17$. The point (2.58, 17) is on the graph. $\log_3 17 \doteq 2.58$.
b) $f(x) = 36$. The point (3.26, 36) is on the graph. $\log_3 36 \doteq 3.26$.
c) $f(x) = 112$. The point (4.29, 112) is on the graph. $\log_3 112 \doteq 4.29$.
d) $f(x) = 143$. The point (4.52, 143) is on the graph. $\log_3 143 \doteq 4.52$.
8. a) $4^x = 32$; by guess and check $x \doteq 2.50$
b) $6^x = 115$; by guess and check $x \doteq 2.65$
c) $3^x = 212$; by guess and check $x \doteq 4.88$
d) $11^x = 896$; by guess and check $x \doteq 2.83$
9. a) $\log_3 3^5; 3^x = 3^5; x = 5$
b) Given that $\log_5 25 = 2$, $5^{\log_5(25)} = 5^2 = 25$
c) Given that $\log_4 \frac{1}{16} = -2$, $4^{\log_4 \frac{1}{16}} = 4^{-2} = \frac{1}{16}$
d) The base m log of a value is the exponent you need to raise m to in order to get that value. You need to raise m to the n power to get m^n. Therefore, $\log_m m^n = n$.
e) The expression $\log_a b$ means what power do you need to raise a to in order to get b. If you substitute that answer into the expression $a^{\log_a b}$ the result is b.

f) $\log_n 1 = 0$ for all non zero values for n.

10. $\log_2 16^{\frac{1}{3}} = \log_2 (2^{(4)})^{(\frac{1}{3})}$

$$\log_2 2^{\frac{4}{3}} = \frac{4}{3}$$

11. $40(10^x) = 2000$

$$10^x = 50$$

$$\log_{10} 50 = x$$

$$x \doteq 1.7 \text{ weeks}$$

12. To determine amount of decay, use $\left(\frac{1}{2}\right)^{\frac{n}{1620}}$ where n is the number of years the radium has been decaying.

a) $5\left(\frac{1}{2}\right)^{\frac{150}{1620}} = 4.68\ g$

b) $5\left(\frac{1}{2}\right)^{\frac{n}{1620}} = 4\ g$

$$\left(\frac{1}{2}\right)^{\frac{n}{1620}} = \frac{4}{5}$$

$$\log_{0.5} \frac{4}{5} = \frac{n}{1620}$$

$$1620 \log_{0.5} \frac{4}{5} = n$$

$$n = 522 \text{ years}$$

13. A: $s(0.0625) = 0.159 + 0.118 \log(0.0625)$

Slope $= 0.017$

B: $s(1) = 0.159 + 0.118 \log(1)$

Slope $= 0.159$

B has a steeper slope

14. a) $s(50) = 93 \log(50) + 65$

$$s(50) \doteq 233 \text{ mph}$$

b) $250 = 93 \log d + 65$

$$185 = 93 \log d$$

$$1.99 = \log d$$

$$10^{1.99} = d$$

$$d = 98 \text{ miles}$$

15. If $\log 365 = \frac{3}{2} \log 150 - 0.7$, then Kepler's equation gives a good approximation of the time it takes for Earth to revolve around the sun.

$$\log 365 = 2.562$$

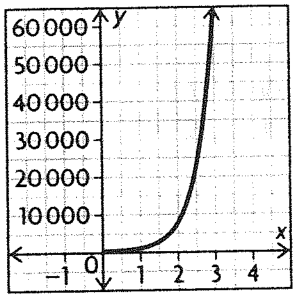
$$\frac{3}{2} \log 150 - 0.7 = 2.564$$

16. a) $\frac{3}{2} \log 2854 - 0.7 \doteq 83$ years

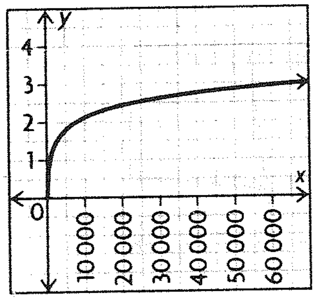
b) $\frac{3}{2} \log 4473 - 0.7 \doteq 164$ years

17. a) $y = 100(2)^{\frac{x}{0.32}}$

b)



c)



d) $y = 0.32 \log_2\left(\frac{x}{100}\right)$; this equation tells how many hours, y, it will take for the number of bacteria to reach x.

e) Evaluate the inverse function for $x = 450$.

$$y = 0.32 \log_2\left(\frac{450}{100}\right)$$

$$y \doteq 0.69 \text{ h}$$

18. a) $\log_5 5 = \frac{\log 5}{\log 5}$

$$= 1.0000$$

b) $\log_2 10 = \frac{\log 10}{\log 2}$

$$= 3.3219$$

c) $\log_5 45 = \frac{\log 45}{\log 5}$

$$= 2.3652$$

d) $\log_8 92 = \frac{\log 92}{\log 8}$

$$= 2.1745$$

e) $\log_4 0.5 = \frac{\log 0.5}{\log 4}$

$$= -0.5000$$

f) $\log_7 325 = \frac{\log 325}{\log 7}$

$$= 2.9723$$

19. a) positive for all values $x > 1$

b) negative for all values $0 < x < 1$

c) undefined for all values $x \leq 0$

20. a) $3^3 + 10^3 = 27 + 1000 = 1027$

b) $5^{1.292} - 3^{3.24} = 8 - 35.14 = -27.14$

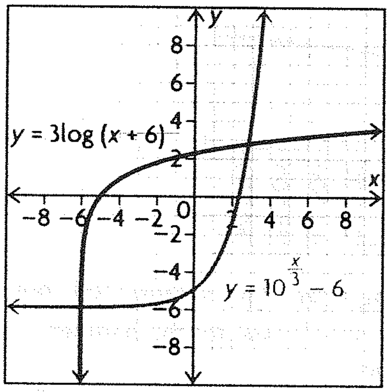
21. a) $y = x^3$

b) $\frac{\sqrt{2}}{3}$

c) $\sqrt[x-2]{0.5}$

d) $2^{\frac{x-2}{3}} + 3$

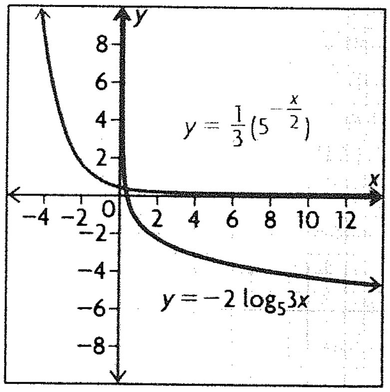
22. a)



function: $y = 3 \log (x + 6)$
$D = \{x \in \mathbf{R} | x > -6\}$
$R = \{y \in \mathbf{R}\}$
asymptote: $x = -6$

inverse: $y = 10^{\frac{x}{3}} - 6$
$D = \{x \in \mathbf{R}\}$
$R = \{y \in \mathbf{R} | y > -6\}$
asymptote: $y = -6$

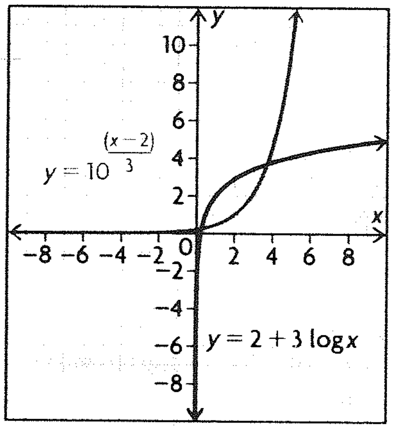
b)



function: $y = -2 \log_5 3x$
$D = \{x \in \mathbf{R} | x > 0\}$
$R = \{y \in \mathbf{R}\}$
asymptote: $x = 0$

inverse: $y = \frac{1}{3}(5^{-\frac{x}{2}})$
$D = \{x \in \mathbf{R}\}$
$R = \{y \in \mathbf{R} | y > 0\}$
asymptote: $y = 0$

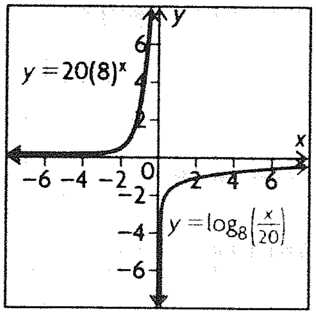
c)



function: $y = 2 + 3 \log x$
$D = \{x \in \mathbf{R} | x > 0\}$
$R = \{y \in \mathbf{R}\}$
asymptote: $x = 0$

inverse: $y = 10^{\frac{(x-2)}{3}}$
$D = \{x \in \mathbf{R}\}$
$R = \{y \in \mathbf{R} | y > 0\}$
asymptote: $y = 0$

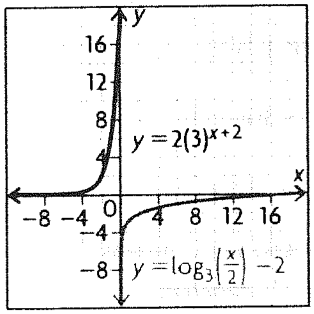
d)



function: $y = 20(8)^x$
$D = \{x \in \mathbf{R}\}$
$R = \{y \in \mathbf{R} \mid y > 0\}$
asymptote: $y = 0$

inverse: $y = \log_8\left(\frac{x}{20}\right)$
$D = \{x \in \mathbf{R} \mid x > 0\}$
$R = \{y \in \mathbf{R}\}$
asymptote: $x = 0$

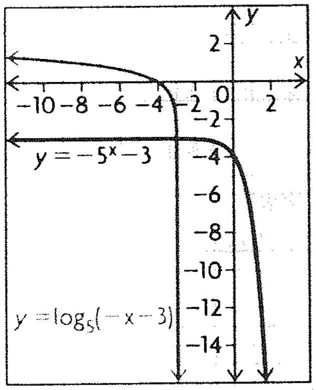
e)



function: $y = 2(3)^{x+2}$
$D = \{x \in \mathbf{R}\}$
$R = \{y \in \mathbf{R} \mid y > 0\}$
asymptote: $y = 0$

inverse: $y = \log_3\left(\frac{x}{2}\right) - 2$
$D = \{x \in \mathbf{R} \mid x > 0\}$
$R = \{y \in \mathbf{R}\}$
asymptote: $x = 0$

f)



function: $y = -5^x - 3$
$D = \{x \in \mathbf{R}\}$
$R = \{y \in \mathbf{R} \mid y < -3\}$
asymptote: $y = -3$

inverse:
$y = \log_5(-x - 3)$
$D = \{x \in \mathbf{R} \mid x < -3\}$
$R = \{y \in \mathbf{R}\}$
asymptote: $x = -3$

23. Given the constraints, two integer values are possible for y, either 1 or 2. If $y = 3$, then x must be 1000, which is not permitted.

8.4 Laws of Logarithms, pp. 475–476

1. a) $\log 45 + \log 68$
b) $\log_m p + \log_m q$
c) $\log 123 - \log 31$
d) $\log_m p - \log_m q$
e) $\log_2 14 + \log_2 9$
f) $\log_4 81 - \log_4 30$
2. a) $\log(7 \times 5) = \log 35$
b) $\log_3 \frac{4}{2} = \log_3 2$
c) $\log_m ab$
d) $\log \frac{x}{y}$
e) $\log_6(7 \times 8 \times 9) = \log_6 504$
f) $\log_4\left(\frac{(10 \times 12)}{20}\right) = \log_4 6$
3. a) $2 \log 5$
b) $-1 \log 7$
c) $q \log_m p$
d) $\frac{1}{3} \log 45$
e) $\frac{1}{2} \log_7 36$
f) $\frac{1}{5} \log_5 125$
4. a) $\log_3 135 - \log_3 5 = \log_3 \frac{135}{5}$
$= \log_3 27$
$= 3$
b) $\log_5 10 + \log_5 2.5 = \log_5(10 \times 2.5)$
$= \log_5 25$
$= 2$

c) $$\begin{aligned}\log 50 + \log 2 &= \log(50 \times 2)\\ &= \log 100\\ &= 2\end{aligned}$$

d) $$\begin{aligned}\log_4 4^7 &= 7 \log_4 4\\ &= 7 \times 1\\ &= 7\end{aligned}$$

e) $$\begin{aligned}\log_2 224 - \log_2 7 &= \log_2 \frac{224}{7}\\ &= \log_2 32\\ &= 5\end{aligned}$$

f) $$\begin{aligned}\log \sqrt{10} &= \frac{1}{2} \log 10\\ &= \left(\frac{1}{2}\right)(1)\\ &= \frac{1}{2}\end{aligned}$$

5. $y = \log_2(4x) = \log_2 x + \log_2 4 = \log_2 x + 2$, so $y = \log_2(4x)$ vertically shifts $y = \log_2 x$ up 2 units; $y = \log_2(8x) = \log_2 x + \log_2 8 = \log_2 x + 3$, so $y = \log_2(8x)$ vertically shifts $y = \log_2 x$ up 3 units; $y = \log_2\left(\frac{x}{2}\right) = \log_2 x - \log_2 2 = \log_2 x - 1$, so $y = \log_2\left(\frac{x}{2}\right)$ vertically shifts $y = \log_2 x$ down 1 unit

6. a) $\log_{25} 5^3 = 3 \log_{25} 5$

$\log_{25} 5$; $25^x = 5$; $x = 0.5$

Therefore $\log_{25} 5^3 = 3 \log_{25} 5 = (3)(0.5) = 1.5$

b) $$\begin{aligned}\log_6 54 + \log_6 2 - \log_6 3 &= \log_6 \frac{(54 \times 2)}{3}\\ &= \log_6 36\\ &= 2\end{aligned}$$

c) $$\begin{aligned}\log_6 6\sqrt{6} &= \log_6 6 + \log_6 \sqrt{6}\\ &= 1 + 0.5\\ &= 1.5\end{aligned}$$

d) $$\begin{aligned}\log_2 \sqrt{36} - \log_2 \sqrt{72} &= \log_2 \frac{\sqrt{36}}{\sqrt{72}}\\ &= \log_2 \sqrt{\frac{1}{2}}\\ &= \log_2 2^{-0.5}\\ &= -0.5\end{aligned}$$

e) $$\begin{aligned}\log_3 54 + \log_3\left(\frac{3}{2}\right) &= \log_3 54 + \log_3 3 - \log_3 2\\ &= \log_3 54 - \log_3 2 + 1\\ &= \log_3 \frac{54}{2} + 1\\ &= \log_3 27 + 1\\ &= 3 + 1\\ &= 4\end{aligned}$$

f) $\log_8 2 + 3 \log_8 2 + \frac{1}{2} \log_8 16$

$$\begin{aligned}&= \log_8 2 + \log_8 2^3 + \log_8 \sqrt{16}\\ &= \log_8 2 + \log_8 8 + \log_8 4\\ &= \log_8 2 + \log_8 4 + 1\\ &= \log_8 (2 \times 4) + 1\\ &= \log_8 8 + 1\\ &= 1 + 1\\ &= 2\end{aligned}$$

7. a) $\log_b x + \log_b y + \log_b z$

b) $\log_b z - (\log_b x + \log_b y)$

c) $\log_b x^2 + \log_b y^3 = 2 \log_b x + 3 \log_b y$

d) $\log_b \sqrt{x^5yz^3} = \frac{1}{2} \log_b x^5yz^3$

$$\begin{aligned}&= \frac{1}{2}(\log_b x^5 + \log_b y + \log_b z^3)\\ &= \frac{1}{2}(5 \log_b x + \log_b y + 3 \log_b z)\end{aligned}$$

8. $\log_5 3$ means $5^x = 3$ and $\log_5 \frac{1}{3}$ means $5^y = \frac{1}{3}$; since $\frac{1}{3} = 3^{-1}$, $5^y = 5^{x(-1)}$; therefore, $\log_5 3 + \log_5 \frac{1}{3} = x + x(-1) = 0$

9. a) $$\begin{aligned}3 \log_5 2 + \log_5 7 &= \log_5 2^3 + \log_5 7\\ &= \log_5 8 + \log_5 7\\ &= \log_5 (8 \times 7)\\ &= \log_5 56\end{aligned}$$

b) $$\begin{aligned}2 \log_3 8 - 5 \log_3 2 &= \log_3 8^2 - \log_3 2^5\\ &= \log_3 64 - \log_3 32\\ &= \log_3 \frac{64}{32}\\ &= \log_3 2\end{aligned}$$

c) $$\begin{aligned}2 \log_2 3 + \log_2 5 &= \log_2 3^2 + \log_2 5\\ &= \log_2 9 + \log_2 5\\ &= \log_2 (9 \times 5)\\ &= \log_2 45\end{aligned}$$

d) $$\begin{aligned}\log_3 12 + \log_3 2 - \log_3 6 &= \log_3 \frac{(12 \times 2)}{6}\\ &= \log_3 4\end{aligned}$$

e) $\log_4 3 + \frac{1}{2} \log_4 8 - \log_4 2$

$$\begin{aligned}&= \log_4 3 + \log_4 \sqrt{8} - \log_4 2\\ &= \log_4 \frac{(3\sqrt{8})}{2}\\ &= \log_4 (3\sqrt{2})\end{aligned}$$

f) $2 \log 8 + \log 9 - \log 36$

$$\begin{aligned}&= \log 8^2 + \log 9 - \log 36\\ &= \log 64 + \log 9 - \log 36\\ &= \log \frac{(64 \times 9)}{36}\\ &= \log 16\end{aligned}$$

10. a)
$$\begin{aligned}\log_2 x &= \log_2 7^2 + \log_2 5\\ \log_2 x &= \log_2 49 + \log_2 5\\ \log_2 x &= \log_2(49 \times 5)\\ \log_2 x &= \log_2 245\\ x &= 245\end{aligned}$$

b)
$$\begin{aligned}\log x &= \log 4^2 + \log 3^3\\ \log x &= \log 16 + \log 27\\ \log x &= \log(16 \times 27)\\ \log x &= \log 432\\ x &= 432\end{aligned}$$

c)
$$\begin{aligned}\log_4 x &= \log_4 48 - \log_4 12\\ \log_4 x &= \log_4 \frac{48}{12}\\ \log_4 x &= \log_4 4\\ x &= 4\end{aligned}$$

d)
$$\begin{aligned}\log_7 x &= \log_7 25^2 - \log_7 5^3\\ \log_7 x &= \log_7 625 - \log_7 125\\ \log_7 x &= \log_7 \frac{625}{125}\\ \log_7 x &= \log_7 5\\ x &= 5\end{aligned}$$

e)
$$\begin{aligned}\log_3 x &= \log_3 10^2 - \log_3 25\\ \log_3 x &= \log_3 100 - \log_3 25\\ \log_3 x &= \log_3 \frac{100}{25}\\ \log_3 x &= \log_3 4\\ x &= 4\end{aligned}$$

f)
$$\begin{aligned}\log_5 x &= \log_5 6 + 3\log_5 2 + \log_5 8\\ \log_5 x &= \log_5 6 + \log_5 2^3 + \log_5 8\\ \log_5 x &= \log_5 6 + \log_5 8 + \log_5 8\\ \log_5 x &= \log_5(6 \times 8 \times 8)\\ \log_5 x &= \log_5 384\\ x &= 384\end{aligned}$$

11. a) $\log_2 xyz$

b) $\log_5 \frac{uw}{v}$

c) $\log_6 a - \log_6 bc = \log_6 \frac{a}{bc}$

d) $\log_2 \frac{x^2y^2}{xy} = \log_2 xy$

e) $\log_3 3 + \log_3 x^2 = \log_3 3x^2$

f)
$$\begin{aligned}\log_4 x^3 + \log_4 x^2 - \log_4 y &= \log_4 \frac{x^3x^2}{y}\\ &= \log_4 \frac{x^5}{y}\end{aligned}$$

12. $\frac{1}{2}\log_a x = \log_a \sqrt{x}$; $\frac{1}{2}\log_a y = \log_a \sqrt{y}$;
$\frac{3}{4}\log_a z = \log_a \sqrt[4]{z^3}$;

So, $\frac{1}{2}\log_a x + \frac{1}{2}\log_a y - \frac{3}{4}\log_a x$
$$\begin{aligned}&= \log_a \sqrt{x} + \log_a \sqrt{y} - \log_a \sqrt[4]{z^3}\\ &= \log_a \frac{\sqrt{x}\sqrt{y}}{\sqrt[4]{z^3}}\end{aligned}$$

13. vertical stretch by a factor of 3 (the exponent that x is raised to); vertical shift 3 units up (the coefficient 8 divided by the base of 2)

14. Answers may vary. For example,
$f(x) = 2\log x - \log 12$
$g(x) = \log \frac{x^2}{12}$
$$\begin{aligned}2\log x - \log 12 &= \log x^2 - \log 12\\ &= \log \frac{x^2}{12}\end{aligned}$$

So, $f(x)$ and $g(x)$ have the same graph.

15. Answers may vary. For example, any number can be written as a power with a given base. The base of the logarithm is 3. Write each term in the quotient as a power of 3. The laws of logarithms make it possible to evaluate the expression by simplifying the quotient and noting the exponent.

16. $\log_x x^{m-1} + 1 = m - 1 + 1 = m$

17.
$$\begin{aligned}\log_b x\sqrt{x} &= \log_b x + \log_b \sqrt{x}\\ &= \log_b x + \frac{1}{2}\log_b x\\ &= 0.3 + 0.3\left(\frac{1}{2}\right)\\ &= 0.45\end{aligned}$$

18. The two functions have different domains. The first function has a domain of $x > 0$. The second function has a domain of all real numbers except 0, since x is squared.

19. Answers may vary; for example,

Product law
$$\begin{aligned}\log_{10} 10 + \log_{10} 10 &= 1 + 1\\ &= 2\\ &= \log_{10} 100\\ &= \log_{10}(10 \times 10)\end{aligned}$$

Quotient law
$$\begin{aligned}\log_{10} 10 - \log_{10} 10 &= 1 - 1\\ &= 0\\ &= \log_{10} 1\\ &= \log_{10}\left(\frac{10}{10}\right)\end{aligned}$$

Power law
$$\begin{aligned}\log_{10} 10^2 &= \log_{10} 100\\ &= 2\\ &= 2\log_{10} 10\end{aligned}$$

Mid-Chapter Review, p. 479

1. a) $\log_5 y = x$

b) $\log_{\frac{1}{3}} y = x$

c) $\log x = y$

d) $\log_p m = q$

2. a) $3^y = x$

b) $10^y = x$

c) $10^k = m$

d) $s^t = r$

3. a) $a = 2; c = -4$; vertical stretch by a factor of 2, vertical translation 4 units down

b) $a = -1; k = 3$; reflection in the x-axis, horizontal compression by a factor of $\frac{1}{3}$

c) $d = -5; c = 1$; vertical compression by a factor of $\frac{1}{4}$, horizontal stretch by a factor of 4

d) $k = 2; d = 2$; horizontal compression by a factor of $\frac{1}{2}$, horizontal translation 2 units to the right

e) $a = \frac{1}{4}; k = \frac{1}{4}$; horizontal translation 5 units to the left, vertical translation 1 unit up

f) $a = 5; k = -1; c = -3$; vertical stretch by a factor of 5, reflection in the y-axis, vertical translation 3 units down

4. a) $y = -4 \log_3 x$

b) $y = \log_3 (x + 3) + 1$

c) $y = \frac{2}{3} \log_3\left(\frac{1}{2}x\right)$

d) $y = 3 \log_3 [-(x - 1)]$

5. a) A vertical stretch by a factor of 4 followed by a reflection in the x-axis takes (9, 2) to (9, −8).

b) A horizontal translation 3 units to the left followed by a translation 1 unit up takes (9, 2) to (6, 3).

c) A vertical compression by a factor of $\frac{2}{3}$ followed by a horizontal stretch by a factor of 2 takes (9, 2) to $(18, \frac{4}{3})$.

d) A vertical stretch by a factor of 3 followed by a reflection in the y-axis and a horizontal translation 1 unit to the right takes (9, 2) to (9, 6), then to (−9, 6), and finally to (−8, 6).

6. It is vertically stretched by a factor of 2 and vertically shifted up 2.

7. a) $3^x = 81; x = 4$

b) $4^x = \frac{1}{16}; x = -2$

c) $5^x = 1; x = 0$

d) $\frac{2^x}{3} = \frac{27}{8}; x = -3$

8. Using the base 10 log key on the calculator

a) 0.602

b) 1.653

c) 2.130

d) 2.477

9. Using a guess and check strategy on the graphing calculator to estimate the result

a) $2^x = 21; x \doteq 4.392$

b) $5^x = 117; x \doteq 2.959$

c) $7^x = 141; x \doteq 2.543$

d) $11^x = 356; x \doteq 2.450$

10. a) $\log (7 \times 4) = \log 28$

b) $\log \frac{5}{2} = \log 2.5$

c) $\log_3 \frac{(11 \times 4)}{6} = \log_3 \frac{22}{3}$

d) $\log_p (q \times q) = \log_p q^2$

11. a) $\log_{11} \frac{33}{3} = \log_{11} 11 = 1$

b) $\log_7 (14 \times 3.5) = \log_7 49 = 2$

c) $\log_5\left(100 \times \frac{1}{4}\right) = \log_5 25 = 2$

d) $\log_{\frac{1}{2}} \frac{72}{9} = \log_{\frac{1}{2}} 8$

$$\left(\frac{1}{2}\right)^x = 8$$

$$x = -3$$

e) $\frac{1}{3} \log_4 16 = \left(\frac{1}{3}\right)(2) = \frac{2}{3}$

f)
$$\begin{aligned}\log_3 9\sqrt{27} &= \log_3 9 + \log_3 \sqrt{27}\\ &= 2 + \frac{1}{2}\log_3 27\\ &= 2 + \left(\frac{1}{2}\right)(3)\\ &= 3.5\end{aligned}$$

12. Compared with the graph of $y = \log x$, the graph of $y = \log x^3$ is vertically stretched by a factor of 3.

13. Use the log button on a calculator to evaluate each expression.

a) $\log 4^8 = 4.82$

b) $\log 200 \div \log 50 = 1.35$

c) $\log\sqrt{40} = 0.80$

d) $(\log 20)^2 = 1.69$

e) $\log 9^4 = 3.82$

f) $5 \log 5 = 3.49$

8.5 Solving Exponential Equations, pp. 485–486

1. a) $x = 4$

b) $(2^2)^{2x} = 2^{5-x}$
$2^{4x} = 2^{5-x}$
$4x = 5 - x$
$5x = 5$
$x = 1$

c) $(3^2)^{x+1} = (3^3)^{2x-3}$
$3^{2x+2} = 3^{6x-9}$
$2x + 2 = 6x - 9$
$11 = 4x$
$x = \frac{11}{4}$

d) $(2^3)^{x-1} = (2^4)^{\frac{1}{3}}$
$2^{3x-3} = 2^{\frac{4}{3}}$
$3x - 3 = \frac{4}{3}$
$9x - 9 = 4$
$9x = 13$
$x = \frac{13}{9}$

e) $2^{3x} = 2^{-1}$
$3x = -1$
$x = -\frac{1}{3}$

f) $4^{2x} = 4^{-2}$
$2x = -2$
$x = -1$

2. a) $\log 2^x = \log 17$
$x \log 2 = \log 17$
$0.301x = 1.230$
$x = 4.088$

b) $\log 6^x = \log 231$
$x \log 6 = \log 231$
$0.778x = 2.364$
$x = 3.037$

c) $5^x = 5$
$x = 1$

d) $5.25 = 1.5^x$
$\log 1.5^x = \log 5.25$
$x \log 1.5 = \log 5.25$
$0.176x = 0.720$
$x = 4.092$

e) $\log 5^{1-x} = \log 10$
$(1 - x)\log 5 = \log 10$
$(1 - x)(0.699) = 1$
$0.699 - 0.699x = 1$
$-0.699x = 0.301$
$x = -0.431$

f) $\log 6^{\frac{x}{3}} = \log 30$
$\frac{x}{3}\log 6 = \log 30$
$\frac{x}{3}(0.778) = 1.477$
$x = 5.695$

3. a) $3^x = 243$
$3^x = 3^5$
$x = 5$

b) $6^x = 216$
$6^x = 6^3$
$x = 3$

c) $5^x = 5\sqrt{5}$
$5^x = 5^{1.5}$
$x = 1.5$

d) $2^x = \sqrt[5]{8}$
$2^x = \sqrt[5]{2^3}$
$2^x = 2^{\frac{3}{5}}$
$x = \frac{3}{5}$

e) $2^x = \frac{1}{4}$
$2^x = 2^{-2}$
$x = -2$

f) $3^x = \frac{1}{\sqrt{3}}$
$3^x = 3^{-\frac{1}{2}}$
$x = -\frac{1}{2}$

4. a) $200 = 300\left(\frac{1}{2}\right)^{\frac{t}{8}}$
$\frac{2}{3} = \left(\frac{1}{2}\right)^{\frac{t}{8}}$
$\log\frac{2}{3} = \log\left(\frac{1}{2}\right)^{\frac{t}{8}}$
$\log\frac{2}{3} = \frac{t}{8}\log\left(\frac{1}{2}\right)$
$-0.176 = \frac{t}{8}(-0.301)$
$t = 4.68$ h

b) $100 = 300\left(\frac{1}{2}\right)^{\frac{t}{8}}$

$\frac{1}{3} = \left(\frac{1}{2}\right)^{\frac{t}{8}}$

$\log\frac{1}{3} = \log\left(\frac{1}{2}\right)^{\frac{t}{8}}$

$\log\frac{1}{3} = \frac{t}{8}\log\left(\frac{1}{2}\right)$

$-0.477 = \frac{t}{8}(-0.301)$

$t = 12.68$ h

c) $75 = 300\left(\frac{1}{2}\right)^{\frac{t}{8}}$

$\frac{1}{4} = \left(\frac{1}{2}\right)^{\frac{t}{8}}$

$\log\frac{1}{4} = \log\left(\frac{1}{2}\right)^{\frac{t}{8}}$

$\log\frac{1}{4} = \frac{t}{8}\log\left(\frac{1}{2}\right)$

$-0.602 = \frac{t}{8}(-0.301)$

$t = 16$ h

d) $20 = 300\left(\frac{1}{2}\right)^{\frac{t}{8}}$

$\frac{1}{15} = \left(\frac{1}{2}\right)^{\frac{t}{8}}$

$\log\frac{1}{15} = \log\left(\frac{1}{2}\right)^{\frac{t}{8}}$

$\log\frac{1}{15} = \frac{t}{8}\log\left(\frac{1}{2}\right)$

$-1.176 = \frac{t}{8}(-0.301)$

$t = 31.26$ h

5. a) $(7^2)^{x-1} = 7^{1.5}$

$7^{2x-2} = 7^{1.5}$

$2x - 2 = 1.5$

$2x = 3.5$

$x = 1.75$

b) $2^{3x-4} = 2^{-2}$

$3x - 4 = -2$

$x = \frac{2}{3}$

c) $(2^{-2})^{x+4} = 2^{\frac{3}{2}}$

$2^{-2x-8} = 2^{\frac{3}{2}}$

$-2x - 8 = \frac{3}{2}$

$-4x - 16 = 3$

$-4x = 19$

$x = -4.75$

d) $36^{2x+4} = 36^x$

$2x + 4 = x$

$x = -4$

e) $2^{2x+2} = 64$

$2^{2x+2} = 2^6$

$2x + 2 = 6$

$x = 2$

f) $(3^2)^{2x+1} = (3^4)(3^3)^x$

$3^{4x+2} = 3^{3x+4}$

$4x + 2 = 3x + 4$

$x = 2$

6. a) $1000 = 500(1.08)^t$

$2 = 1.08^t$

$\log 2 = \log 1.08^t$

$\log 2 = t\log 1.08$

$0.301 = 0.033t$

$t = 9.12$ years

b) $5000 = 1000(1.01)^t$

$5 = 1.01^t$

$\log 5 = \log 1.01^t$

$\log 5 = t\log 1.01$

$0.699 = 0.0043t$

$t = 162.6$ months $= 13.5$ years

c) $7500 = 5000(1.025)^t$

$1.5 = 1.025^t$

$\log 1.5 = \log 1.025^t$

$\log 1.5 = t\log 1.025$

$0.176 = 0.0107t$

$t = 16.44$ quarters or 4.1 years

d) $1500 = 500(1.0023)^t$

$3 = 1.0023^t$

$\log 3 = \log 1.0023^t$

$\log 3 = t\log 1.0023$

$0.477 = 0.000998t$

$t = 477.9$ weeks or 9.2 years

7. $20(2^t) = 163\,840$

$2^t = 8192$

$\log 2^t = \log 8192$

$t\log 2 = \log 8192$

$0.301t = 3.913$

$t = 13$ quarter hours or 3.25 h

8. a) $4^x(4 + 1) = 160$

$4^x = 32$

$\log 4^x = \log 32$

$x\log 4 = \log 32$

$$\begin{aligned} 0.602x &= 1.505 \\ x &= 2.5 \end{aligned}$$

b)
$$\begin{aligned} 2^x(2^2 + 1) &= 320 \\ 2^x &= 64 \\ \log 2^x &= \log 64 \\ x \log 2 &= \log 64 \\ 0.301x &= 1.806 \\ x &= 6 \end{aligned}$$

c)
$$\begin{aligned} 2^x(2^2 - 1) &= 96 \\ 2^x &= 32 \\ \log 2^x &= \log 32 \\ x \log 2 &= \log 32 \\ 0.301x &= 1.505 \\ x &= 5 \end{aligned}$$

d)
$$\begin{aligned} 10^x(10 - 1) &= 9000 \\ 10^x &= 1000 \\ \log 10^x &= \log 1000 \\ x \log 10 &= \log 1000 \\ x &= 3 \end{aligned}$$

e)
$$\begin{aligned} 3^x(3^2 + 1) &= 30 \\ 3^x &= 3 \\ x &= 1 \end{aligned}$$

f)
$$\begin{aligned} 4^x(4^3 - 1) &= 63 \\ 4^x &= 1 \\ x &= 0 \end{aligned}$$

9. a) Solve using logarithms. Both sides can be divided by 225, leaving only a term with a variable in the exponent on the left. This can be solved using logarithms.

b) Solve by factoring out a power of 3 and then simplifying. Logarithms may still be necessary in a situation like this, but the factoring must be done first because logarithms cannot be used on the equation in its current form.

10. a)
$$\begin{aligned} \log 5^{t-1} &= \log 3.92 \\ (t - 1) \log 5 &= \log 3.92 \\ 0.699(t - 1) &= 0.593 \\ t - 1 &= 0.849 \\ t &= 1.849 \end{aligned}$$

b)
$$\begin{aligned} 3^x &= 25 \\ \log 3^x &= \log 25 \\ x \log 3 &= \log 25 \\ 0.477x &= 1.398 \\ x &= 2.931 \end{aligned}$$

c)
$$\begin{aligned} \log 4^{2x} &= \log 5^{2x-1} \\ 2x \log 4 &= (2x - 1) \log 5 \\ 0.602(2x) &= 0.699(2x - 1) \\ 1.204x &= 1.398x - 0.699 \\ -0.194x &= -0.699 \\ x &= 3.606 \end{aligned}$$

d)
$$\begin{aligned} 2^x &= 53.2 \\ \log 2^x &= \log 53.2 \\ x \log 2 &= \log 53.2 \\ 0.301x &= 1.726 \\ x &= 5.734 \end{aligned}$$

11. a) $I_f = I_o(0.95)^t$ where I_f is the final intensity, I_o is the original intensity, and t is the thickness

b)
$$\begin{aligned} 0.6 &= 1(0.95)^t \\ \log 0.6 &= \log 0.95^t \\ \log 0.6 &= t \log 0.95 \\ -0.222 &= -0.0222t \\ t &= 10 \text{ mm} \end{aligned}$$

12.
$$\begin{aligned} 3^{2x} - 5(3^x) + 6 &= 0 \\ (3^x - 3)(3^x - 2) &= 0 \\ 3^x - 3 = 0 \text{ or } 3^x - 2 &= 0 \\ 3^x = 3 \qquad 3^x &= 2 \\ x = 1 \qquad \log 3^x &= \log 2 \\ x \log 3 &= \log 2 \\ 0.477x &= 0.301 \\ x &= 0.631 \end{aligned}$$

13. $a^y = x$, so $\log a^y = \log x$; $y \log a = \log x$;
$$y = \frac{\log x}{\log a}$$
A graphing calculator does not allow logarithms of base 5 to be entered directly. However, $y = \log_5 x$ can be entered for graphing, as $y = \frac{\log x}{\log 5}$.

14. a)
$$\begin{aligned} \log (2^x)^2 &= \log [32(2^{4x})] \\ \log 2x + \log 2x &= \log 32 + \log 2^{4x} \\ x \log 2 + x \log 2 &= \log 32 + 4x \log 2 \\ 0.301x + 0.301x &= 1.505 + 4x(0.301) \\ x &= 2.5 \end{aligned}$$

b)
$$\begin{aligned} 3^{x^2+20} &= (3^{-3})^{3x} \\ x^2 + 20 &= -9x \\ x^2 + 9x + 20 &= 0 \\ (x - 5)(x - 4) &= 0 \\ x &= 5 \text{ or } x = 4 \end{aligned}$$

c)
$$\begin{aligned} \log 2 + \log 3^x &= \log 7 + \log 5^x \\ \log 2 + x \log 3 &= \log 7 + x \log 5 \\ 0.301 + 0.477x &= 0.845 + 0.699x \\ -0.222x &= 0.544 \\ x &= -2.45 \end{aligned}$$

15. Let $\log_a 2 = x$. Then $a^x = 2$. $(a^x)^3 = 2^3$, or $a^{3x} = 8$. Since $\log_a 2 = \log_b 8$, $\log_b 8 = x$. So $b^x = 8$. Since each equation is equal to 8, $a^{3x} = b^x$ and $a^3 = b$.

16.
$$\begin{aligned} 3(5^{2x}) &= 6(4^{3x}) \\ 5^{2x} &= 2(4^{3x}) \\ \log 5^{2x} &= \log 2(4^{3x}) \\ \log 5^{2x} &= \log 2 + \log (4^{3x}) \\ 2x \log 5 &= \log 2 + 3x \log 4 \end{aligned}$$

$$2x(0.699) = 0.301 + 3x(0.602)$$
$$1.398x = 0.301 + 1.806x$$
$$x = -0.737$$

Substitute to determine y

$y = 3(5^{(2\times(-0.737))})$

$y = 3(5^{-1.475})$

$y = 0.279$

17. a) $\log 6^{3x} = \log 4^{2x-3}$
$$3x \log 6 = (2x - 3)\log 4$$
$$3x(0.778) = (2x - 3)(0.602)$$
$$2.33x = 1.204x - 1.806$$
$$1.13x = -1.806$$
$$x = -1.60$$

b) $\log(1.2)^x = \log(2.8)^{x+4}$
$$x \log 1.2 = (x + 4)\log 2.8$$
$$0.079x = (x + 4)(0.447)$$
$$0.079x = 0.447x + 1.789$$
$$-0.368x = 1.789$$
$$x = -4.86$$

c) $\log 3(2)^x = \log 4^{x+1}$
$$\log 3 + \log 2^x = (x + 1)\log 4$$
$$\log 3 + x\log 2 = (x + 1)\log 4$$
$$0.477 + 0.301x = (x + 1)(0.602)$$
$$0.477 + 0.301x = 0.602x + 0.602$$
$$-0.125 = 0.301x$$
$$x = -0.42$$

18. $(2^x)^x = 10$
$$2^{x^2} = 10$$
$$\log 2^{x^2} = \log 10$$
$$x^2 \log 2 = 1$$
$$x^2 = \frac{1}{\log 2}$$
$$x = \pm\sqrt{\frac{1}{\log 2}}$$
$$x = \pm 1.82$$

8.6 Solving Logarithmic Equations, pp. 491–492

1. a) $\log_2 x = \log_2 5^2$
$$x = 25$$

b) $\log_3 x = \log_3 3^4$
$$x = 81$$

c) $\log x = \log 2^3$
$$x = 8$$

d) $\log(x - 5) = \log 10$
$$x - 5 = 10$$
$$x = 15$$

e) $2^x = 8$
$$x = 3$$

f) $\log_2 x = \log_2 \sqrt{3}$
$$x = \sqrt{3}$$

2. a) $x^4 = 625$
$$x^4 = 5^4$$
$$x = 5$$

b) $x^{-\frac{1}{2}} = 6$
$$\frac{1}{\sqrt{x}} = 6$$
$$1 = 6\sqrt{x}$$
$$\frac{1}{6} = \sqrt{x}$$
$$\frac{1}{36} = x$$

c) $5^2 = 2x - 1$
$$25 = 2x - 1$$
$$26 = 2x$$
$$x = 13$$

d) $10^3 = 5x - 2$
$$1000 = 5x - 2$$
$$1002 = 5x$$
$$x = 200.4$$

e) $x^{-2} = 0.04$
$$\frac{1}{x^2} = 0.04$$
$$1 = 0.04x^2$$
$$25 = x^2$$
$$x = 5$$

f) $2x - 4 = 36$
$$2x = 40$$
$$x = 20$$

3. $6.3 = \log\frac{a}{1.6} + 4.2$
$$2.1 = \log\frac{a}{1.6}$$
$$10^{2.1} = \frac{a}{1.6}$$
$$125.89 = \frac{a}{1.6}$$
$$201.43 = a$$

4. a) $x^{\frac{3}{2}} = 3^3$
$$x^3 = (3^3)^2$$
$$x^3 = (3^2)^3$$
$$x^3 = 9^3$$
$$x = 9$$

b) $x^2 = 5$
$$x = \sqrt{5}$$

c) $$\begin{aligned} 3^3 &= 3x + 2\\ 27 &= 3x + 2\\ 25 &= 3x\\ x &= \frac{25}{3} \end{aligned}$$

d) $$\begin{aligned} 10^4 &= x\\ x &= 10\,000 \end{aligned}$$

e) $$\begin{aligned} \left(\frac{1}{3}\right)^x &= 27\\ 3^{-x} &= 3^3\\ -x &= 3\\ x &= -3 \end{aligned}$$

f) $$\begin{aligned} \left(\frac{1}{2}\right)^{-2} &= x\\ x &= 4 \end{aligned}$$

5. a) $$\begin{aligned} \log_2 3x &= 3\\ 2^3 &= 3x\\ 8 &= 3x\\ x &= \frac{8}{3} \end{aligned}$$

b) $$\begin{aligned} \log 3x &= 1\\ 10 &= 3x\\ x &= \frac{10}{3} \end{aligned}$$

c) $$\begin{aligned} \log_5 2x + \log_5 \sqrt{9} &= 2\\ \log_5 [(3)(2x)] &= 2\\ \log_5 6x &= 2\\ 5^2 &= 6x\\ 25 &= 6x\\ x &= \frac{25}{6} \end{aligned}$$

d) $$\begin{aligned} \log_4 \frac{x}{2} &= 2\\ 4^2 &= \frac{x}{2}\\ 16 &= \frac{x}{2}\\ x &= 32 \end{aligned}$$

e) $$\begin{aligned} \log x^3 - \log 3 &= \log 9\\ \log \frac{x^3}{3} &= \log 9\\ \frac{x^3}{3} &= 9\\ x^3 &= 27\\ x^3 &= 3^3\\ x &= 3 \end{aligned}$$

f) $$\begin{aligned} \log_3 \left[\frac{(4x)(5)}{2}\right] &= 4\\ \log_3 10x &= 4\\ 3^4 &= 10x\\ 81 &= 10x\\ x &= 8.1 \end{aligned}$$

6. $$\begin{aligned} \log_6 [x(x-5)] &= 2\\ \log_6 (x^2 - 5x) &= 2\\ 6^2 &= x^2 - 5x\\ 36 &= x^2 - 5x\\ x^2 - 5x - 36 &= 0\\ (x-9)(x+4) &= 0\\ x = 9 \text{ or } x &= -4 \end{aligned}$$

Restrictions: $x > 5$ ($x - 5$ must be positive) so $x = 9$

7. a) $$\begin{aligned} \log_7 [(x+1)(x-5)] &= 1\\ \log_7 (x^2 - 4x - 5) &= 1\\ 7^1 &= x^2 - 4x - 5\\ x^2 - 4x - 12 &= 0\\ (x-6)(x+2) &= 0\\ x = 6 \text{ or } x &= -2 \end{aligned}$$

As x must be >5, -2 is inadmissible. $x = 6$

b) $$\begin{aligned} \log_3 [(x-2)x] &= 1\\ \log_3 (x^2 - 2x) &= 1\\ 3^1 &= x^2 - 2x\\ x^2 - 2x - 3 &= 0\\ (x-3)(x+1) &= 0\\ x = 3 \text{ or } x &= -1 \end{aligned}$$

As x must be >2, -1 is inadmissible. $x = 3$

c) $$\begin{aligned} \log_6 \frac{x}{(x-1)} &= 1\\ 6^1 &= \frac{x}{(x-1)}\\ 6x - 6 &= x\\ 5x &= 6\\ x &= \frac{6}{5} \end{aligned}$$

d) $$\begin{aligned} \log [(2x+1)(x-1)] &= \log 9\\ 2x^2 - x - 1 &= 9\\ 2x^2 - x - 10 &= 0\\ (2x-5)(x+2) &= 0\\ x = 2.5 \text{ or } x &= -2 \end{aligned}$$

As x must be >1, -2 is inadmissible. $x = 2.5$

e) $$\begin{aligned} \log [(x+2)(x-1)] &= 1\\ 10^1 &= x^2 + x - 2\\ x^2 + x - 12 &= 0\\ (x+4)(x-3) &= 0\\ x = -4 \text{ or } x &= 3 \end{aligned}$$

As x must be >1, -4 is inadmissible. $x = 3$

f) $\log_2 x^3 - \log_2 x = 8$

$$\begin{aligned}\log_2 \frac{x^3}{x} &= 8\\ \log_2 x^2 &= 8\\ 2^8 &= x^2\\ (2^4)^2 &= x^2\\ x = 2^4 &= 16\end{aligned}$$

8. a) Use the rules of logarithms to obtain $\log_9 20 = \log_9 x$. Then, because both sides of the equation have the same base, $20 = x$.

b) Use the rules of logarithms to obtain $\log \frac{x}{2} = 3$. Then use the definition of a logarithm to obtain $10^3 = \frac{x}{2}$; $1000 = \frac{x}{2}$; $2000 = x$.

c) Use the rules of logarithms to obtain $\log x = \log 64$. Then, because both sides of the equation have the same base, $x = 64$.

9. a) $50 = 10 \log\left(\frac{I}{10^{-12}}\right)$

$$\begin{aligned}5 &= \log\left(\frac{I}{10^{-12}}\right)\\ 5 &= \log I - \log 10^{-12}\\ 5 &= \log I - (-12)\\ -7 &= \log I\\ 10^{-7} &= I\end{aligned}$$

b) $84 = 10 \log\left(\frac{I}{10^{-12}}\right)$

$$\begin{aligned}8.4 &= \log\left(\frac{I}{10^{-12}}\right)\\ 8.4 &= \log I - \log 10^{-12}\\ 8.4 &= \log (I + 12)\\ -3.6 &= \log I\\ 10^{-3.6} &= I\end{aligned}$$

10. $\log_a \left[\frac{x + 2}{x - 1}\right] = \log_a (8 - 2x)$

$$\begin{aligned}\frac{x + 2}{x - 1} &= 8 - 2x\\ x + 2 &= -2x^2 + 10x - 8\\ -2x^2 + 9x - 10 &= 0\\ 2x^2 - 9x + 10 &= 0\\ (2x - 5)(x - 2) &= 0\\ x = 2.5 \text{ or } x &= 2\end{aligned}$$

11. a) $x = 0.80$

b) $x = -6.91$

c) $x = 3.16$

d) $x = 0.34$

12. $\log_5 [(x - 1)(x - 2)] = \log_5 (x + 6)$

$$\begin{aligned}(x - 1)(x - 2) &= x + 6\\ x^2 - 3x + 2 &= x + 6\\ x^2 - 4x - 4 &= 0\end{aligned}$$

Using the quadratic formula

$$x = \frac{4 \pm \sqrt{16 - (4)(-4)}}{2}$$

$$x = \frac{4 \pm \sqrt{32}}{2}$$

$x = 4.83$ or $x = -0.83$

As x must be >2, -0.83 is extraneous; $x = 4.83$

13. $\log_3 (-8) = x$; $3^x = -8$; Raising positive 3 to any power produces a positive value. If $x \geq 1$, then $3^x \geq 3$. If $0 \leq x < 1$, then $1 \leq x < 3$. If $x < 0$, then $0 < x < 1$.

14. a) $x > 3$

b) If x is 3, we are trying to take the logarithm of 0. If x is less than 3, we are trying to take the logarithm of a negative number.

15. $\frac{1}{2}(\log x + \log y) = \frac{1}{2} \log xy = \log \sqrt{xy}$ so $\frac{x + y}{5} = \sqrt{xy}$ and $x + y = 5\sqrt{xy}$. Squaring both sides gives $(x + y)^2 = 25xy$. Expanding gives $x^2 + 2xy + y^2 = 25xy$; therefore, $x^2 + y^2 = 23xy$.

16. $\log (35 - x^3) = 3 \log (5 - x)$

$$\begin{aligned}\log (35 - x^3) &= \log (5 - x)^3\\ 35 - x^3 &= (5 - x)^3\\ 35 - x^3 &= -x^3 + 15x^2 - 75x + 125\\ 15x^2 - 75x + 90 &= 0\\ x^2 - 5x + 6 &= 0\\ (x - 3)(x - 2) &= 0\\ x = 3 \text{ or } x &= 2\end{aligned}$$

17. $\log_2 a + \log_2 b = 4$; $\log_2 ab = 4$; $2^4 = ab$; $16 = ab$. The values of a and b that satisfy the original equation are pairs that have a product of 16, but a and b must also both be positive. The possible pairs are: 1 and 16, 2 and 8, 4 and 4, 8 and 2, and 16 and 1.

18. $\log_2 (5x + 4) = 3 + \log_2 (x - 1)$

$$\begin{aligned}\log_2 (5x + 4) - \log_2 (x - 1) &= 3\\ \log_2 \frac{5x + 4}{x - 1} &= 3\\ 2^3 &= \frac{5x + 4}{x - 1}\\ 8(x - 1) &= 5x + 4\\ 8x - 8 &= 5x + 4\\ 3x &= 12\\ x &= 4\end{aligned}$$

Substituting 4 in for x in the first equation

$y = \log_2((5)(4) + 4)$
$y = \log_2 24$
$2^y = 24$
$\log 2^y = \log 24$
$y \log 2 = \log 24$
$0.301y = 1.38$
$y = 4.58$

19. a) $5^0 = \log_3 x$
$\log_3 x = 1$
$3^1 = x$
$x = 3$

b) $2^1 = \log_4 x$
$\log_4 x = 2$
$4^2 = x$
$x = 16$

20. $(2^{-1})^{x+y} = 2^4$ $\qquad$ $(x - y)^{-3} = 8$
$2^{-x-y} = 2^4$ $\qquad$ $(x - y)^{-3} = \left(\frac{1}{2}\right)^{-3}$
$-x - y = 4$ $\qquad$ $x - y = \frac{1}{2}$

Adding the two equations gives
$-2y = 4\frac{1}{2}$
$y = -2.25$

Substituting into the first equation
$-x + 2.25 = 4$
$-x = 1.75$
$x = -1.75$

8.7 Solving Problems with Exponential and Logarithmic Functions, pp. 499–501

1. First earthquake: $5.2 = \log x$; $10^{5.2} = 158\,489$
Second earthquake; $6 = \log x$; $10^6 = 1\,000\,000$
Second earthquake is 6.3 times stronger than the first.

2. $\text{pH} = -\log(\text{H}^+)$
$\text{pH} = -\log 6.21 \times 10^{-8}$
$\text{pH} = -(-7.2)$
$\text{pH} = 7.2$

3. $1\,000\,000 \times 10^{-12}\ \text{W/m}^2 = 10^{-6}\ \text{W/m}^2$; the intensity of the sound
$L = 10 \log \frac{10^{-6}}{10^{-12}}$
$L = 10 \log 10^6$
$L = (10)(6) = 60\ \text{dB}$

4. $69 = 10 \log \frac{I}{10^{-12}}$ $\qquad$ $60 = 10 \log \frac{I}{10^{-12}}$
$6.9 = \log I - \log 10^{-12}$ $\qquad$ $6 = \log I - \log 10^{-12}$
$6.9 = \log I + 12$ $\qquad$ $6 = \log I + 12$
$-5.1 = \log I$ $\qquad$ $-6 = \log I$
$10^{-5.1} = I$ $\qquad$ $10^{-6} = I$
$I = 7.9 \times 10^{-6}$ $\qquad$ $I = 1 \times 10^{-6}$

A heavy snore is 7.9 times louder than a normal conversation.

5. a) $9 = -\log \text{H}$
$-9 = \log \text{H}$
$10^{-9} = 0.000\,000\,001 = \text{H}$

b) $6.6 = -\log \text{H}$
$-6.6 = \log \text{H}$
$10^{-6.6} = 0.000\,000\,251 = \text{H}$

c) $7.8 = -\log \text{H}$
$-7.8 = \log \text{H}$
$10^{-7.8} = 0.000\,000\,016 = \text{H}$

d) $13 = -\log \text{H}$
$-13 = \log \text{H}$
$10^{-13} = 0.000\,000\,000\,000\,1 = \text{H}$

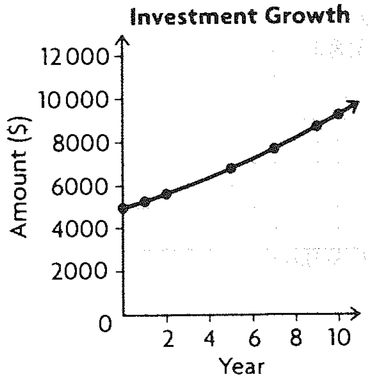
6. a) $\text{pH} = -\log 0.000\,32 = 3.49$
b) $\text{pH} = -\log 0.000\,3 = 3.52$
c) $\text{pH} = -\log 0.000\,045 = 4.35$
d) $\text{pH} = -\log 0.005 = 2.30$

7. a) $\text{pH} = -\log 10^{-7} = 7$
b) Tap water is more acidic than distilled water as it has a lower pH than distilled water (pH 7).

8. $109 = 10 \log \frac{I}{10^{-12}}$ $\qquad$ $118 = 10 \log \frac{I}{10^{-12}}$
$10.9 = \log I - \log 10^{-12}$ $\qquad$ $11.8 = \log I - \log 10^{-12}$
$10.9 = \log I + 12$ $\qquad$ $11.8 = \log I + 12$
$-1.1 = \log I$ $\qquad$ $-0.2 = \log I$
$10^{-1.1} = I$ $\qquad$ $10^{-0.2} = I$
$I = 0.079$ $\qquad$ $I = 0.63$

An amplifier is 7.98 times louder than a lawn mower.

9. a) $y = 5000(1.0642)^t$

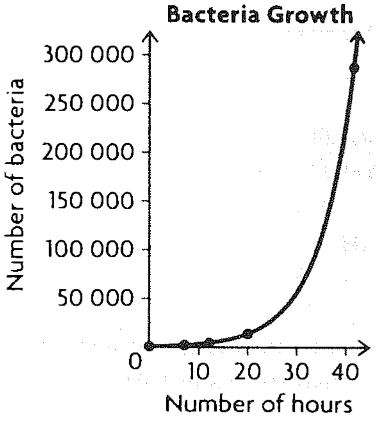


b) 6.42%

c) $10\,000 = 5000(1.0642)^t$
$2 = 1.0642^t$
$\log 2 = \log 1.0642^t$
$\log 2 = t \log 1.0642$
$\frac{\log 2}{\log 1.0642} = t$
$t = 11.14$ years

10. $4.2 = 10^{1-0.13x}$
$\log 4.2 = \log 10^{1-0.13x}$
$\log 4.2 = (1 - 0.13x) \log 10$
$0.623 = 1 - 0.13x$
$-0.376 = -0.13x$
$2.90 = x$

11. a) Average growth is 15%.
Equation is $y = 850(1.15)^x$



b) $1700 = 850(1.15)^x$
$2 = 1.15^x$
$\log 2 = \log 1.15^x$
$\log 2 = x \log 1.15$
$0.301 = x(0.061)$
$x = 4.9$ h

12. a) 1.22, 1.43, 1.69, 2.00, 2.18, 2.35

b) Add the growth factors and divide by 7 = 1.81

c) $w = 5.061\,88(1.0618)^t$

d) $w = 5.061\,88(1.0618)^t$

e) $2 = (1.0618)^t$
$\log 2 = t \log 1.0618$
$0.301 = 0.026t$
$t = 11.5$ °C

13. $0.5 = (0.979)^x$
x = number of cycles
$\log 0.5 = \log 0.979^x$
$\log 0.5 = x \log 0.979$
$-0.301 = -0.009x$
$33 = x$
33 cycles

14. $4000 = 2500(1.065)^t$
$1.6 = (1.065)^t$
$\log 1.6 = \log 1.065^t$
$\log 1.6 = t \log 1.065$
$0.204 = 0.0273t$
$7.4 = t$
7.4 years

15. ~~20~~ $= 80(10^{-0.023t})$
$0.25 = 10^{-0.023t}$
$\log 0.25 = \log 10^{-0.023t}$
$\log 0.25 = (-0.023t) \log 10$
$-0.602 = -0.023t$
$26.2 = t$
~~26.2~~ days 5.4 days

16. Answers may vary. For example: (1) Tom invested \$2000 in an account that accrued interest, compounded annually, at a rate of 6%. How long will it take for Tom's investment to triple? (2) Indira invested \$5000 in a stock that made her \$75 every month. How long will it take her investment to triple?

The first problem could be modelled using an exponential function. Solving this problem would require the use of logarithms. The second problem could be modelled using a linear equation. Solving the second problem would not require the use of logarithms.

17. $70 = 10 \log \frac{I}{10^{-12}}$
$7 = \log I - \log 10^{-12}$
$7 = \log I + 12$
$-5 = \log I$
$10^{-5} = I$
$I = 0.000\,01$

Therefore, the intensity of the sound of the second car is 0.000 02.

$x = 10 \log \frac{0.000\,02}{10^{-12}}$
$x = \log 20\,000\,000$
$x = 73$ dB

18. a) $C = P(1.038)^t$ where P is the present cost of goods and services and t is the number of years.

b) $C = 400(1.038)^{10}$
$C = \$580.80$

c) $47.95 = P(1.038)^{10}$
$47.95 = P(1.45)$
$P = \$33.07$

8.8 Rates of Change in Exponential and Logarithmic Functions, pp. 507–508

1. a) $\dfrac{16 - 75}{10 - 2} = -7.375$

b) $\dfrac{32 - 125}{5 - 1} = -23.25$

c) $\dfrac{10 - 16}{13 - 10} = -2$

2. The instantaneous rate of decline was greatest in year 1. The negative change from year 1 to year 2 was 50, which is greater than the negative change in any other two-year period.

3. Use the equation $y = (96.313)(0.8296)^x$, which was obtained from doing an exponential regression on the data.

a) $y = (96.314)(0.8297)^{1.9}$ $\quad y = (96.314)(0.8297)^{2.1}$
$y = 67.5360$ $\quad y = 65.0593$
$$\frac{67.5520 - 65.0764}{1.9 - 2.1} = -12.378$$

b) $y = (96.314)(0.8297)^{6.9}$ $\quad y = (96.314)(0.8297)^{7.1}$
$y = 26.5610$ $\quad y = 25.5876$
$$\frac{26.5610 - 25.5876}{7.9 - 8.1} = -4.867$$

c) $y = (96.314)(0.8297)^{11.9}$
$y = 10.4436$
$y = (96.314)(0.8297)^{12.1}$
$y = 10.0608$
$$\frac{10.4436 - 10.0608}{11.9 - 12.1} = -1.914$$

4. a) $A(t) = 6000(1.075)^t$

b) $A(t) = 6000(1.075)^{9.9}$ $\quad A(t) = 6000(1.075)^{10.1}$
$y = 12\,277.08$ $\quad y = 12\,455.95$
$$\frac{12\,277.08 - 12\,455.95}{9.9 - 10.1} = 894.35$$

c) $A(t) = 6000(1.0375)^t$
$A(t) = 6000(1.0375)^{19.9}$ $\quad A(t) = 6000(1.0375)^{20.1}$
$$\frac{12\,482.87 - 12\,575.12}{19.9 - 20.1} = 461.25$$

5. a) i) $y = 1000(1.06)^2$
$y = 1123.60$
$$\frac{1123.60 - 1000.00}{2 - 0} = 61.80$$

ii) $y = 1000(1.06)^5$
$y = 1338.23$
$$\frac{1338.23 - 1000.00}{5 - 0} = 67.65$$

iii) $y = 1000(1.06)^{10}$
$y = 1790.85$
$$\frac{1790.85 - 1000.00}{10 - 0} = 79.08$$

b) The rate of change is not constant because the value of the account each year is determined by adding a percent of the previous year's value.

6. a) $M(t) = 500(0.5)^{\frac{24}{5.2}} = 20.40$ g

b) $M(47.9) = 500(0.5)^{\frac{47.9}{5.2}}$ $\quad M(48.1) = 500(0.5)^{\frac{48.1}{5.2}}$
$M(47.9) = 0.8434$ $\quad M(48.1) = 0.8212$
$$\frac{0.8434 - 0.8212}{47.9 - 48.1} = -0.111 \text{ g/h}$$

7. a) $\dfrac{30.21 - 0.0002}{20 - 1} = 1.59$ g/day

b) $y = 0.0017(1.7698)^x$, where x is the number of days after the egg is laid

c) i) $y = 0.0017(1.7698)^{3.9}$ $\quad y = 0.0017(1.7698)^{4.1}$
$y = 0.015\,75$ $\quad y = 0.017\,65$
$$\frac{0.015\,75 - 0.017\,66}{3.9 - 4.1} = 0.0095 \text{ g/day}$$

ii) $y = 0.0017(1.7698)^{11.9}$ $\quad y = 0.0017(1.7698)^{12.1}$
$y = 1.5162$ $\quad y = 1.6995$
$$\frac{1.5162 - 1.6995}{11.9 - 12.1} = 0.917 \text{ g/day}$$

iii) $y = 0.0017(1.7698)^{19.9}$ $\quad y = 0.0017(1.7698)^{20.1}$
$y = 145.93$ $\quad y = 163.58$
$$\frac{145.93 - 163.58}{19.9 - 20.1} = 88.25 \text{ g/day}$$

d)
$$\begin{aligned} 6 &= 0.0017(1.7698)^x \\ 3529 &= 1.7698^x \\ \log 3529 &= \log 1.7698^x \\ \log 3529 &= x \log 1.7698 \\ 3.548 &= 0.248x \\ x &= 14.3 \text{ days} \end{aligned}$$

8. a)
$$\begin{aligned} 50 &= 100(1.2)^{-t} \\ 0.5 &= (1.2)^{-t} \\ \log 0.5 &= \log (1.2)^{-t} \\ \log 0.5 &= -t \log (1.2) \\ -0.301 &= -(0.079)t \\ t &= 3.81 \end{aligned}$$

b) $y = 100(1.2)^{-3.80}$ $\quad y = 100(1.2)^{-3.82}$
$y = 50.02$ $\quad y = 49.83$
$$\frac{50.02 - 49.83}{3.80 - 3.82} = 9.5\,\%/\text{year}$$

9. a) $y = 12\,000(0.982)^t$

b) $y = 12\,000(0.982)^{9.9}$ $\quad y = 12\,000(0.982)^{10.1}$

$y = 10\,025.01$ $\quad y = 9988.66$

$$\frac{10\,025.01 - 9988.66}{9.9 - 10.1} = -181.7 \text{ people/year}$$

c) $6000 = 12\,000(0.982)^t$

$0.5 = (0.982)^t$

$\log 0.5 = \log(0.982)^t$

$\log 0.5 = t \log 0.982$

$-0.301 = -0.0079t$

$38.1 = t$ (yrs for pop to decrease by half)

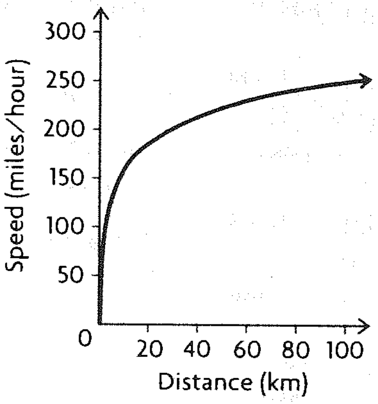
$y = 12\,000(0.982)^{38.0}$ $\quad y = 12\,000(0.982)^{38.2}$

$y = 6017.5$ $\quad y = 5995.7$

$$\frac{6017.5 - 5995.7}{38.0 - 38.2} = -109 \text{ people/year}$$

10. Both functions approach a horizontal asymptote. Each change in x yields a smaller and smaller change in y. Therefore, the instantaneous rate of change grows increasingly small, toward 0, as x increases.

11. a)



b) $S(d) = 93 \log 10 + 65$; $S(d) = 158$ mph

$S(d) = 93 \log 100 + 65$; $S(d) = 251$ mph

251 mph − 158 mph = 93 mph

93 mph/90 m = 1.03 miles/hour/hour

c) $S(d) = 93 \log 9.9 + 65$ $\quad S(d) = 93 \log 10.1 + 65$

$S(9.9) = 157.59$ $\quad S(10.1) = 158.40$

$$\frac{157.59 - 158.40}{9.9 - 10.1} = 4.03 \text{ miles/hour/hour}$$

$S(d) = 93 \log 99.9 + 65$ $\quad S(d) = 93 \log 100.1 + 65$

$S(99.9) = 250.96$ $\quad S(100.1) = 251.04$

$$\frac{250.96 - 251.04}{99.9 - 100.1} = 0.403 \text{ miles/hour/hour}$$

d) The rate at which the wind changes during shorter distances is much greater than the rate at which the wind changes at farther distances. As the distance increases, the rate of change approaches 0.

12. To calculate the instantaneous rate of change for a given point, use the exponential function to calculate the values of y that approach the given value of x. Do this for values on either side of the given value of x. Determine the average rate of change for these values of x and y. When the average rate of change has stabilized to a constant value, this is the instantaneous rate of change.

13. a) and **b)** Only a and k affect the instantaneous rate of change. Increases in the absolute value of either parameter tend to increase the instantaneous rate of change.

Chapter Review, pp. 510–511

1. a) $x = 4^y$; $y = \log_4 x$

b) $x = a^y$; $y = \log_a x$

c) $x = \left(\frac{3}{4}\right)^y$; $y = \log_{\frac{3}{4}} x$

d) $q = P^m$; $m = \log_p q$

2. a) $a = -3$; $k = 2$; vertical stretch by a factor of 3, reflection in the x-axis, horizontal compression by a factor of $\frac{1}{2}$

b) $d = 5$; $c = 2$; horizontal translation 5 units to the right, vertical translation 2 units up

c) $a = \frac{1}{2}$; $k = 5$; vertical compression by a factor of $\frac{1}{2}$, horizontal compression by a factor of $\frac{1}{5}$

d) $k = -\frac{1}{3}$; $c = -3$; horizontal stretch by a factor of 3, reflection in the y-axis, vertical shift 3 units down

3. a) $a = \frac{2}{5}$; $c = -3$; $y = \frac{2}{5}\log x - 3$

b) $a = -1$; $k = \frac{1}{2}$; $d = 3$; $y = -\log\left[\frac{1}{2}(x - 3)\right]$

c) $a = 5$; $k = -2$; $y = 5\log(-2x)$

d) $k = -1$; $d = 4$; $y = \log(-x - 4) - 2$

4. Compared to $y = \log x$, $y = 3 \log (x - 1) + 2$ is vertically stretched by a factor of 3, horizontally translated 1 unit to the right, and vertically translated 2 units up.

5. a) $7^x = 343$; $7^x = 7^3$; $x = 3$

b) $\left(\frac{1}{2}\right)^x = 25$; $\left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^{-2}$; $x = -2$

c) $19^x = 1$; $19^x = 19^0$; $x = 0$

d) $4^x = \frac{1}{256}$; $4^x = 4^{-4}$; $x = -4$

6. a) $3^x = 53$

$\log 3^x = \log 53$

$x \log 3 = \log 53$

$0.477x = 1.724$
$x = 3.615$

b) $4^x = \frac{1}{10}$
$\log 4^x = \log \frac{1}{10}$
$x \log 4 = \log \frac{1}{10}$
$0.602x = -1$
$x = -1.661$

c) $6^x = 159$
$\log 6^x = \log 159$
$x \log 6 = \log 159$
$0.778x = 2.201$
$x = 2.829$

d) $15^x = 1456$
$\log 15^x = \log 1456$
$x \log 15 = \log 1456$
$1.176x = 3.163$
$x = 2.690$

7. a) $\log[(5)(11)] = \log 55$

b) $\log \frac{20}{4} = \log 5$

c) $\log_5 \frac{(6)(8)}{12} = \log_5 4$

d) $\log 3^2 + \log 2^4 = \log 9 + \log 16$
$= \log[(9)(16)]$
$= \log 128$

8. a) $\log_6 \frac{42}{7} = \log_6 6$
$= 1$

b) $\log_3 \frac{(5)(18)}{10} = \log_3 9$
$= 2$

c) $\frac{1}{3} \log_7 49$; $\log_7 49$ in exponential form: $7^x = 49$, $x = 2$, so $\frac{1}{3} \log_7 49 = \left(\frac{1}{3}\right)(2) = \frac{2}{3}$

d) $\log_4 8^2 = \log_4 64$
$= 3$

9. It is shifted 4 units up.

10. a) $5^x = 5^5$; $x = 5$

b) $\log 4^x = \log 16 + \log \sqrt{128}$
$x \log 4 = \log 16 + \frac{1}{2} \log 128$
$0.602x = 1.204 + 1.054$
$0.602x = 2.258$
$x = 3.75$

c) $4^{5x} = (4^2)^{2x-1}$
$4^{5x} = 4^{4x-2}$
$5x = 4x - 2$
$x = -2$

d) $(3^{5x})(3^2)^2 = 3^3$
$3^{5x+4} = 3^3$
$5x + 4 = 3$
$x = -0.2$

11. a) $\log 6^x = \log 78$
$x \log 6 = \log 78$
$0.778x = 1.892$
$x = 2.432$

b) $\log (5.4)^x = \log 234$
$x \log (5.4) = \log 234$
$0.732x = 2.369$
$x = 3.237$

c) $\log (8)(3^x) = \log 132$
$\log 8 + \log 3^x = \log 132$
$\log 8 + x \log 3 = \log 132$
$0.903 + 0.477x = 2.121$
$0.477x = 1.218$
$x = 2.553$

d) $(1.23)^x = 2.7$
$\log (1.23)^x = \log 2.7$
$x \log 1.23 = \log 2.7$
$0.0899x = 0.4314$
$x = 4.799$

12. a) Multiple through by 4^x: $4^{2x} + 6 = 5(4^x)$
$4^{2x} - 5(4^x) + 6 = 0$
$(4^x - 3)(4^x - 2) = 0$
$4^x - 3 = 0 \qquad 4^x - 2 = 0$
$4^x = 3 \qquad 4^x = 2$
$x \log 4 = \log 3 \qquad x \log 4 = \log 2$
$0.602x = 0.477 \qquad 0.602x = 0.301$
$x = 0.79$ or $x = 0.5$

b) $8(5^{2x}) + 8(5^x) - 6 = 0$
$(4(5^x) - 2)(2(5^x) + 3) = 0$
$4(5^x) - 2 = 0 \qquad 2(5^x) + 3 = 0$
$4(5^x) = 2 \qquad 2(5^x) = -3$
$5^x = \frac{1}{2} \qquad 5^x = -\frac{3}{2}$
$\log 5^x = \log \frac{1}{2}$ (no solution)
$x \log 5 = \log \frac{1}{2}$
$0.699x = -0.301$
$x = -0.43$

13. $7 = 20(0.5)^t$
$0.35 = (0.5)^t$
$\log 0.35 = t \log 0.5$
$-0.456 = -0.301t$
$t = (1.52)(3.6) = 5.45$ days

14. a) $5^3 = 2x - 1$
$125 = 2x - 1$
$x = 63$
b) $10^4 = 3x$
$10\,000 = 3x$
$x = \dfrac{10\,000}{3}$
c) $\log_4(3x - 5) = \log_4[(11)(2)]$
$\log_4(3x - 5) = \log_4 22$
$3x - 5 = 22$
$3x = 27$
$x = 9$
d) $\log(4x - 1) = \log[2(x + 1)]$
$\log(4x - 1) = \log(2x + 2)$
$4x - 1 = 2x + 2$
$2x = 3$
$x = 1.5$
15. a) $\log\dfrac{(x + 9)}{x} = 1$
$10^1 = \dfrac{(x + 9)}{x}$
$10x = x + 9$
$9x = 9$
$x = 1$
b) $\log[x(x - 3)] = 1$
$\log(x^2 - 3x) = 1$
$10^1 = x^2 - 3x$
$x^2 - 3x - 10 = 0$
$(x - 5)(x + 2) = 0$
$x = 5$ or $x = -2$
As x must be >3, $x = -2$ is inadmissible: $x = 5$
c) $\log(x - 1)(x + 2) = 1$
$\log(x^2 + x - 2) = 1$
$10^1 = x^2 + x - 2$
$x^2 + x - 12 = 0$
$(x + 4)(x - 3) = 0$
$x = -4$ or $x = 3$
As x must be >1, $x = -4$ is inadmissible: $x = 3$
d) $\dfrac{1}{2}\log(x^2 - 1) = 2$
$\log(x^2 - 1) = 4$
$10^4 = x^2 - 1$
$10\,001 = x^2$
$x = \pm\sqrt{10\,001}$
16. $100 = 10\log\left(\dfrac{I}{10^{-12}}\right)$
$10 = \log\left(\dfrac{I}{10^{-12}}\right)$
$10 = \log I - \log 10^{-12}$
$10 = \log I + 12$
$\log I = -2$
$I = 10^{-2}\ \text{W/m}^2$
17. $82 = 10\log\left(\dfrac{I}{10^{-12}}\right)$
$8.2 = \log\left(\dfrac{I}{10^{-12}}\right)$
$8.2 = \log I - \log 10^{-12}$
$8.2 = \log I + 12$
$\log I = -3.8$
$I = 10^{-3.8}\ \text{W/m}^2$
18. $\dfrac{10^{6.2}}{10^{5.5}} = 5$ times
19. $\dfrac{10^7}{10^{6.4}} = 3.9$ times
20. $\dfrac{10^{4.7}}{10^{2.3}} = 251.2$ $\qquad \dfrac{10^{12.5}}{10^{10.1}} = 251.2$

The relative change in each case is the same. Each change produces a solution with concentration 251.2 times the original solution.

21. It is exponential as there is a common ratio ($\doteq 5$) when comparing the y-values.
Doing an exponential regression gives $y = 3(2.25^x)$
22. $15\,000 = 20\,000(0.984)^t$
$0.75 = 0.984^t$
$\log 0.75 = \log 0.984^t$
$\log 0.75 = t\log 0.984$
$-0.125 = -0.007t$
$t = 17.8$ years
23. a) $\dfrac{514\,013 - 132\,459}{1994 - 1950} = 8671$ people per year
b) $\dfrac{345\,890 - 132\,459}{1980 - 1950} = 7114$

The rate of growth for the first 30 years is slower than the rate of growth for the entire period.

c) Doing an exponential regression
$y = 134\,322(1.03^x)$, where x is the number of years after 1950
d) i) $y = 134\,322(1.03^{19.9})$ $\quad y = 134\,322(1.03^{20.1})$
$y = 241\,884$ $\quad y = 243\,318$
$\dfrac{241\,884 - 243\,318}{19.9 - 20.1} = 7171$ people per year
ii) $y = 134\,322(1.03^{39.9})$ $\quad y = 134\,322(1.03^{40.1})$
$y = 436\,870$ $\quad y = 439\,460$
$\dfrac{436\,870 - 439\,460}{39.9 - 40.1} = 12\,950$ people per year

24. a) The increases from year to year are not the same, so a linear model is not best. Doing an exponential regression gives $y = 23(1.17^x)$, where x is the number of years since 1998.

b) $y = 23(1.17^{17})$

331 808 DVD player owners

c) Answers may vary. For example, I assumed that the rate of growth would be the same through 2015. This is not reasonable. As more people buy the players, there will be fewer people remaining to buy them, or newer technology may replace them.

d) $\frac{43 - 27}{4 - 1} = 5.3$ about 5300 DVD players per year

e) $y = 23(1.17^{1.9})$ $\quad y = 23(1.17^{2.1})$

$y = 30.99$ $\quad y = 31.98$

$\frac{30.99 - 31.98}{1.9 - 2.1} = 4.95$ about 4950 DVD players per year

f) Answers may vary. For example, the prediction in part e) makes sense because the prediction is for a year covered by the data given. The prediction made in part b) does not make sense because the prediction is for a year that is beyond the data given, and conditions may change, making the model invalid.

Chapter Self-Test, p. 512

1. a) $x = 4^y$; $\log_4 x = y$

b) original function in exponential form: $x = 6^y$

inverse $y = 6^x$; $\log_6 y = x$

2. a) $k = 2$; $d = 4$; $c = 3$; horizontal compression by a factor of $\frac{1}{2}$, horizontal translation 4 units to the right, vertical translation 3 units up

b) $a = -\frac{1}{2}$; $d = -5$; $c = -1$; vertical compression by a factor of $\frac{1}{2}$, reflection in the x-axis, horizontal translation 5 units to the left, vertical translation 1 unit down

3. a) $3^x = \frac{1}{9}$; $3^x = 3^{-2}$; $x = -2$

b) $\log_5 \frac{100}{4}$

$\log_5 25$

$5^x = 25$

$x = 5$

4. a) $\log \frac{(15 \times 40)}{6}$

$\log 100 = 2$

b) $\log_7 343 + \log_7 49^2$

$\log_7 (343)(2401)$

$\log_7 823\,543$

$7^x = 823\,543$

$7^x = 7^7$

$x = 7$

5. $\log_4 x^2 + \log_4 \left[(y)\left(\frac{1}{3}\right)^{3}\cancel{(3)}\right] - \log_4 x$

$= \log_4 x^2 + \log_4 \frac{y^3}{27} - \log_4 x$

$= \log_4 \frac{x^2y}{x}$

$= \log_4 \cancel{xy} \left(\frac{x \cdot y^3}{27}\right)$

6. $\log 5^{x+2} = \log 6^{x+1}$

$(x + 2)\log 5 = (x + 1)\log 6$

$0.699(x + 2) = 0.778(x + 1)$

$x + 2 = 1.113x + 1.113$

$0.887 = 0.113x$

$x = 7.85$

7. a) $\log_4 (x + 2)(x - 1) = 1$

$\log_4 (x^2 + x - 2) = 1$

$4^1 = x^2 + x - 2$

$4 = x^2 + x - 2$

$x^2 + x - 6 = 0$

$(x + 3)(x - 2) = 0$

$x = -3$ or $x = 2$

As x must be > 1, $x = -3$ is inadmissible; $x = 2$

b) $\log_3 [(8x - 2)(x - 1)] = 2$

$\log_3 (8x^2 - 10x + 2) = 2$

$3^2 = 8x^2 - 10x + 2$

$8x^2 - 10x - 7 = 0$

$x = \frac{10 \pm \sqrt{100 - (4)(8)(-7)}}{16}$

$x = \frac{28}{16} = 1\frac{3}{4}$ or $x = -\frac{1}{2}$

As x must be > 1, $x = -\frac{1}{2}$ is inadmissible; $x = 1\frac{3}{4}$

8. a) 50g

b) $A(t) = 100(0.5)^{\frac{t}{5730}}$

c) $80 = 100(0.5)^{\frac{t}{5730}}$

$0.8 = (0.5)^{\frac{t}{5730}}$

$\log 0.8 = \log (0.5)^{\frac{t}{5730}}$

$\log 0.8 = \frac{t}{5730}\log (0.5)$

$-0.0969 = \left(\frac{t}{5730}\right)(-0.301)$

$t = 1844$ years

d) $A(t) = 100(0.5)^{\frac{99.9}{5730}}$ $A(t) = 100(0.5)^{\frac{100.1}{5730}}$

$A(t) = 98.799$ $A(t) = 98.796$

$$\frac{98.799 - 98.796}{99.9 - 100.1} = -0.015 \text{ g/year}$$

9. a) $t = \log\left(\frac{35 - 22}{75}\right) \div \log(0.75)$

$$t = \frac{-0.761}{-0.125} = 6 \text{ min}$$

b) $0 = \log\left(\frac{T - 22}{75}\right) \div \log(0.75)$

$$0 = \log\left(\frac{T - 22}{75}\right)$$

$$10^0 = \frac{T - 22}{75}$$

$$75 = T - 22$$

$$T = 97°$$

CHAPTER 9

Combinations of Functions

Getting Started, p. 516

1. a)
$$\begin{aligned} f(-1) &= (-1)^3 - 3(-1)^2 - 10(-1) + 24 \\ &= -1 - 3(1) + 10 + 24 \\ &= -1 - 3 + 10 + 24 \\ &= 30 \end{aligned}$$

$$\begin{aligned} f(4) &= (4)^3 - 3(4)^2 - 10(4) + 24 \\ &= 64 - 3(16) - 40 + 24 \\ &= 64 - 48 - 40 + 24 \\ &= 0 \end{aligned}$$

b)
$$\begin{aligned} f(-1) &= \frac{4(-1)}{1 - (-1)} \\ &= \frac{-4}{1 + 1} \\ &= \frac{-4}{2} \\ &= -2 \end{aligned}$$

$$\begin{aligned} f(4) &= \frac{4(4)}{1 - (4)} \\ &= \frac{16}{1 - 4} \\ &= \frac{16}{-3} \\ &= -5\frac{1}{3} \end{aligned}$$

c) $f(-1) = 3\log_{10}(-1)$

Since you cannot take the log of a negative number, the expression is undefined.

$$\begin{aligned} f(4) &= 3\log_{10}(4) \\ &\doteq 3(0.6021) \\ &\doteq 1.81 \end{aligned}$$

d)
$$\begin{aligned} f(-1) &= -5(0.5^{(-1-1)}) \\ &= -5(0.5^{-2}) \\ &= -5(4) \\ &= -20 \end{aligned}$$

$$\begin{aligned} f(4) &= -5(0.5^{(4-1)}) \\ &= -5(0.5^3) \\ &= -5(0.125) \\ &= -0.625 \end{aligned}$$

2. The domain is the x-values. From the graph, the domain is $\{x \in \mathbf{R} \mid x \neq 1\}$.

The range is the y-values. From the graph, the range is $\{y \in \mathbf{R} \mid y \neq 2\}$.

There is no minimum or maximum value.

The function is never increasing.

The function is decreasing from $(-\infty, 1)$ and $(1, \infty)$.

The function approaches $-\infty$ as x approaches 1 from the left and ∞ as x approaches 1 from the right.

The vertical asymptote is $x = 1$.

The horizontal asymptote is $y = 2$.

3. a) The vertical stretch turns the function into $y = 2|x|$.

The translation 3 units to the right turns the function into $y = 2|x - 3|$.

b) The reflection in the x-axis turns the function into $y = -\cos(x)$.

The horizontal compression by a factor of $\frac{1}{2}$ turns the function into $y = -\cos(2x)$.

c) The reflection in the y-axis turns the function into $y = \log_3(-x)$.

The translation 4 units left makes the function $y = \log_3(-(x + 4))$ or $\log_3(-x - 4)$.

The translation 1 unit down turns the function into $y = \log_3(-x - 4) - 1$.

d) The vertical stretch of 4 turns the function into $y = \frac{4}{x}$.

The reflection in the x-axis turns the function into $y = -\frac{4}{x}$.

The vertical translation 5 units down turns the function into $y = -\frac{4}{x} - 5$.

4. a) $2x^3 - 7x^2 - 5x + 4 = 0$

$$\begin{array}{r|rrrr} -1 & 2 & -7 & -5 & 4 \\ & \downarrow & -2 & 9 & -4 \\ \hline & 2 & -9 & 4 & 0 \end{array}$$

$$\begin{aligned} 2x^2 - 9x + 4 &= 0 \\ (2x - 1)(x - 4) &= 0 \end{aligned}$$

$$2x - 1 = 0 \text{ or } x - 4 = 0$$
$$2x = 1 \text{ or } x = 4$$
$$x = \frac{1}{2} \text{ or } x = 4$$

So, the solutions are $x = -1, \frac{1}{2}$, and 4

b)

$$\frac{2x+3}{x+3}+\frac{1}{2}=\frac{x+1}{x-1}$$
$$2(x+3)(x-1)\left(\frac{2x+3}{x+3}+\frac{1}{2}=\frac{x+1}{x-1}\right)$$
$$2(x-1)(2x+3)+(x+3)(x-1)=2(x+3)(x+1)$$
$$2(2x^2+x-3)+x^2+2x-3=2(x^2+4x+3)$$
$$4x^2+2x-6+x^2+2x-3=2x^2+8x+6$$
$$5x^2+4x-9=2x^2+8x+6$$
$$3x^2-4x-15=0$$
$$(3x+5)(x-3)=0$$
$$x=-\frac{5}{3}\text{ or }x=3$$

c) $\log x+\log(x-3)=1$

$$\log(x(x-3))=1$$
$$10^1=x(x-3)$$
$$10=x^2-3x$$
$$0=x^2-3x-10$$
$$0=(x-5)(x+2)$$
$$x=5\text{ or }x=-2$$

Cannot take the log of a negative number, so $x = 5$.

d) $10^{-4x}-22=978$

$$10^{-4x}=1000$$
$$10^{-4x}=10^3$$
$$-4x=3$$
$$x=-\frac{3}{4}$$

e) $5^{x+3}-5^x=0.992$

$$5^3(5^x)-5^x=0.992$$
$$5^x(5^3-1)=0.992$$
$$5^x(125-1)=0.992$$
$$5^x(124)=0.992$$
$$5^x=0.008$$
$$5^x=5^{-3}$$
$$x=-3$$

f)

$$2\cos^2 x=\sin x+1$$
$$2\,(\sin^2 x-1)=\sin x+1$$
$$2\sin^2 x-2=\sin x+1$$
$$2\sin^2 x-\sin x-3=0$$
$$(2\sin x-3)(\sin x+1)=0$$
$$2\sin x-3=0\text{ or }\sin x+1=0$$
$$\sin x=\frac{3}{2}\text{ or }\sin x=-1$$

Since $\sin x$ cannot be greater than 1, the first equation does not give a solution.

$\sin x=-1$

$x=270°$

---

5. a) $x^3-x^2-14x+24<0$

Find the critical points by solving $x^3-x^2-14x+24=0$.

$x^3-x^2-14x+24=0$

$$\begin{array}{r|rrrr} 2 & 1 & -1 & -14 & 24 \\ & \downarrow & 2 & 2 & -24 \\ \hline & 1 & 1 & -12 & 0 \end{array}$$

So, 2 is a critical value.

$$x^2+x-12=0$$
$$(x+4)(x-3)=0$$
$$x+4=0\text{ or }x-3=0$$
$$x=-4\text{ or }x=3$$

The critical values are -4, 2, and 3.

Test points that are in the intervals created by the critical values.

$(x-2)(x-3)(x+4)$

Test -5: $(-)(-)(-)=(-)<0$

Test 0: $(-)(-)(+)=(+)>0$

Test 2.5: $(+)(-)(+)=(-)<0$

Test 4: $(+)(+)(+)=(+)>0$

So, the solution is $(-\infty,-4)\cup(2,3)$

b) $\dfrac{(2x-3)(x-4)}{(x+2)}\geq 0$

Find the critical values.

$$2x-3=0$$
$$2x=3$$
$$x=\frac{3}{2}$$
$$x-4=0$$
$$x=4$$
$$x+2=0$$
$$x=-2$$

The critical values are $\frac{3}{2}$, 4, and -2

Test -3: $(-)(-)\div(-)=(-)<0$

Test 0: $(-)(-)\div(+)=(+)>0$

Test 2: $(+)(-) \div (+) = (-) < 0$
Test 5: $(+)(+) \div (+) = (+) > 0$
The solution is $(-2, \frac{3}{2}) \cup [4, \infty)$

6. a) $f(x) = 2\sin(x - \pi)$
$$f(-x) = 2\sin(-x - \pi)$$
$$= -2\sin(x - \pi)$$
So, it is odd.

b) $f(x) = \dfrac{3}{4 - x}$
$$f(-x) = \frac{3}{4 - (-x)}$$
$$= \frac{3}{4 + x}$$
So, it is neither.

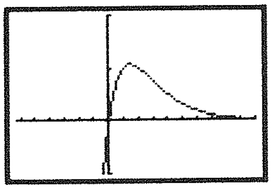
c) $f(x) = 4x^4 - 3x^2$
$$f(-x) = 4(-x)^4 - 3(-x)^2$$
$$= 4x^4 - 3x^2$$
So, it is even.

d) $f(x) = 2^{3x-1}$
$$f(-x) = 2^{3(-x)-1}$$
$$= 2^{-3x-1}$$
So, it is neither.

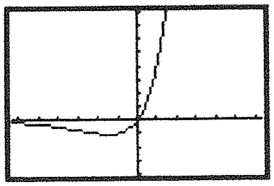
7. Polynomial, logarithmic, and exponential functions are continuous. Rational and trigonometric functions are sometimes continuous and sometimes not.

9.1 Exploring Combinations of Functions, p. 520

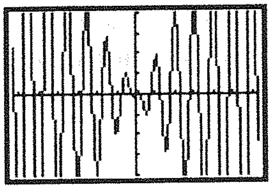
1. Answers may vary. For example, the graph of $y = ((\frac{1}{2})^x)(2x)$ is



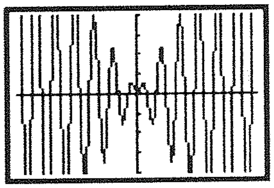
2. a) A function with a vertical asymptote and a horizontal asymptote:
If the functions $y = 2^x$ and $y = 2x$ are multiplied, the resulting function will have a vertical asymptote and a horizontal asymptote.
Answers may vary; for example, $y = (2^x)(2x)$;



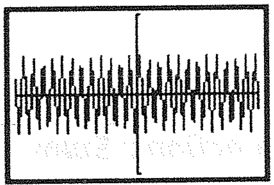
b) A function that is even:
$y = 2x$ is odd
$y = \cos(2\pi x)$ is odd
The product of the two functions will be even.
Answers may vary; for example,
$y = (2x)(\cos(2\pi x))$;



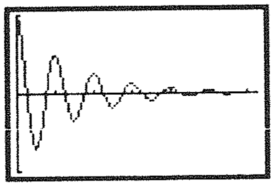
c) A function that is odd:
$y = 2x$ is odd
$y = \sin(2\pi x)$ is even
The product of the two functions is odd.
Answers may vary; for example,
$y = (2x)(\sin(2\pi x))$;



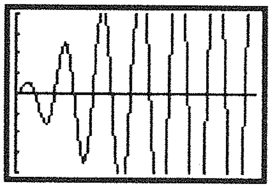
d) A function that is periodic:
$y = \sin 2\pi x$ is periodic
$y = \cos 2\pi x$ is periodic
The product of the two functions is periodic.
Answers may vary; for example,
$y = (\sin 2\pi x)(\cos 2\pi x)$;



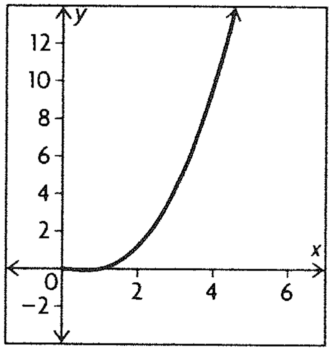
e) A function that resembles a periodic function with decreasing maximum values and increasing minimum values:
$y = \cos 2\pi x$ is periodic
$y = (\frac{1}{2})^x$ is decreasing
The product of the two functions will be a function that resembles a periodic function with decreasing maximum values and increasing minimum values
Answers may vary; for example, $y = (\frac{1}{2})^x(\cos 2\pi x)$ where $0 \le x \le 2\pi$;



f) A function that resembles a periodic function with increasing maximum values and decreasing minimum values:
$y = \sin 2\pi x$ is periodic
$y = 2x$ is increasing
The product of the two functions will be a function that resembles a periodic function with increasing maximum values and decreasing minimum values. Answers may vary; for example, $y = 2x \sin 2\pi x$ where $0 \leq x \leq 2\pi$;



3. Answers will vary. For example,
$y = x^2$
$y = \log x$
The product will be $y = x^2 \log x$.
Graph:



9.2 Combining Two Functions: Sums and Differences, pp. 528–530

1. a) $f + g = \{(-4, 4 + 2), (-2, 4 + 1), (1, 3 + 2), (4, 6 + 4)\}$
$= \{(-4, 6), (-2, 5), (1, 5), (4, 10)\}$

b) $g + f = \{(-4, 2 + 4), (-2, 1 + 4), (1, 2 + 3), (4, 4 + 6)\}$
$= \{(-4, 6), (-2, 5), (1, 5), (4, 10)\}$

c) $f - g = \{(-4, 4 - 2), (-2, 4 - 1), (1, 3 - 2), (4, 6 - 4)\}$
$= \{(-4, 2), (-2, 3), (1, 1), (4, 2)\}$

d) $g - f = \{(-4, 2 - 4), (-2, 1 - 3), (1, 2 - 3), (4, 4 - 6)\}$
$= \{(-4, -2), (-2, -3), (1, -1), (4, -2)\}$

e) $f + f = \{(-4, 4 + 4), (-2, 4 + 4), (1, 3 + 3), (3, 5 + 5), (4, 6 + 6)\}$
$= \{(-4, 8), (-2, 8), (1, 6), (3, 10), (4, 12)\}$

f) $g - g = \{(-4, 2 - 2), (-2, 1 - 1), (0, 2 - 2), (1, 2 - 2), (2, 2 - 2), (4, 4 - 4)\}$
$= \{(-4, 0), (-2, 0), (0, 0), (1, 0), (2, 0), (4, 0)\}$

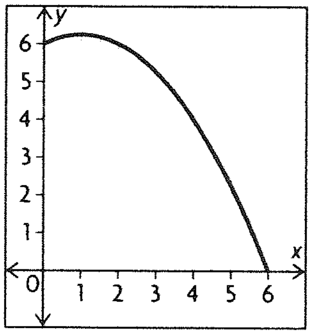
2. a)
$$f(4) = 4^2 - 3 = 16 - 3 = 13$$
$$g(4) = -\frac{6}{4 - 2} = -\frac{6}{2} = -3$$
So, $(f + g)(4) = 13 + -3 = 10$

b) 2; $(f + g)(x)$ is undefined at $x = 2$ because $g(x)$ is undefined at $x = 2$.

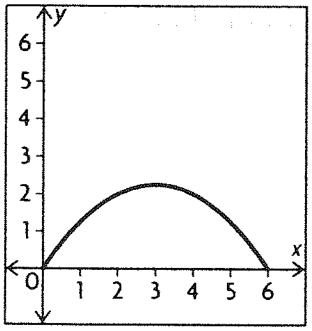
c) Since 2 cannot be part of the domain, the domain is $\{x \in \mathbf{R} | x \neq 2\}$

3. The domain of $f(x)$ is $x \geq -1$. The domain of $g(x)$ is $x < 1$. So the domain of $f - g$ is $\{x \in \mathbf{R} | -1 \leq x < 1\}$.

4. To find the graph of $f + g$, add corresponding y-coordinates.
So, the graph should be:



To find the graph of $f - g$, subtract the corresponding y-coordinates.



5. a) $f + g = |x| + x$

b) $(f + g)(x) = |x| + x$

$(f + g)(-x) = |-x| + -x$

$= |x| - x$

The function is neither even or odd.

6. a) $f + g = \{(-6, 1 + 6), (-3, 7 + 3)\}$

$= \{(-6, 7), (-3, 10)\}$

b) $g + f = \{(-6, 6 + 1), (-3, 3 + 7)\}$

$= \{(-6, 7), (-3, 10)\}$

c) $f - g = \{(-6, 1 - 6), (-3, 7 - 3)\}$

$= \{(-6, -5), (-3, 4)\}$

d) $g - f = \{(-6, 6 - 1), (-3, 3 - 7)\}$

$= \{(-6, 5), (-3, -4)\}$

e) $f - f = \{(-9, -2 - (-2)), (-8, 5 - 5),$
$(-6, 1 - 1), (-3, 7 - 7),$
$(-1, -2 - (-2)), (0, -10 - (-10))\}$

$= \{(-9, 0), (-8, 0), (-6, 0), (-3, 0),$
$(-1, 0), (0, 0)\}$

f) $g + g = \{(-7, 7 + 7), (-6, 6 + 6),$
$(-5, 5 + 5), (-4, 4 + 4), (-3, 3 + 3)\}$

$= \{(-7, 14), (-6, 12), (-5, 10), (-4, 8),$
$(-3, 6)\}$

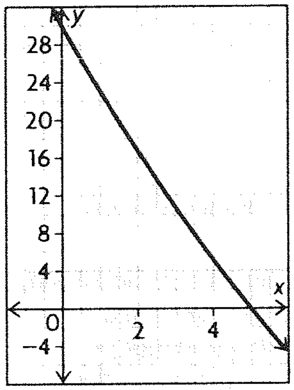
7. a)
$$(f + g)(x) = \frac{1}{3x + 4} + \frac{1}{x - 2}$$
$$= \frac{x - 2}{(3x + 4)(x - 2)} + \frac{3x + 4}{(3x + 4)(x - 2)}$$
$$= \frac{4x + 2}{(3x + 4)(x - 2)}$$
$$= \frac{2(2x + 1)}{3x^2 - 2x - 8}$$

b) The denominator cannot be 0, so $3x + 4 \neq 0$ or $x \neq -\frac{4}{3}$ and $x - 2 \neq 0$ or $x \neq 2$.
So, the domain is $\{x \in \mathbf{R} | x \neq -\frac{4}{3} \text{ or } 2\}$

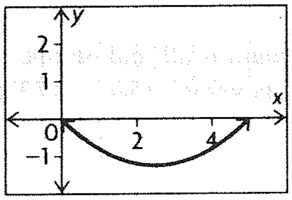
c)
$$(f + g)(8) = \frac{2(2(8) + 1)}{3(8)^2 - 2(8) - 8}$$
$$= \frac{2(17)}{3(64) - 16 - 8}$$
$$= \frac{34}{192 - 16 - 8}$$
$$= \frac{34}{168}$$
$$= \frac{17}{84}$$

d)
$$(f - g)(8) = \frac{1}{3(8) + 4} - \frac{1}{(8) - 2}$$
$$= \frac{1}{24 + 4} - \frac{1}{6}$$
$$= \frac{1}{28} - \frac{1}{6}$$
$$= \frac{3}{84} - \frac{14}{84}$$
$$= -\frac{11}{84}$$

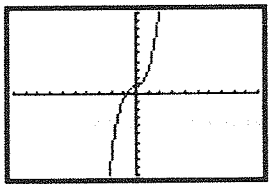
8. To find the graph of $f + g$, add corresponding y-coordinates.
So, the graph should be:



To find the graph of $f - g$, subtract corresponding y-coordinates.
So, the graph should be:

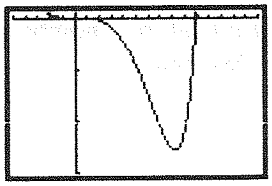


9. a) $f(x) + g(x) = 2^x + x^3$



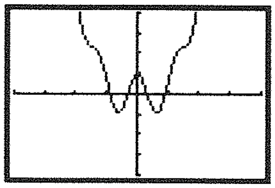
symmetry: The function is not symmetric.
increasing/decreasing: The function is always increasing.
zeros: $x = -0.8262$
maximum/minimum: no maximum or minimum
period: N/A
domain and range: The domain is all real numbers. The range is all real numbers.

$f(x) - g(x) = 2^x - x^3$



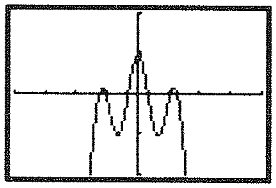
symmetry: The function is not symmetric.
increasing/decreasing: The function is always decreasing.
zeros: $x = 1.3735$
maximum/minimum: no maximum or minimum
period: N/A
domain and range: The domain is all real numbers. The range is all real numbers.

b) $f(x) + g(x) = \cos(2\pi x) + x^4$



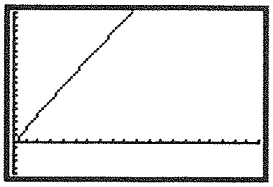
symmetry: The function is symmetric across the line $x = 0$.
increasing/decreasing: The function is decreasing from $-\infty$ to -0.4882 and 0 to 0.4882 and increasing from -0.4882 to 0 and 0.4882 to ∞.
zeros: $x = -0.7092, -0.2506, 0.2506, 0.7092$
maximum/minimum: relative maximum at $x = 0$ and relative minimums at $x = -0.4882$ and $x = 0.4882$
period: N/A
domain and range: The domain is all real numbers. The range is all real numbers greater than -0.1308.

$f(x) - g(x) = \cos(2\pi x) - x^4$



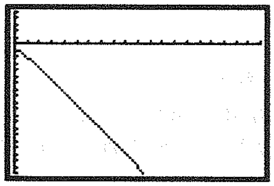
symmetry: The function is symmetric across the line $x = 0$.
increasing/decreasing: The function is increasing from $-\infty$ to -0.9180 and -0.5138 to 0 and 0.5138 to 0.9180; decreasing from -0.9180 to -0.5138 and 0 to 0.5138 and 0.9180 to ∞.
zeros: $x = -1, -0.8278, -0.2494, 0.2494, 0.8278, 1$
maximum/minimum: relative maxima at -0.9180, 0, and 0.9180; relative minima at -0.5138 and 0.5138
period: N/A
domain and range: The domain is all real numbers. The range is all real numbers less than 1.

c) $f(x) + g(x) = \log(x) + 2x$



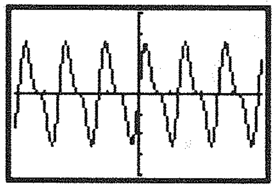
symmetry: The function is not symmetric.
increasing/decreasing: The function is increasing from 0 to ∞.
zeros: none
maximum/minimum: none
period: N/A
domain and range: The domain is all real numbers greater than 0. The range is all real numbers.

$f(x) - g(x) = \log(x) - 2x$



symmetry: The function is not symmetric.
increasing/decreasing: The function is increasing from 0 to approximately 0.2 and decreasing from approximately 0.2 to ∞.
zeros: none
maximum/minimum: maximum at $x \doteq 0.2$
period: N/A
domain and range: The domain is all real numbers greater than 0. The range is all real numbers less than or equal to approximately -1.1.

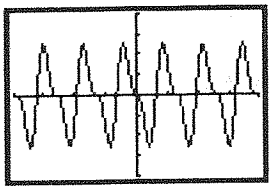
d) $f(x) + g(x) = \sin(2\pi x) + 2\sin(\pi x)$



symmetry: The function is symmetric about the origin.
increasing/decreasing: The function is increasing from $-0.33 + 2k$ to $0.33 + 2k$ and decreasing from $0.33 + 2k$ to $1.67 + 2k$.
zeros: k
maximum/minimum: minimum at $x = -0.33 + 2k$ and maximum at $x = 0.33 + 2k$

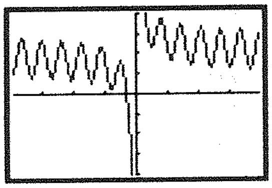
period: 2
domain and range: The domain is all real numbers. The range is all real numbers between −2.598 and 2.598.

$f(x) - g(x) = \sin(2\pi x) - 2\sin(\pi x)$



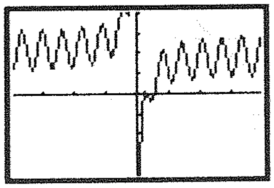
symmetry: The function is symmetric about the origin.
increasing/decreasing: 0.67 + 2*k* to 1.33 + 2*k* and decreasing from −0.67 + 2*k* to 0.67 + 2*k*
zeros: *k*
maximum/minimum: minimum at 0.67 + 2*k* and maximum at 1.33 + 2*k*
period: 2
domain and range: The domain is all real numbers. The range is all real numbers between −2.598 to 2.598.

e) $f(x) + g(x) = \sin(2\pi x) + \frac{1}{x}$



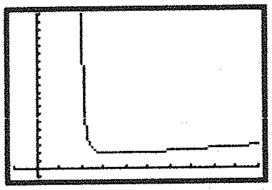
symmetry: The function is not symmetric.
increasing/decreasing: The function is increasing and decreasing at irregular intervals.
zeros: the zeros are changing at irregular intervals.
maximum/minimum: the maximums and minimums are changing at irregular intervals
period: N/A
domain and range: The domain is all real numbers except 0. The range is all real numbers.

$f(x) - g(x) = \sin(2\pi x) - \frac{1}{x}$



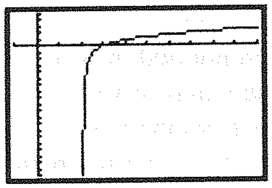
symmetry: The function is not symmetric.
increasing/decreasing: The function is increasing and decreasing at irregular intervals.
zeros: The zeros are changing at irregular intervals.
maximum/minimum: The maximums and minimums are changing at irregular intervals.
period: N/A
domain and range: The domain is all real numbers except 0. The range is all real numbers.

f) $f(x) + g(x) = \sqrt{x - 2} + \frac{1}{x - 2}$



symmetry: The function is not symmetric.
increasing/decreasing: The function is increasing from 3.5874 to ∞ and decreasing from 2 to 3.5874.
zeros: none
maximum/minimum: minimum at $x = 3.5874$
period: N/A
domain and range: The domain is all real numbers greater than 2. The range is all real numbers greater than 1.8899.

$f(x) = g(x) = \sqrt{x - 2} - \frac{1}{x - 2}$



symmetry: The function is not symmetric.
increasing/decreasing: The function is increasing from 2 to ∞.
zeros: $x = 3$
maximum/minimum: none
period: N/A
domain and range: The domain is all real numbers greater than 2. The range is all real numbers.

10. a) The sum of two even functions will be even because replacing x with $-x$ will still result in the original function.

b) The sum of two odd functions will be odd because replacing x with $-x$ will still result in the opposite of the original function.

c) The sum of an even and an odd function will result in neither an even or an odd function because replacing x with $-x$ will not result in the same function or in the opposite of the function.

11. a) $R(t) = 5000 - 25t - 1000\cos\left(\frac{\pi}{6}t\right)$;
it is neither odd nor even; it is increasing during the first 6 months of each year and decreasing during the last 6 months of each year; it has one zero, which is the point at which the deer population has become extinct; it has a maximum value of 3850 and a minimum value of 0, so its range is $\{R(t) \in \mathbf{R} \mid 0 \le R(t) \le 3850\}$.

b) $$R(t) = 5000 - 25t - 1000\cos\left(\frac{\pi}{6}t\right)$$
$$0 = 5000 - 25t - 1000\cos\left(\frac{\pi}{6}t\right)$$
Use a graphing calculator to find the zero of the function.
The deer population is extinct after about 167 months, or 13 years and 11 months.

12. The stopping distance can be defined by the function $s(x) = 0.006x^2 + 0.21x$.
If the vehicle is travelling at 90 km/h, the stopping distance is:
$$\begin{aligned} s(90) &= 0.006(90)^2 + 0.21(90) \\ &= 0.006(8100) + 18.9 \\ &= 48.6 + 18.9 \\ &= 67.5 \text{ m} \end{aligned}$$

13. $f(x) = \sin(\pi x)$; $g(x) = \cos(\pi x)$

14. The function is neither even nor odd; it is not symmetrical with respect to the y-axis or with respect to the origin; it extends from the third quadrant to the first quadrant; it has a turning point between $-n$ and 0 and another turning point at 0; it has zeros at $-n$ and 0; it has no maximum or minimum values; it is increasing when $x \in (-\infty, -n)$ and when $x \in (0, \infty)$; when $x \in (-n, 0)$, it increases, has a turning point, and then decreases; its domain is $\{x \in \mathbf{R}\}$, and its range is $\{y \in \mathbf{R}\}$.

15. a) $f(x) = 0$; $g(x) = 0$

b) $f(x) = x^2$; $g(x) = x^2$
$$\begin{aligned} (f + g)(x) &= x^2 + x^2 \\ &= 2x^2 \end{aligned}$$
It is a vertical stretch of 2 from the original function.

c) $f(x) = \dfrac{1}{x - 2}$; $g(x) = \dfrac{1}{x - 2} + 2$
$$\begin{aligned} \text{So, } (f - g)(x) &= \frac{1}{x - 2} - \left(\frac{1}{x - 2} + 2\right) \\ &= -2 \end{aligned}$$

16. $$\begin{aligned} h(x) &= x^2 - nx + 5 + mx^2 + x - 3 \\ &= (1 + m)x^2 + (1 - n)x + 2 \end{aligned}$$
$$\begin{aligned} 3 &= (1 + m)(1)^2 + (1 - n)(1) + 2 \\ 3 &= 1 + m + 1 - n + 2 \\ 3 &= 4 + m - n \\ -1 &= m - n \text{ EQUATION 1} \\ 18 &= (1 + m)(-2)^2 + (1 - n)(-2) + 2 \\ 18 &= (1 + m)(4) - 2 + 2n + 2 \\ 18 &= 4 + 4m + 2n \\ 14 &= 4m + 2n \text{ EQUATION 2} \\ -1 &= m - n \\ 14 &= 4m + 2n \\ 2(-1 &= m - n) \\ 14 &= 4m + 2n \\ -2 &= 2m - 2n \\ 14 &= 4m + 2n \\ 12 &= 6m \\ 2 &= m \\ -1 &= 2 - n \\ -3 &= -n \\ 3 &= n \end{aligned}$$

9.3 Combining Two Functions: Products, pp. 537–539

1. a) $$\begin{aligned} (f \times g)(x) &= \{(0, 2 \times -1), (1, 5 \times -2), \\ &\quad (2, 7 \times 3), (3, 12 \times 5)\} \\ &= \{(0, -2), (1, -10), \\ &\quad (2, 21), (3, 60)\} \end{aligned}$$

b) $$\begin{aligned} (f \times g)(x) &= \{(0, 3 \times 4), (2, 10 \times -2)\} \\ &= \{(0, 12), (2, -20)\} \end{aligned}$$

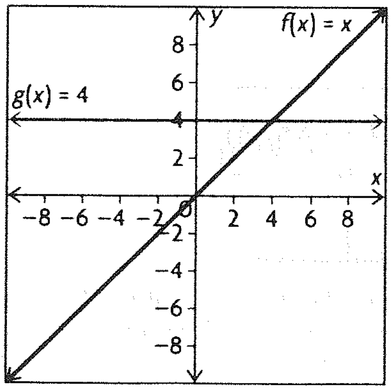
c) $$\begin{aligned} (f \times g)(x) &= x(4) \\ &= 4x \end{aligned}$$

d) $$\begin{aligned} (f \times g)(x) &= x(2x) \\ &= 2x^2 \end{aligned}$$

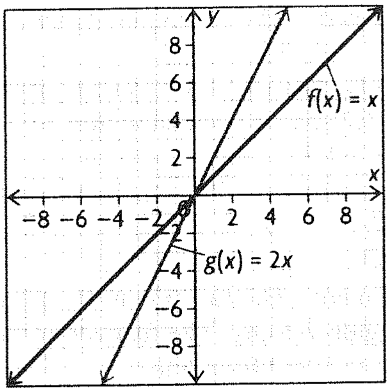
e) $$\begin{aligned} (f \times g)(x) &= (x + 2)(x^2 - 2x + 1) \\ &= x^3 - 2x^2 + x + 2x^2 - 4x + 2 \\ &= x^3 - 3x + 2 \end{aligned}$$

f) $$\begin{aligned} (f \times g)(x) &= 2^x\left(\sqrt{x - 2}\right) \\ &= 2^x\sqrt{x - 2} \end{aligned}$$

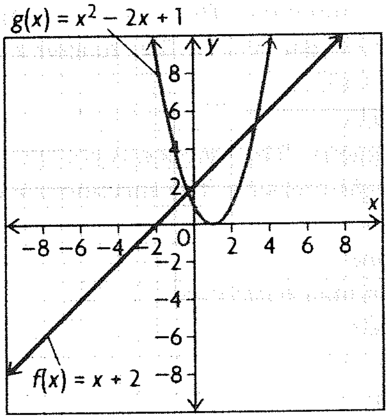
2. a) 1(c):



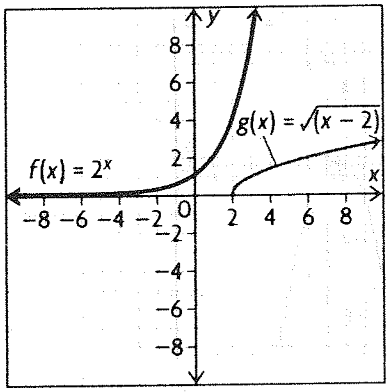
1(d):



1(e):



1(f):



b) 1(c): f: $\{x \in \mathbf{R}\}$; g: $\{x \in \mathbf{R}\}$

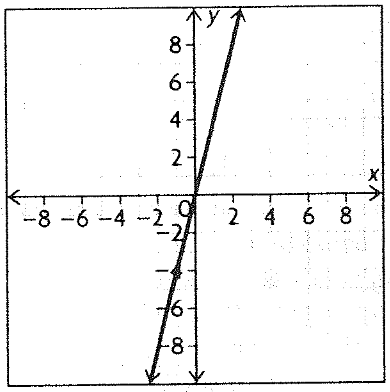
1(d): f: $\{x \in \mathbf{R}\}$; g: $\{x \in \mathbf{R}\}$

1(e): f: $\{x \in \mathbf{R}\}$; g: $\{x \in \mathbf{R}\}$

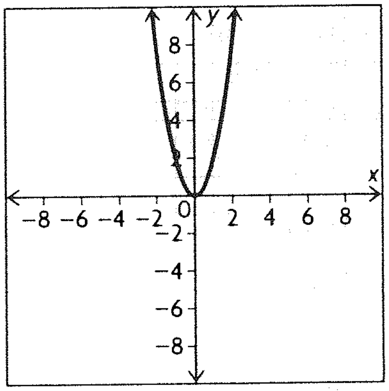
1(f): f: $\{x \in \mathbf{R}\}$; g: $\{x \in \mathbf{R} | x \geq 2\}$

c) The graph of $f \times g$ can be found by multiplying corresponding y-coordinates.

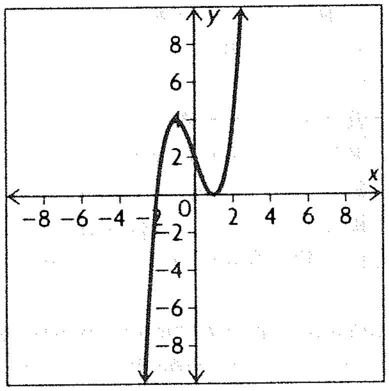
1(c):



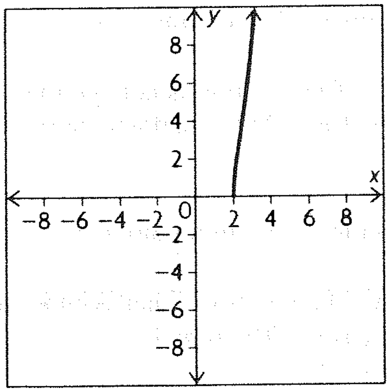
1(d):



1(e):



1(f):



d) 1(c): $\{x \in \mathbf{R}\}$
1(d): $\{x \in \mathbf{R}\}$
1(e): $\{x \in \mathbf{R}\}$
1(f): $\{x \in \mathbf{R} \mid x \geq 2\}$

3. $(f \times g)(x) = \left(\sqrt{1 + x}\right)\left(\sqrt{1 - x}\right)$
So, $x \geq -1$ and $x \leq 1$ since the radicand must be greater than or equal to 0.
So, the domain is $\{x \in \mathbf{R} \mid -1 \leq x \leq 1\}$

4. a) $(f \times g)(x) = (x - 7)(x + 7)$
$= x^2 - 49$

b) $(f \times g)(x) = \left(\sqrt{x + 10}\right)\left(\sqrt{x + 10}\right)$
$= x + 10$

c) $(f \times g)(x) = 7x^2(x - 9)$
$= 7x^3 - 63x^2$

d) $(f \times g)(x) = (-4x - 7)(4x + 7)$
$= -16x^2 - 28x - 28x - 49$
$= -16x^2 - 56x - 49$

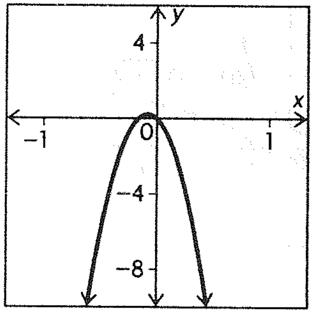
e) $(f \times g)(x) = 2 \sin x\left(\dfrac{1}{x - 1}\right)$
$= \dfrac{2 \sin x}{x - 1}$

f) $(f \times g)(x) = \log(x + 4)(2^x)$
$= 2^x \log(x + 4)$

5. 4(a): $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} \mid y \geq -49\}$
4(b): $D = \{x \in \mathbf{R} \mid x \geq -10\}$;
$R = \{y \in \mathbf{R} \mid y \geq 0\}$
4(c): $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R}\}$
4(d): $D = \{x \in \mathbf{R}\}$; $R = \{y \in \mathbf{R} \mid y \leq 0\}$
4(e): $D = \{x \in \mathbf{R} \mid x \neq -1\}$; $R = \{y \in \mathbf{R}\}$
4(f): $D = \{x \in \mathbf{R} \mid x > -4\}$; $R = \{y \in \mathbf{R} \mid y \geq 0\}$

6. 4(a): symmetry: The function is symmetric about the line $x = 0$.
increasing/decreasing: The function is increasing from 0 to ∞. The function is decreasing from $-\infty$ to 0.
zeros: $x = -7, 7$
maximum/minimum: The minimum is at $x = 0$.
period: N/A
4(b): symmetry: The function is not symmetric.
increasing/decreasing: The function is increasing from -10 to ∞.
zeros: $x = -10$
maximum/minimum: The minimum is at $x = -10$.
period: N/A
4(c): symmetry: The function is not symmetric.
increasing/decreasing: The function is increasing from $-\infty$ to 0 and from 6 to ∞.
zeros: $x = 0, 9$
maximum/minimum: The relative minimum is at $x = -6$. The relative maximum is at $x = 0$.
period: N/A
4(d): symmetry: The function is symmetric about the line $x = -1.75$.
increasing/decreasing: The function is increasing from $-\infty$ to -1.75 and is decreasing from -1.75 to ∞.
zero: $x = -1.75$
maximum/minimum: The maximum is at $x = -1.75$.
period: N/A
4(e): symmetry: The function is not symmetric.
increasing/decreasing: The function is increasing from $-\infty$ to 0 and from 6 to ∞.
zeros: $x = 0, 9$
maximum/minimum: The relative minima are at $x = -4.5336$ and 4.4286. The relative maximum is at $x = -1.1323$.
period: N/A
4(f): symmetry: The function is not symmetric.
increasing/decreasing: The function is increasing from -4 to ∞.
zeros: none
maximum/minimum: none
period: N/A

7. $f(x) = -4x$
$g(x) = 6x + 1$
$(f \times g)(x) = -4x(6x + 1)$
$= -24x^2 - 4x$



8. a) $\left\{x \in \mathbf{R} \,\middle|\, x \neq -2, 7, \frac{\pi}{2}, \text{or } \frac{3\pi}{2}\right\}$
b) $\{x \in \mathbf{R} \mid x > 8\}$
c) $\{x \in \mathbf{R} \mid x \geq -81 \text{ and } x \neq 0, \pi, \text{or } 2\pi\}$
d) $\{x \in \mathbf{R} \mid x \leq -1 \text{ or } x \geq 1 \text{ and } x \neq -3\}$

9. $(f \times p)(t)$ represents the total energy consumption in a particular country at time t

10. a) $R(x) = (20\,000 - 750x)(25 + x)$ or $R(x) = 500\,000 + 1250x - 750x^2$, where x is the increase in the admission fee in dollars
b) Yes, it's the product of the function $P(x) = 20\,000 - 750x$, which represents the number of daily visitors, and $F(x) = 25 + x$, which represents the admission fee.

c) Use a graphing calculator. The ticket price that will maximize revenue is $25.83.

11. $m(t) = ((0.9)^t)(650 + 300t)$

Use a graphing calculator to estimate.

The amount of contaminated material is at its greatest after about 7.3 s.

12. The statement is false. If $f(x)$ and $g(x)$ are odd functions, then their product will always be an even function. The reason is because when you multiply a function that has an odd degree with another function that has an odd degree, you add the exponents, and when you add two odd numbers together, you get an even number.

13. $h(x) = (mx^2 + 2x + 5)(2x^2 - nx - 2)$

$$-40 = (m(1)^2 + 2(1) + 5) \times (2(1)^2 - n(1) - 2)$$
$$-40 = (m + 2 + 5)(2 - n - 2)$$
$$-40 = (m + 7)(-n)$$
$$\frac{40}{m+7} = n \text{ EQUATION 1}$$
$$24 = (m(-1)^2 + 2(-1) + 5) \times (2(-1)^2 - n(-1) - 2)$$
$$24 = (m - 2 + 5)(2 + n - 2)$$
$$24 = (m + 3)(n)$$
$$\frac{24}{m+3} = n \text{ EQUATION 2}$$
$$\frac{40}{m+7} = \frac{24}{m+3}$$
$$40(m + 3) = 24(m + 7)$$
$$40m + 120 = 24m + 168$$
$$16m = 48$$
$$m = 3$$
$$\frac{24}{3+3} = n$$
$$\frac{24}{6} = n$$
$$4 = n$$

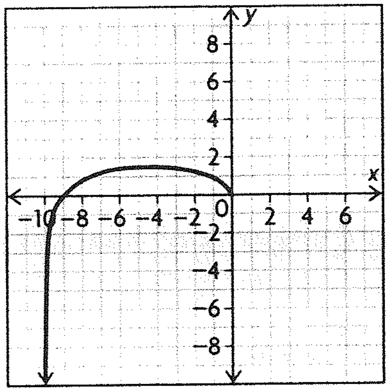
So, the equations are $f(x) = 3x^2 + 2x + 5$ and $g(x) = 2x^2 - 4x - 2$.

14. a) $(f \times g)(x) = \sqrt{-x}\log(x + 10)$

The domain is $\{x \in \mathbf{R} \mid -10 < x \leq 0\}$.

b) One strategy is to create a table of values for $f(x)$ and $g(x)$ and to multiply the corresponding y-values together. The resulting values could then be graphed. Another strategy is to graph $f(x)$ and $g(x)$ and to then create a graph for $(f \times g)(x)$ based on these two graphs. The first strategy is probably better than the second strategy, since the y-values for $f(x)$ and $g(x)$ will not be round numbers and will not be easily discernable from the graphs of $f(x)$ and $g(x)$.

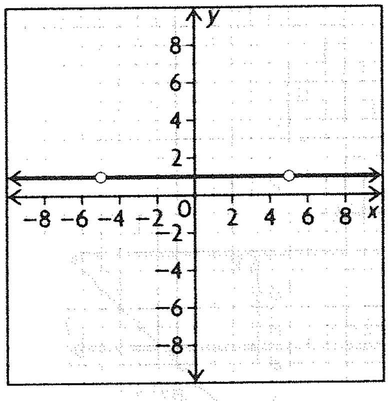
c)



15. a) $f(x) \times \frac{1}{f(x)} = (x^2 - 25) \times \frac{1}{x^2 - 25}$

$= 1$

b) The domain of the function is $\{x \in \mathbf{R} \mid x \neq -5 \text{ or } 5\}$.



c) The range will always be 1. If f is of odd degree, there will always be at least one value that makes the product undefined and which is excluded from the domain. If f is of even degree, there may be no values that are exlcuded from the domain.

16. a)
$$f(x) = 2^x$$
$$g(x) = x^2 + 1$$
$$(f \times g)(x) = 2^x(x^2 + 1)$$

b)
$$f(x) = x$$
$$g(x) = \sin(2\pi x)$$
$$(f \times g)(x) = x\sin(2\pi x)$$

17. a) $4x^2 - 91 = (2x + 9)(2x - 9)$
$$f(x) = (2x + 9)$$
$$g(x) = (2x - 9)$$

b) $8\sin^3 x + 27 = (2\sin x + 3) \times (4\sin^2 x - 6\sin x + 9)$
$$f(x) = (2\sin x + 3)$$
$$g(x) = (4\sin^2 x - 6\sin x + 9)$$

c) $4x^{\frac{5}{2}} - 3x^{\frac{3}{2}} + x^{\frac{1}{2}} = x^{\frac{1}{2}}(4x^5 - 3x^3 + 1)$
$$f(x) = x^{\frac{1}{2}}$$
$$g(x) = (4x^5 - 3x^3 + 1)$$

d) $\frac{6x - 5}{2x + 1} = \frac{1}{2x + 1} \times 6x - 5$

$f(x) = \frac{1}{2x + 1}$

$g(x) = 6x - 5$

9.4 Exploring Quotients of Functions, p. 542

1. a) $(f \div g)(x) = \frac{5}{x}, x \neq 0$

b) $(f \div g)(x) = \frac{4x}{2x - 1}, x \neq \frac{1}{2}$

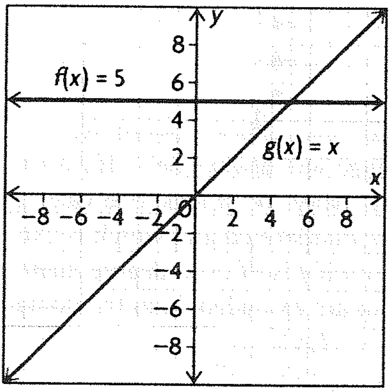
c) $(f \div g)(x) = \frac{4x}{x^2 + 4}$

d) $(f \div g)(x) = \frac{(x + 2)(\sqrt{x - 2})}{x - 2}, x > 2$

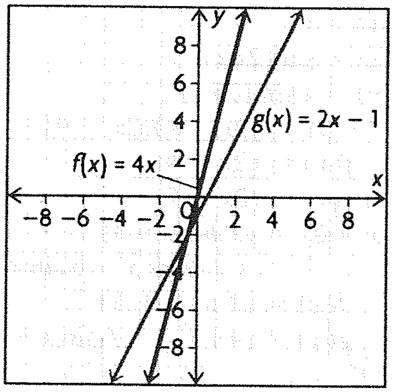
e) $(f \div g)(x) = \frac{8}{1 + \left(\frac{1}{2}\right)^x}$

f) $(f \div g)(x) = \frac{x^2}{\log(x)}, x > 0$

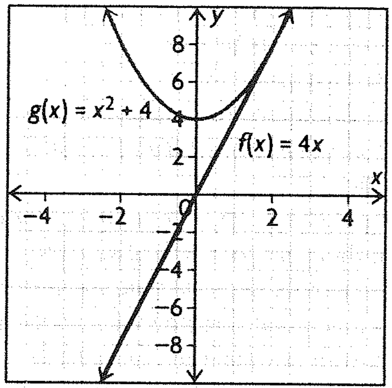
2. a) 1(a):



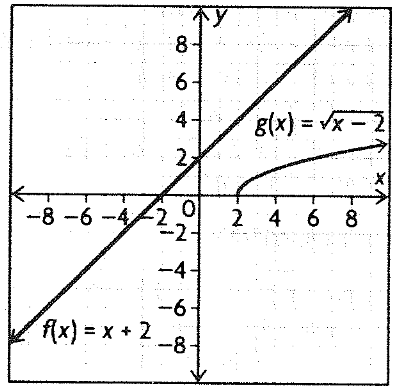
1(b):



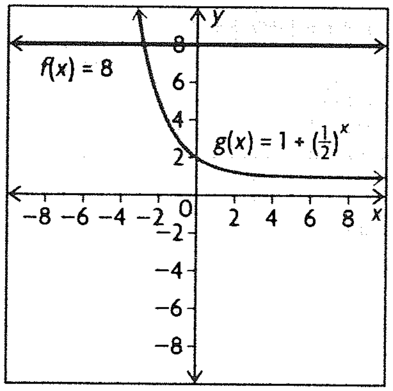
1(c):



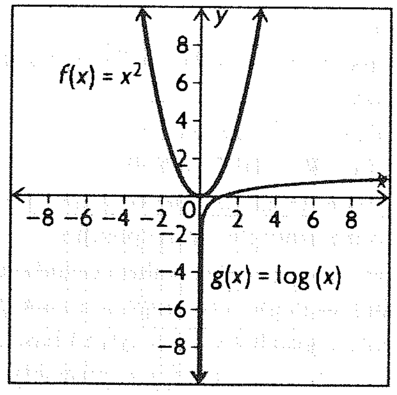
1(d):



1(e):

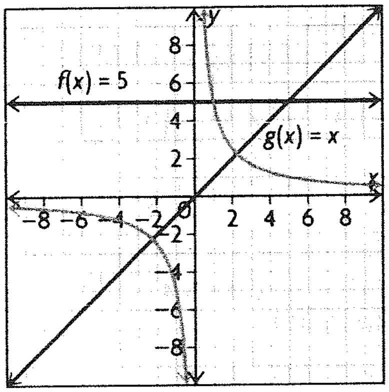


1(f):



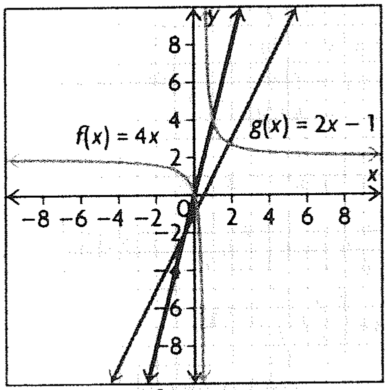
b) 1(a): domain of f: $\{x \in \mathbf{R}\}$; domain of g: $\{x \in \mathbf{R}\}$
1(b): domain of f: $\{x \in \mathbf{R}\}$; domain of g: $\{x \in \mathbf{R}\}$
1(c): domain of f: $\{x \in \mathbf{R}\}$; domain of g: $\{x \in \mathbf{R}\}$
1(d): domain of f: $\{x \in \mathbf{R}\}$; domain of g: $\{x \in \mathbf{R} \mid x \geq 2\}$
1(e): domain of f: $\{x \in \mathbf{R}\}$; domain of g: $\{x \in \mathbf{R}\}$
1(f): domain of f: $\{x \in \mathbf{R}\}$; domain of g: $\{x \in \mathbf{R} \mid x > 0\}$

c) 1(a):



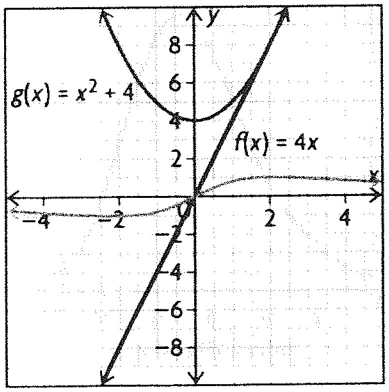
$y = \left(\frac{f}{g}\right)(x) = \frac{5}{x}$

1(b):



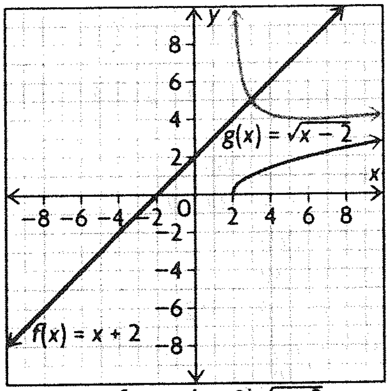
$y = \left(\frac{f}{g}\right)(x) = \frac{4x}{2x - 1}$

1(c):



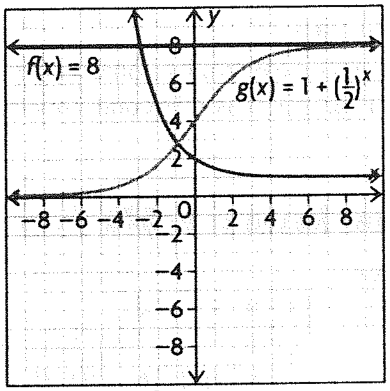
$y = \left(\frac{f}{g}\right)(x) = \frac{4x}{x^2 + 4}$

1(d):



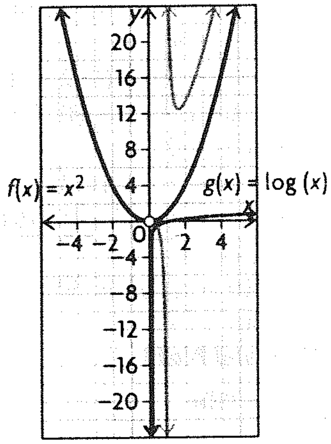
$y = \left(\frac{f}{g}\right)(x) = \frac{(x + 2)\sqrt{x - 2}}{x - 2}$

1(e):



$y = \left(\frac{f}{g}\right)(x) = \frac{8}{1 + \left(\frac{1}{2}\right)^x}$

1(f):



$y = \left(\frac{f}{g}\right)(x) = \frac{x^2}{\log(x)}$

d) 1(a): domain of $(f \div g)$: $\{x \in \mathbf{R} \mid x \neq 0\}$
1(b): domain of $(f \div g)$: $\left\{x \in \mathbf{R} \mid x \neq \frac{1}{2}\right\}$
1(c): domain of $(f \div g)$: $\{x \in \mathbf{R}\}$
1(d): domain of $(f \div g)$: $\{x \in \mathbf{R} \mid x > 2\}$
1(e): domain of $(f \div g)$: $\{x \in \mathbf{R}\}$
1(f): domain of $(f \div g)$: $\{x \in \mathbf{R} \mid x > 0\}$

3. a) $\frac{260}{1 + 24(0.9)^t} = \frac{260}{1 + 24(0.9)^{20}}$
$\doteq 66$ cm

The rate of change is $[66 - (260 \div 25)] \div 20$ or 2.798 cm/day.

b) The maximum height is 260, so half of 260 is 130 cm.

$$130 = \frac{260}{1 + 24(0.9)^t}$$
$$130 + 3120(0.9)^t = 260$$
$$3120(0.9)^t = 130$$
$$(0.9)^t = 130 \div 3120$$
$$t \log 0.9 = \log(130 \div 3120)$$
$$t \doteq 30 \text{ days}$$

c) $\left(\dfrac{260}{1 + 24(0.9)^{30.1}} - \dfrac{260}{1 + 24(0.9)^{30}}\right) \div 0.1$

$= 6.848$ cm/day

d) It slows down and eventually comes to zero. This is seen on the graph as it becomes horizontal at the top.

Mid-Chapter Review, p. 544

1. multiplication

2. a) $(f + g)(x) = \{(-9, -2 + 4), (-6, -3 + -6), (0, 2 + 12)\}$
$= \{(-9, 2), (-6, -9), (0, 14)\}$

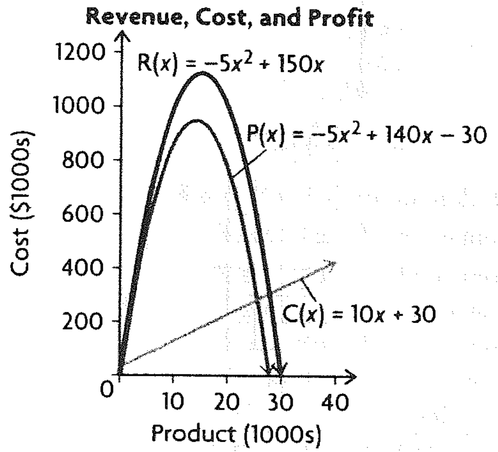
b) $(g + f)(x) = \{(-9, 4 + -2), (-6, -6 + -3), (0, 12 + 2)\}$
$= \{(-9, 2), (-6, -9), (0, 14)\}$

c) $(f - g)(x) = \{(-9, -2 - 4), (-6, -3 - -6), (0, 2 - 12)\}$
$= \{(-9, -6), (-6, 3), (0, -10)\}$

d) $(g - f)(x) = \{(-9, 4 - -2), (-6, -6 - -3), (0, 12 - 2)\}$
$= \{(-9, 6), (-6, -3), (0, 10)\}$

3. a) $P(x) = R(x) - C(x)$
$= -5x^2 + 150x - (10x + 30)$
$= -5x^2 + 140x - 30$

b)



c) $= -5x^2 + 140x - 30$
$= -5(7.5)^2 + 140(7.5) - 30$
$=$ \$738.75 thousand
$=$ \$738 750

4. a) $R(h) = 24.39h$

b) $N(h) = 24.97h$

c) $W(h) = 24.78h$

d) $S(h) = 24.39h + 0.58h + 0.39h$
$= 25.36h$

e) $25.36(8) + 1.5(25.36)(3) = \$317$

5. a) $(f \times g)(x) = \left(x + \dfrac{1}{2}\right)\left(x + \dfrac{1}{2}\right)$
$= x^2 + x + \dfrac{1}{4}$

$D = \{x \in \mathbf{R}\}$

b) $(f \times g)(x) = \sin(3x)(\sqrt{x - 10})$
$D = \{x \in \mathbf{R} | x \geq 10\}$

c) $(f \times g)(x) = 11x^3 \times \dfrac{2}{x + 5}$
$= \dfrac{22x^3}{x + 5}$

$D = \{x \in \mathbf{R} | x \neq -5\}$

d) $(f \times g)(x) = (90x - 1)(90x + 1)$
$= 8100x^2 - 1$

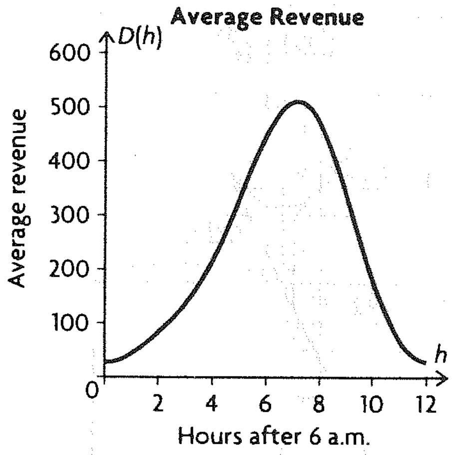
$D = \{x \in \mathbf{R}\}$

6. a) $C(h) \times D(h) = R(h)$

$= 90 \cos\left(\dfrac{\pi}{6}h\right) \sin\left(\dfrac{\pi}{6}h\right) - 102 \sin\left(\dfrac{\pi}{6}h\right)$

$- 210 \cos\left(\dfrac{\pi}{6}h\right) + 238$

b)



c) $$\begin{aligned} R(2) &= 90\cos\left(\frac{\pi}{6}(2)\right)\sin\left(\frac{\pi}{6}(2)\right) \\ &\quad - 102\sin\left(\frac{\pi}{6}(2)\right) - 210\cos\left(\frac{\pi}{6}(2)\right) + 238 \\ &\doteq \$470.30 \end{aligned}$$

7. a) $$\begin{aligned} (f \div g)(x) &= 240 \div 3x \\ &= \frac{80}{x} \end{aligned}$$

$D = \{x \in \mathbf{R} \mid x \neq 0\}$

b) $$\begin{aligned} (f \div g)(x) &= \frac{10x^2}{x^3 - 3x} \\ &= \frac{10x^2}{x(x^2 - 3)} \\ &= \frac{10x^2}{x^2 - 3} \end{aligned}$$

$D = \{x \in \mathbf{R} \mid x \neq \pm\sqrt{3}\}$

c) $(f \div g)(x) = \dfrac{x + 8}{\sqrt{x - 8}}$

$D = \{x \in \mathbf{R} \mid x > 8\}$

d) $(f \div g)(x) = \dfrac{7x^2}{\log x}$

$D = \{x \in \mathbf{R} \mid x > 0\}$

8. $\csc x$, $\sec x$, $\cot x$

9.5 Composition of Functions, pp. 552–554

1. a) $$\begin{aligned} g(0) &= 1 - 0^2 \\ &= 1 \\ f(g(0)) &= f(1) \\ &= 2(1) - 3 \\ &= 2 - 3 \\ &= -1 \end{aligned}$$

b) $$\begin{aligned} f(4) &= 2(4) - 3 \\ &= 8 - 3 \\ &= 5 \\ g(f(4)) &= g(5) \\ &= 1 - 5^2 \\ &= 1 - 25 \\ &= -24 \end{aligned}$$

c) $$\begin{aligned} g(-8) &= 1 - (-8)^2 \\ &= 1 - 64 \\ &= -63 \\ (f \circ g)(-8) &= f(-63) \\ &= 2(-63) - 3 \\ &= -126 - 3 \\ &= -129 \end{aligned}$$

d) $$\begin{aligned} g\left(\frac{1}{2}\right) &= 1 - \left(\frac{1}{2}\right)^2 \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \\ (g \circ g)\left(\frac{1}{2}\right) &= g\left(\frac{3}{4}\right) \\ &= 1 - \left(\frac{3}{4}\right)^2 \\ &= 1 - \frac{9}{16} \\ &= \frac{7}{16} \end{aligned}$$

e) $(f \circ f^{-1})(1) = 1$

f) $$\begin{aligned} g(2) &= 1 - 2^2 \\ &= 1 - 4 \\ &= -3 \\ (g \circ g)(2) &= g(-3) \\ &= 1 - (-3)^2 \\ &= 1 - 9 \\ &= -8 \end{aligned}$$

2. a) $$\begin{aligned} (g \circ f)(2) &= g(5) \\ &= 3 \end{aligned}$$

b) $$\begin{aligned} (f \circ f)(1) &= f(2) \\ &= 5 \end{aligned}$$

c) $$\begin{aligned} (f \circ g)(5) &= f(3) \\ &= 10 \end{aligned}$$

d) $(f \circ g)(0)$ is undefined because there is no $g(0)$.

e) $(f \circ f^{-1})(2) = 2$

f) $$\begin{aligned} (g^{-1} \circ f)(1) &= g^{-1}(2) \\ &= 4 \end{aligned}$$

3. a) $$\begin{aligned} f(g(2)) &= f(5) \\ &= 5 \end{aligned}$$

b) $$\begin{aligned} g(f(4)) &= g(3) \\ &= 5 \end{aligned}$$

c) $$\begin{aligned} (g \circ g)(-2) &= g(1) \\ &= 4 \end{aligned}$$

d) $(f \circ f)(2) = f(-1)$

$f(-1)$ does not exist, so $(f \circ f)(2)$ is undefined.

4. a) $$\begin{aligned} d(5) &= 80(5) \\ &= 400 \\ C(d(5)) &= C(400) \\ &= 0.09(400) \\ &= 36 \end{aligned}$$

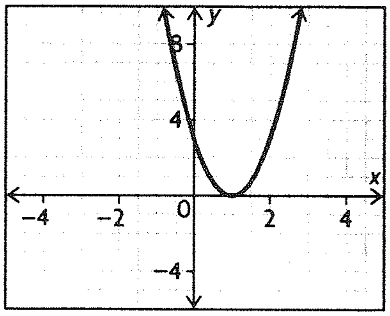
It costs \$36 to travel for 5 hours.

b) $C(d(t))$ represents the relationship between the time driven and the cost of gasoline.

5. a) $f(g(x)) = f(x - 1)$

$$\begin{aligned} &= 3(x - 1)^2 \\ &= 3(x^2 - 2x + 1) \\ &= 3x^2 - 6x + 3 \end{aligned}$$

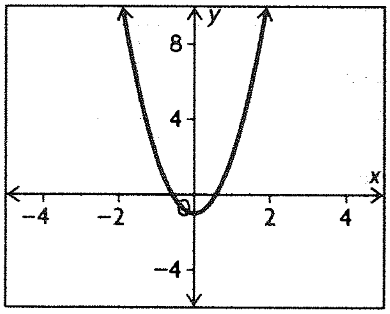
The domain is $\{x \in \mathbf{R}\}$.



$g(f(x)) = g(3x^2)$

$$= 3x^2 - 1$$

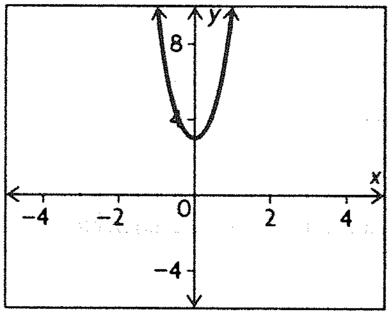
The domain is $\{x \in \mathbf{R}\}$.



b) $f(g(x)) = f(x^2 + 1)$

$$\begin{aligned} &= 2(x^2 + 1)^2 + (x^2 + 1) \\ &= 2(x^4 + 2x^2 + 1) + x^2 + 1 \\ &= 2x^4 + 4x^2 + 2 + x^2 + 1 \\ &= 2x^4 + 5x^2 + 3 \end{aligned}$$

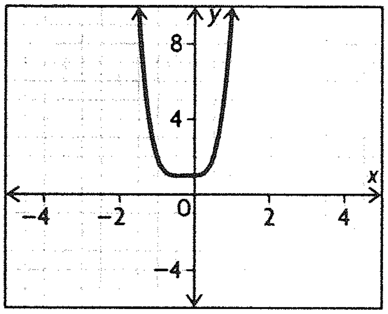
The domain is $\{x \in \mathbf{R}\}$.



$g(f(x)) = g(2x^2 + x)$

$$\begin{aligned} &= (2x^2 + x)^2 + 1 \\ &= 4x^4 + 4x^3 + x^2 + 1 \end{aligned}$$

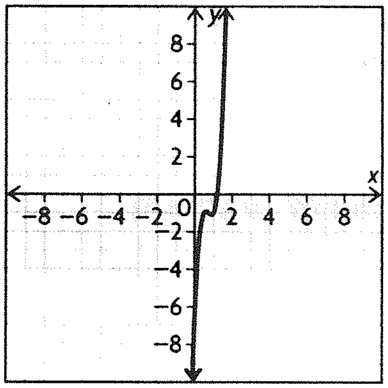
The domain is $\{x \in \mathbf{R}\}$.



c) $f(g(x)) = f(2x - 1)$

$$\begin{aligned} &= 2(2x - 1)^3 - 3(2x - 1)^2 \\ &\quad + (2x - 1) - 1 \\ &= 2(8x^3 - 12x^2 + 6x - 1) \\ &\quad - 3(4x^2 - 4x + 1) + 2x - 1 - 1 \\ &= 16x^3 - 24x^2 + 12x - 2 \\ &\quad - 12x^2 + 12x - 3 + 2x - 2 \\ &= 16x^3 - 36x^2 + 26x - 7 \end{aligned}$$

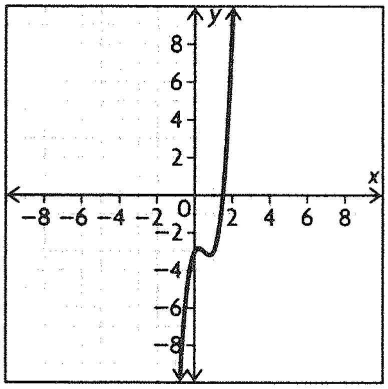
The domain is $\{x \in \mathbf{R}\}$.



$g(f(x)) = g(2x^3 - 3x^2 + x - 1)$

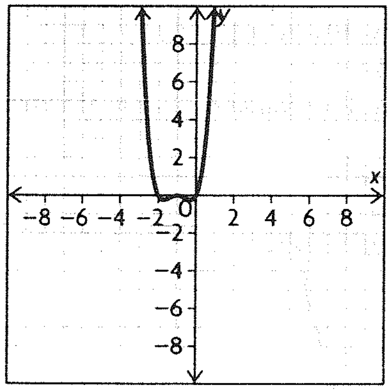
$$\begin{aligned} &= 2(2x^3 - 3x^2 + x - 1) - 1 \\ &= 4x^3 - 6x^2 + 2x - 2 - 1 \\ &= 4x^3 - 6x^2 + 2x - 3 \end{aligned}$$

The domain is $\{x \in \mathbf{R}\}$.



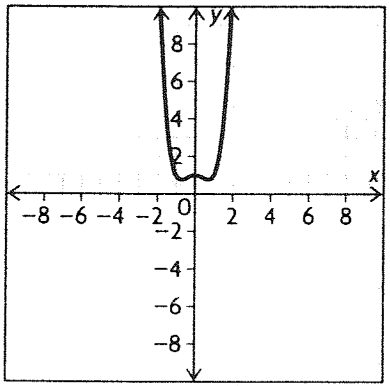
d) $$\begin{aligned} f(g(x)) &= f(x+1) \\ &= (x+1)^4 - (x+1)^2 \\ &= (x+1)^2((x+1)^2 - 1) \\ &= (x^2+2x+1)(x^2+2x+1-1) \\ &= (x^2+2x+1)(x^2+2x) \\ &= x^4+2x^3+2x^3+4x^2+x^2+2x \\ &= x^4+4x^3+5x^2+2x \end{aligned}$$

The domain is $\{x \in \mathbf{R}\}$.



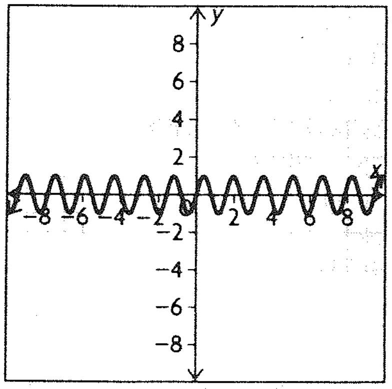
$$\begin{aligned} g(f(x)) &= g(x^4 - x^2) \\ &= x^4 - x^2 + 1 \end{aligned}$$

The domain is $\{x \in \mathbf{R}\}$.



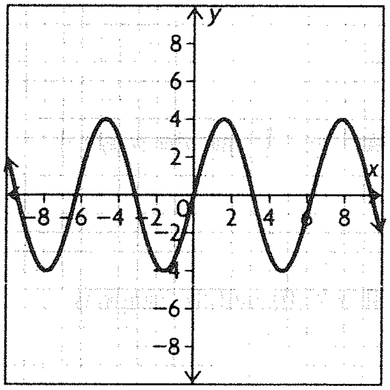
e) $$\begin{aligned} f(g(x)) &= f(4x) \\ &= \sin 4x \end{aligned}$$

The domain is $\{x \in \mathbf{R}\}$.



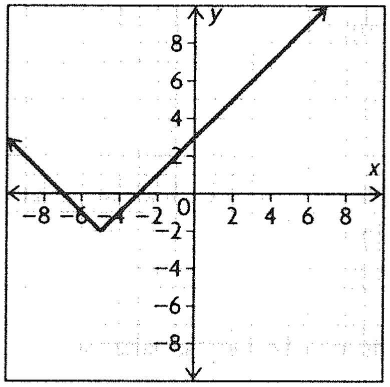
$$\begin{aligned} g(f(x)) &= g(\sin x) \\ &= 4 \sin x \end{aligned}$$

The domain is $\{x \in \mathbf{R}\}$.



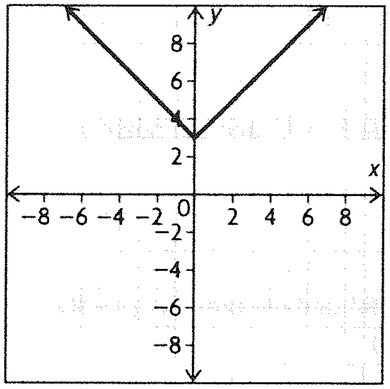
f) $$\begin{aligned} f(g(x)) &= f(x+5) \\ &= |x+5| - 2 \end{aligned}$$

The domain is $\{x \in \mathbf{R}\}$.



$$\begin{aligned} g(f(x)) &= g(|x| - 2) \\ &= |x| - 2 + 5 \\ &= |x| + 3 \end{aligned}$$

The domain is $\{x \in \mathbf{R}\}$.



6. a) $$\begin{aligned} f \circ g &= f(\sqrt{x-4}) \\ &= 3\sqrt{x-4} \end{aligned}$$

The domain is $\{x \in \mathbf{R} \mid x \geq 4\}$ and the range is $\{y \in \mathbf{R} \mid y \geq 0\}$.

$$\begin{aligned} g \circ f &= g(3x) \\ &= \sqrt{3x - 4} \end{aligned}$$

The domain is $\{x \in \mathbf{R} | x \geq \frac{4}{3}\}$ and the range is $\{y \in \mathbf{R} | y \geq 0\}$.

b) $$\begin{aligned} f \circ g &= f(3x + 1) \\ &= \sqrt{3x + 1} \end{aligned}$$

The domain is $\{x \in \mathbf{R} | x \geq -\frac{1}{3}\}$ and the range is $\{y \in \mathbf{R} | y \geq 0\}$.

$$\begin{aligned} g \circ f &= g(\sqrt{x}) \\ &= 3\sqrt{x} + 1 \end{aligned}$$

The domain is $\{x \in \mathbf{R} | x \geq 0\}$ and the range is $\{y \in \mathbf{R} | y \geq 1\}$.

c) $$\begin{aligned} f \circ g &= f(x^2) \\ &= \sqrt{4 - (x^2)^2} \\ &= \sqrt{4 - x^4} \end{aligned}$$

The domain is $\{x \in \mathbf{R} | -\sqrt{2} \leq x \leq \sqrt{2}\}$ and the range is $\{y \in \mathbf{R} | y \geq 0\}$.

$$\begin{aligned} g \circ f &= g(\sqrt{4 - x^2}) \\ &= (\sqrt{4 - x^2})^2 \\ &= 4 - x^2 \end{aligned}$$

The domain is $\{x \in \mathbf{R} | -2 \leq x \leq 2\}$ and the range is $\{y \in \mathbf{R} | 0 < y < 2\}$.

d) $$\begin{aligned} f \circ g &= f(\sqrt{x - 1}) \\ &= 2\sqrt{x - 1} \end{aligned}$$

The domain is $\{x \in \mathbf{R} | x \geq 1\}$ and the range is $\{y \in \mathbf{R} | y \geq 1\}$.

$$\begin{aligned} g \circ f &= g(2^x) \\ &= \sqrt{2^x - 1} \end{aligned}$$

The domain is $\{x \in \mathbf{R} | x \geq 0\}$ and the range is $\{y \in \mathbf{R} | y \geq 0\}$.

e) $$\begin{aligned} f \circ g &= f(\log x) \\ &= 10^{\log x} \\ &= x \end{aligned}$$

The domain is $\{x \in \mathbf{R} | x > 0\}$ and the range is $\{y \in \mathbf{R}\}$.

$$\begin{aligned} g \circ f &= g(10^x) \\ &= \log 10^x \\ &= x \end{aligned}$$

The domain is $\{x \in \mathbf{R}\}$ and the range is $\{y \in \mathbf{R}\}$.

f) $$\begin{aligned} f \circ g &= f(5^{2x} + 1) \\ &= \sin(5^{2x} + 1) \end{aligned}$$

The domain is $\{x \in \mathbf{R}\}$ and the range is $\{y \in \mathbf{R} | -1 \leq y \leq 1\}$.

$$\begin{aligned} g \circ f &= g(\sin x) \\ &= 5^{2 \sin x} + 1 \end{aligned}$$

The domain is $\{x \in \mathbf{R}\}$ and the range is $\{y \in \mathbf{R} | \frac{26}{25} \leq y \leq 26\}$.

7. a) Answers may vary. For example, $f(x) = \sqrt{x}$ and $g(x) = x^2 + 6$

b) Answers may vary. For example, $f(x) = x^6$ and $g(x) = 5x - 8$

c) Answers may vary. For example, $f(x) = 2^x$ and $g(x) = 6x + 7$

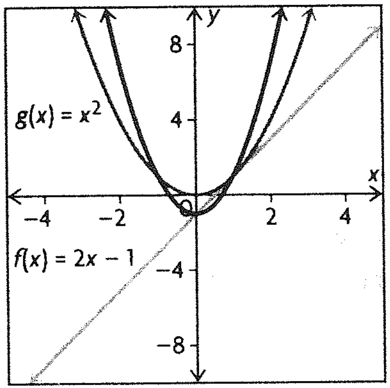
d) Answers may vary. For example, $f(x) = \frac{1}{x}$ and $g(x) = x^3 - 7x + 2$

e) Answers may vary. For example, $f(x) = \sin^2 x$ and $g(x) = 10x + 5$

f) Answers may vary. For example, $f(x) = \sqrt[3]{x}$ and $g(x) = (x + 4)^2$

8. a) $$\begin{aligned} (f \circ g)(x) &= f(x^2) \\ &= 2x^2 - 1 \end{aligned}$$

b)



$(f \circ g)(x) = 2x^2 - 1$

c) It is compressed by a factor of 2 and translated down 1 unit.

9. a) $$\begin{aligned} f(g(x)) &= f(3x + 2) \\ &= 2(3x + 2) - 1 \\ &= 6x + 4 - 1 \\ &= 6x + 3 \end{aligned}$$

The slope of $g(x)$ has been multiplied by 2, and the y-intercept of $g(x)$ has been vertically translated 1 unit up.

b) $$\begin{aligned} g(f(x)) &= g(2x - 1) \\ &= 3(2x - 1) + 2 \\ &= 6x - 3 + 2 \\ &= 6x - 1 \end{aligned}$$

The slope of $f(x)$ has been multiplied by 3.

10. $$\begin{aligned} D(p) &= 0.80(975 + 39.95p) \\ &= 780 + 31.96p \end{aligned}$$

11. $$\begin{aligned} f(g(x)) &= f(0.75x) \\ &= 0.08(0.75x) \\ &= 0.06x \end{aligned}$$

12. a) $d(s) = \sqrt{16 + s^2}$; $s(t) = 560t$

b) $d(s(t)) = \sqrt{16 + 313\,600t^2}$, where t is the time in hours and $d(s(t))$ is the distance in kilometres

13. $c(v(t)) = \left(\frac{40 + 3t + t^2}{500} - 0.1\right)^2 + 0.15;$
The car is running most economically 2 hours into the trip.

14. Graph A(k); $f(x)$ is vertically compressed by a factor of 0.5 and reflected in the x-axis.
Graph B(b); $f(x)$ is translated 3 units to the right.
Graph C(d); $f(x)$ is horizontally compressed by a factor of $\frac{1}{2}$.
Graph D(1); $f(x)$ is translated 4 units down.
Graph E(g); $f(x)$ is translated 3 units up.
Graph F(c); $f(x)$ is reflected in the y-axis.

15. Sum: $y = f + g$
$f(x) = \frac{4}{x - 3}; g(x) = 1$
Product: $y = f \times g$
$f(x) = x - 3;$
$g(x) = \frac{x + 1}{(x - 3)^2}$
Quotient: $y = f \div g$
$f(x) = 1 + x; g(x) = x - 3$
Composition: $y = f \circ g$
$f(x) = \frac{4}{x} + 1; g(x) = x - 3$

16. a) $f(k) = f(x(t(k)))$
$= f(x(3k - 2))$
$= f(3(3k - 2) + 2)$
$= f(9k - 6 + 2)$
$= f(9k - 4)$
$= 3(9k - 4) - 2$
$= 27k - 12 - 2$
$= 27k - 14$

b) $f(k) = f(x(t(k)))$
$= f(x(3k - 5))$
$= f\left(\sqrt{3(3k - 5) - 1}\right)$
$= f\left(\sqrt{9k - 15 - 1}\right)$
$= f\left(\sqrt{9k - 16}\right)$
$= 2\sqrt{9k - 16} - 5$

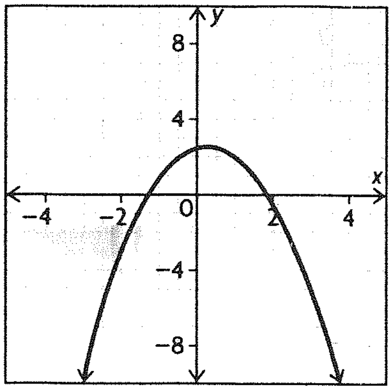
9.6 Techniques for Solving Equations and Inequalities, pp. 560–562

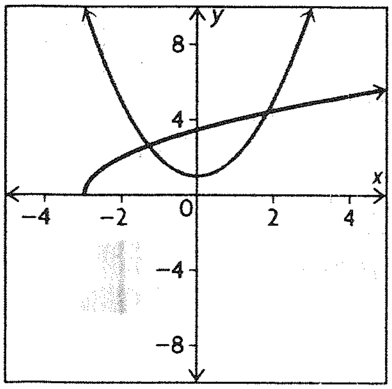
1. Use the graph to find the solutions.
a) i) $x = \frac{1}{2}, 2, \text{ or } \frac{7}{2}$
ii) $x = -1 \text{ or } 2$
b) i) $\frac{1}{2} < x < 2 \text{ or } x > \frac{7}{2}$
ii) $-1 < x < 2$
c) i) $x \leq \frac{1}{2}; 2 \leq x \leq \frac{7}{2}$
ii) $x \leq -1 \text{ or } x \geq 2$
d) i) $\frac{1}{2} \leq x \leq 2 \text{ or } x \geq \frac{7}{2}$
ii) $-1 \leq x \leq 2$

2. a) $3 = 2^{2x}$
Try $x = 1$: $3 = 2^2$
$3 = 4$ Too high
Try $x = 0.5$: $3 = 2^1$
$3 = 2$ Too low
Try 0.6: $3 = 2^{1.2}$
$3 = 2.3$ Too low
Try 0.8: $3 = 2^{1.6}$
$3 = 3.03$
So, $x \doteq 0.8$
b) $0 = \sin(0.25x^2)$
Try $x = 0$: $0 = \sin 0$
$0 = 0$ Correct
Try $x = 2$: $0 = \sin(0.25(4))$
$0 = 0.84$ Too high
Try $x = 3$: $0 = \sin(0.25(9))$
$0 = 0.78$ Too high
Try $x = 3.5$: $0 = \sin(0.25(12.25))$
$0 = 0.08$ Close
Try $x = 3.6$: $0 = \sin(0.25(12.96))$
$0 = -0.1$
So, $x = 0$ and 3.5
c) $3x = 0.5x^3$
Try $x = -2$: $3(-2) = 0.5(-2)^3$
$-6 = -4$
Try $x = -3$: $3(-3) = 0.5(-3)^3$
$-9 = -13.5$
Try $x = -2.5$: $3(-2.5) = 0.5(-2.5)^3$
$-7.5 = -7.8$
Try $x = -2.4$: $3(-2.4) = 0.5(-2.4)^3$
$-7.2 = -6.9$
So, $x \doteq -2.4$
d) $\cos x = x$
Try $x = 0$: $\cos 0 = 0$
$1 = 0$
Try 0.5: $\cos 0.5 = 0.5$
$0.8 = 0.5$
Try 0.6: $\cos 0.6 = 0.6$
$0.8 = 0.6$

Try 0.7: $\cos 0.7 = 0.7$
$0.76 = 0.7$
So, $x \doteq 0.7$

3. Graph and use the graph to find the solutions.
$x = -1.3$ or 1.8





4. Use the graph to find the solutions.
$f(x) < g(x)$: $1.3 < x < 1.6$
$f(x) = g(x)$: $x = 0$ or 1.3
$f(x) > g(x)$: $0 < x < 1.3$ or $1.6 < x < 3$

5. a) $5 \sec x = -x^2$
Try $x = 2$: $5 \sec 2 = -2^2$
$-12.0 = -4$
Try $x = 2.5$: $5 \sec 2.5 = -2.5^2$
$-6.24 = -6.25$
So, $x \doteq 2.5$

b) $\sin^3 x = \sqrt{x} - 1$
Try $x = 2$: $\sin^3 2 = \sqrt{2} - 1$
$0.75 = 0.41$
Try $x = 2.2$: $\sin^3 2.2 = \sqrt{2.2} - 1$
$0.53 = 0.48$
So, $x \doteq 2.2$

c) $5^x = x^5$
Try $x = 1$: $5^1 = 1^5$
$5 = 1$
Try $x = 2$:
$5^2 = 2^5$
$25 = 32$
Try $x = 1.9$: $5^{1.9} = 1.9^5$
$21.28 = 24.76$
Try $x = 1.8$: $5^{1.8} = 1.8^5$
$18.12 = 18.90$
So, $x \doteq 1.8$

d) $\cos x = \dfrac{1}{x}$

Try $x = -2$: $\cos -2 = \dfrac{1}{-2}$

$-0.42 = -0.5$

Try $x = -2.1$: $\cos -2.1 = \dfrac{1}{-2.1}$

$-0.50 = -0.58$
So, $x \doteq -2.1$

e) $\log x = (x - 10)^2 + 1$
Try $x = 9$: $\log 9 = (9 - 10)^2 + 1$
$0.95 = 0$
Try $x = 10$: $\log 10 = (10 - 10)^2 + 1$
$1 = 1$
So, $x = 10$.

f) $\sin(2\pi x) = -4x^2 + 16x - 12$
Try $x = 0$: $\sin 0 = -4(0)^2 + 16(0) - 12$
$0 = -12$
Try $x = 1$: $\sin 2\pi = -4(1)^2 + 16(1) - 12$
$0 = 0$
Try $x = 3$: $\sin 6\pi = -4(3)^2 + 16(3) - 12$
$0 = 0$
So, $x = 1$ or 3

6. Use a graphing calculator to estimate the solutions.
a) $x = -1.81$ or 0.48
b) $x = -1.38$ or 1.6
c) $x = -1.38$ or 1.30
d) $x = -0.8, 0,$ or 0.8
e) $x = 0.21$ or 0.74
f) $x = 0, 0.18, 0.38,$ or 1

7. Since the graph crosses the x-axis at $x = 0.7$, the x-coordinate of the solution is 0.7. Use $x = 0.7$ to find the y-coordinate.
$y = -3x^2$
$y = -3(0.7)^2$
$y = -1.47$
So, the coordinates are $(0.7, -1.5)$.

8. $2.3(0.96)^t = 1.95(0.97)^t$
Use a graphing calculator to estimate the solution.
$t \doteq 15$
So, they will be about the same in 1997 + 15 or 2012.

9. Use a graphing calculator to estimate the solutions.

a) $x \in (-0.57, 1)$
b) $x \in [0, 0.58]$
c) $x \in (-\infty, 0)$
d) $x \in (0.17, 0.83)$
e) $x \in (0.35, 1.51)$
f) $x \in (0.1, 0.5)$

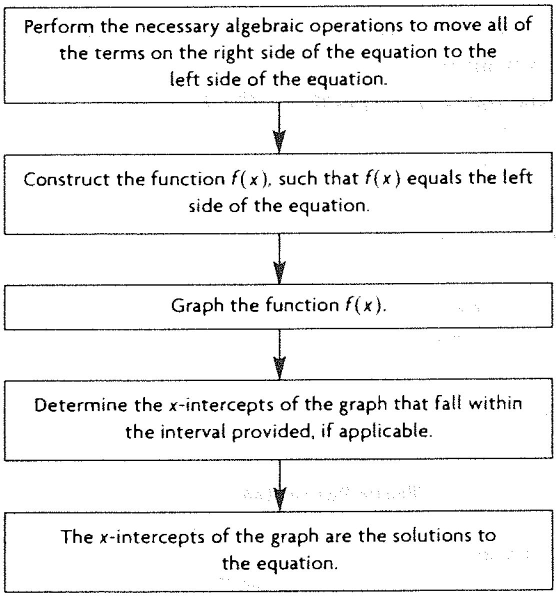
10. Answers may vary. For example, $f(x) = x^3 + 5x^2 + 2x - 8$ and $g(x) = 0$

11. Answers may vary. For example, $f(x) = -x^2 + 25$ and $g(x) = -x + 5$

12.

$$\begin{aligned} a\cos x &= bx^3 + 6 \\ a\cos(-1.2) &= b(-1.2)^3 + 6 \\ a\cos(-0.7) &= b(-0.7)^3 + 6 \\ 0.36a &= -1.728b + 6 \\ 0.76a &= -0.343b + 6 \\ \frac{-1.728b + 6}{0.36} &= \frac{-0.343b + 6}{0.76} \\ -1.31328b + 4.56 &= -0.12348b + 2.16 \\ -1.1898b &= -2.4 \\ b &\doteq 2 \\ \text{So, } a &= \frac{-1.728(2) + 6}{0.36} \\ &\doteq 7 \end{aligned}$$

13. Answers may vary. For example:


Perform the necessary algebraic operations to move all of the terms on the right side of the equation to the left side of the equation.

Construct the function $f(x)$, such that $f(x)$ equals the left side of the equation.

Graph the function $f(x)$.

Determine the x-intercepts of the graph that fall within the interval provided, if applicable.

The x-intercepts of the graph are the solutions to the equation.

14. Use a graphing calculator to determine the solutions.

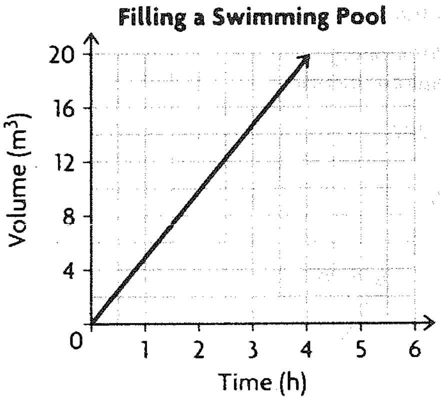
$x = 0 \pm 2n$, $x = -0.67 \pm 2n$ or $x = 0.62 \pm 2n$, where $n \in \mathbf{I}$

15. Use a graphing calculator to determine the solutions.

$x \in (2n, 2n + 1)$, where $n \in \mathbf{I}$

9.7 Modelling with Functions, pp. 569–574

1. a) The equation of the graph is $6.25\pi\left(\frac{x}{4}\right)$



b) $y = 6.25\pi\left(\frac{x}{4}\right)$

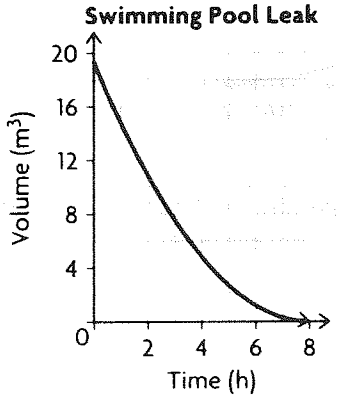
c) $8 = 6.25\pi\left(\frac{x}{4}\right)$

$8 = 1.5625\pi x$

$1.6 = x$

So, it will take about 1.6 hours.

2. a) $y = \frac{6.25\pi}{64}(x - 8)^2$



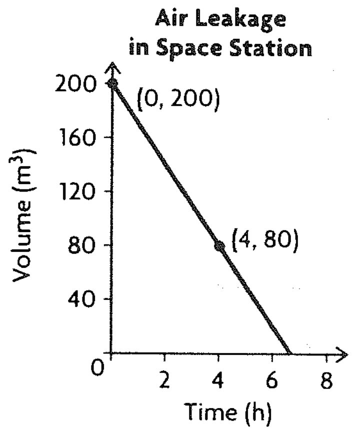
b) $V(t) = \frac{6.25\pi}{64}(t - 8)^2$

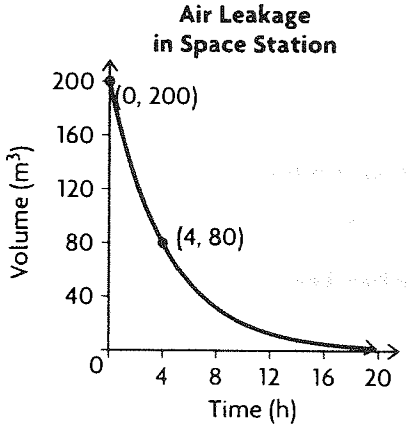
c)
$$\begin{aligned} V(2) &= \frac{6.25\pi}{64}(2 - 8)^2 \\ &= \frac{6.25\pi}{64}(-6)^2 \\ &= \frac{6.25\pi}{64}(36) \\ &\doteq 11 \text{ m}^3 \end{aligned}$$

d) The initial volume is 19.6 m^3.
So, the rate of change is $(11 - 19.6) \div 2$ or -4.3 m^3/hr

e) As time elapses, the pool is losing less water in the same amount of time.

3. a) Answers may vary. For example:


Air Leakage in Space Station


Air Leakage in Space Station

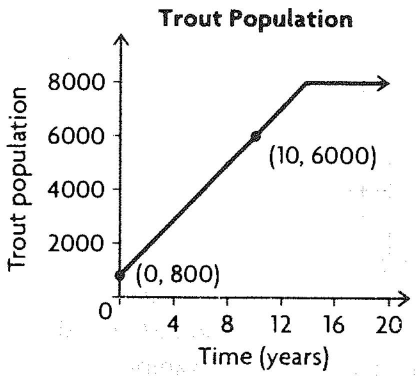
b) From the linear graph, the y-intercept is $(0, 200)$ and the slope is -30, so the equation is

$$\begin{aligned} V(t) &= -30t + 200 \\ V(t) &= -30t + 200 \\ 0 &= -30t + 200 \\ -200 &= -30t \\ 6.7 &\doteq t \end{aligned}$$

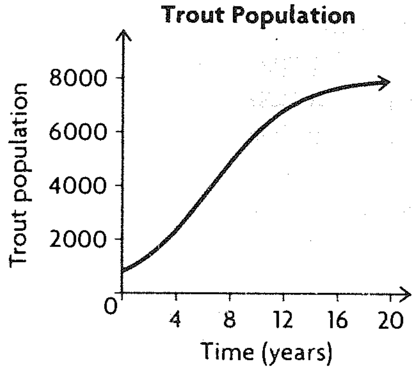
c) Using a graphing calculator, the equation that fits the model is $V(t) = 200(0.795)^t$

$$\begin{aligned} V(t) &= 200(0.795)^t \\ 20 &= 200(0.795)^t \\ 0.1 &= 0.795^t \\ \log 0.1 &= t \log 0.795 \\ 10 &\doteq t \end{aligned}$$

4. a)


Trout Population

b) $P(t) = \dfrac{8000}{1 + 9(0.719)^t}$


Trout Population

c) $P(4) = \frac{8000}{1 + 9(0.719)^4} \doteq 2349$ trout four years after restocking

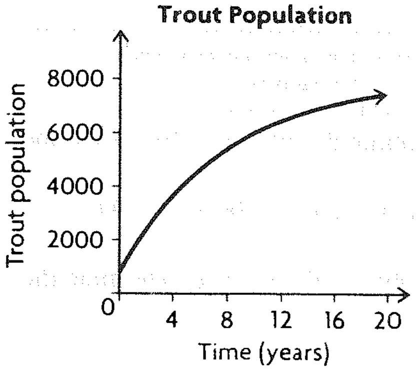
d) The rate of change is $(2349 - 800) \div 4$ or 387.25 trout per year.

5. a) the carrying capacity of the lake; 8000

b) Use $(0, 800)$ and $(10, 6000)$.

$$\begin{aligned} 800 &= 8000 - a(b)^0 \\ 800 &= 8000 - a \\ -7200 &= -a \\ 7200 &= a \\ 6000 &= 8000 - 7200(b)^{10} \\ -2000 &= -7200(b)^{10} \\ 0.278 &= b^{10} \\ 0.278^{\frac{1}{10}} &= b \\ 0.88 &\doteq b \end{aligned}$$

c)


Trout Population

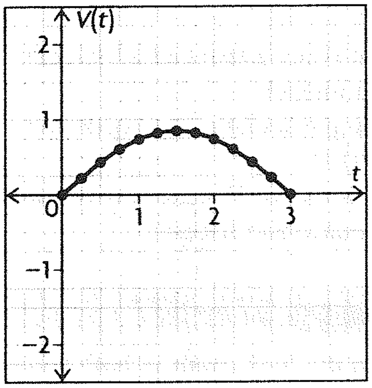
d) $P(4) = 8000 - 7200(0.88)^4$
$= 8000 - 4317.81$
$\doteq 3682$

e) The rate of change is $(3682 - 800) \div 4$ or 720.5 trout per year.

f) In the model in the previous problem, the carrying capacity of the lake is divided by a number that gets smaller and smaller, while in this model, a number that gets smaller and smaller is subtracted from the carrying capacity of the lake.

6. Answers may vary. For example, the first model more accurately calculates the current price of gasoline because prices are rising quickly.

7. a) $V(t) = 0.85 \cos\left(\frac{\pi}{3}(t - 1.5)\right)$



b) The scatter plot and the graph are very close to being the same, but they are not exactly the same.

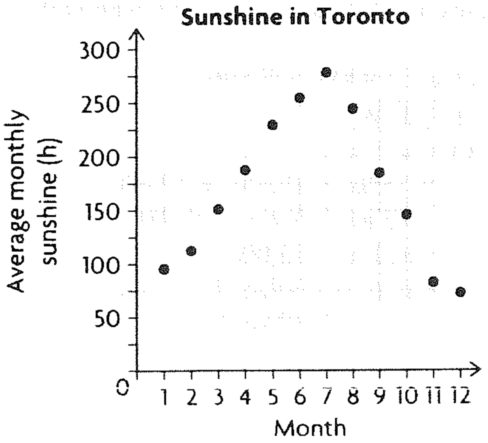
c) $V(6) = 0.85 \cos\left(\frac{\pi}{3}(6 - 1.5)\right)$
$= 0.85 \cos\left(\frac{\pi}{3}(4.5)\right)$
$= 0$ L/s

d) From the graph, the rate of change appears to be at its smallest at $t = 1.5$ s.

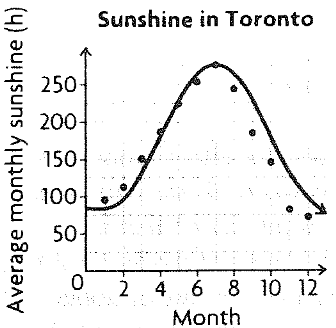
e) It is the maximum of the function.

f) From the graph, the rate of change appears to be greatest at $t = 0$ s.

8. a)



b)



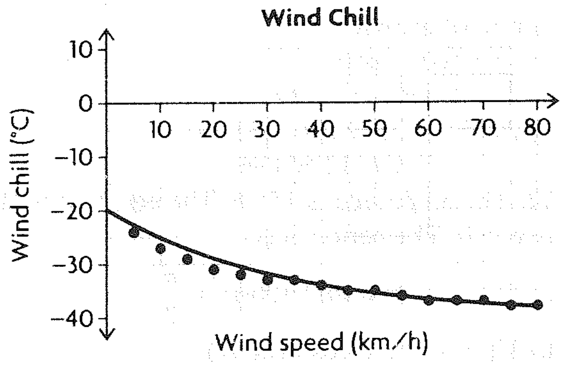
c) The equation is

$S(t) + -97 \cos\left(\frac{\pi}{6}(t - 1)\right) + 181.$

d) From the model, the maximum will be at $t = 7$ and the minimum will be at $t = 1$.

e) It doesn't fit it perfectly, because, actually, the minimum is not at $t = 1$, but at $t = 12$.

9. a)



b) Answers may vary. For example, $C(s) = -38 + 14(0.97)^s$

c) $C(0) = -38 + 14(0.97)^0$
$= -38 + 14$
$= -24°\text{C}$
$C(100) = -38 + 14(0.97)^{100}$
$= -38 + 0.666$
$\doteq -37.3°\text{C}$
$C(200) = -38 + 14(0.97)^{200}$
$= -38 + 0.032$
$\doteq -38°\text{C}$

These answers don't appear to be very reasonable, because the wind chill for a wind speed of 0 km/h should be −20°C, while the wind chills for wind speeds of 100 km/h and 200 km/h should be less than −38°C. The model only appears to be somewhat accurate for wind speeds of 10 to 70 km/hr.

10. a) Answers will vary; for example, one polynomial model is $P(t) = 1.4t^2 + 3230$, while an exponential model is $P(t) = 3230(1.016)^t$. While neither model is perfect, it appears that the polynomial model fits the data better.

b) $P(155) = 1.4(155)^2 + 3230$
$\doteq 36\ 865$
$P(155) = 3230(1.016)^{155}$
$\doteq 37\ 820$

c) A case could be made for either model. The polynomial model appears to fit the data better, but population growth is usually exponential.

d) According to the polynomial model, in 2000, the population was increasing at a rate of about 389 000 per year, while according to the exponential model, in 2000, the population was increasing at a rate of about 465 000 per year.

11. a) $P(t) = 3339.18(1.132\ 25)^t$

b) $75 = 3339.18(1.132\ 25)^t$

Use a graphing calculator to solve.

$t \doteq -31$

So, they were introduced around the year 1924.

c) rate of growth
$= (93\ 100 - 650) \div (35 - 0)$
$\doteq 2641$ rabbits per year.

d) $P(65) = 3339.18(1.132\ 25)^{65}$
$\doteq 10\ 712\ 509.96$

12. The amplitude is 155.6. The equation of the axis is $y = 0$. The period is $\frac{1}{60}$ s.

a) $V(t) = 155.6 \sin\left(120\pi t + \dfrac{\pi}{2}\right)$

b) $V(t) = 155.6 \cos(120\pi t)$

c) The cosine function was easier to determine. The cosine function is at its maximum when the argument is 0, so no horizontal translation was necessary.

13. a) Answers will vary; for example, a linear model is $P(t) = -9t + 400$; a quadratic model is $P(t) = \frac{23}{90}(t - 30)^2 + 170$; an exponential model is $P(t) = 400(0.972)^t$.

The exponential model fits the data far better than the other two models.

b) $P(60) = -9(60) + 400$
$= -540 + 400$
$= -140$ kPa

$$\begin{aligned} P(60) &= \frac{23}{90}(60 - 30)^2 + 170 \\ &= \frac{23}{90}(30)^2 + 170 \\ &= \frac{23}{90}(900) + 170 \\ &= 230 + 170 \\ &= 400 \text{ kPa} \end{aligned}$$

$P(60) = 400(0.972)^{60}$
$\doteq 73$ kPa

c) The exponential model gives the most realistic answer, because it fits the data the best. Also, the pressure must be less than 170 kPa, but it cannot be negative.

14. As a population procreates, the population becomes larger, and thus, more and more organisms exist that can procreate some more. In other words, the act of procreating enables even more procreating in the future.

15. a) linear, quadratic, or exponential

b) linear or quadratic

c) exponential

16. a) $T(n) = \dfrac{1}{6}n^3 + \dfrac{1}{2}n^2 + \dfrac{1}{3}n$

b) $47\ 850 = \dfrac{1}{6}n^3 + \dfrac{1}{2}n^2 + \dfrac{1}{3}n$

So, $n \doteq 64.975$. So, it is not a tetrahedral number because n must be an integer.

17. a) Using a graphing calculator, the equation is $P(t) = 30.75(1.008\ 418)^t$

b) In 2000, the growth rate of Canada was less than the growth rate of Ontario and Alberta.

Chapter Review, pp. 576–577

1. The only operation that will result in both vertical and horizontal asymptotes is division.

2. a) Since shop 1 contains $-1.4t^2$, it sales are decreasing after 2000. Since shop 2 contains all addition, its sales are increasing after 2000.

b) $S_{1+2} = 700 - 1.4t^2 + t^3 + 3t^2 + 500$
$= t^3 + 1.6t^2 + 1200$

c) $t = 6$: $t^3 + 1.6t^2 + 1200$
$= (6)^3 + 1.6(6)^2 + 1200$
$= 216 + 1.6(36) + 1200$
$= 216 + 57.6 + 1200$
$= 1473.6$ thousand or 1 473 600

d) The owner should close the first shop, because the sales are decreasing and will eventually reach zero.

3. a) $C(x) = 9.45x + 52\ 000$

b) $I(x) = 15.8x$

c) $P(x) = I - C$
$= 15.8x - (9.45x + 52\ 000)$
$= 15.8x - 9.45x - 52\ 000$
$= 6.35x - 52\ 000$

4. a) $(f \times g)(x) = 3\tan(7x) \times 4\cos(7x)$
$= 12\sin(7x)$

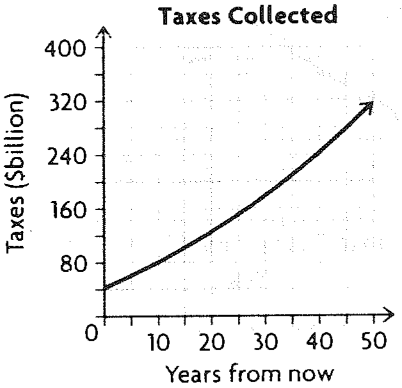
b) $$\begin{aligned}(f \times g)(x) &= \sqrt{3x^2} \times 3\sqrt{3x^2}\\ &= 3(3x^2)\\ &= 9x^2\end{aligned}$$

c) $$\begin{aligned}(f \times g)(x) &= (11x - 7)(11x + 7)\\ &= 121x^2 - 49\end{aligned}$$

d) $$\begin{aligned}(f \times g)(x) &= ab^x \times 2ab^{2x}\\ &= 2a^2b^{3x}\end{aligned}$$

5. a) $$\begin{aligned}C \times A &= (15\,000\,000(1.01)^t)(2850 + 200t)\\ &= 42\,750\,000\,000(1.01)^t\\ &\quad + 3\,000\,000\,000t(1.01)^t\end{aligned}$$

b)



d) $42\,750\,000\,000(1.01)^{26} + 3\,000\,000\,000t(1.01)^{26}$
$\doteq \$156\,402\,200\,032.31$

6. a) $$\begin{aligned}(f \div g)(x) &= 105x^3 \div 5x^4\\ &= \frac{21}{x}\end{aligned}$$

b) $$\begin{aligned}(f \div g)(x) &= (x - 4) \div (2x^2 + x - 36)\\ &= (x - 4) \div (x - 4)(2x + 9)\\ &= \frac{1}{2x + 9}\end{aligned}$$

c) $$\begin{aligned}(f \div g)(x) &= \sqrt{x + 15} \div (x + 15)\\ &= \frac{\sqrt{x + 15}}{x + 15}\end{aligned}$$

d) $$\begin{aligned}(f \div g)(x) &= 11x^5 \div 22x^2 \log x\\ &= \frac{x^3}{2 \log x}\end{aligned}$$

7. a) $\{x \in \mathbf{R} | x \neq 0\}$

b) $\left\{x \in \mathbf{R} \middle| x \neq 4, x \neq -\frac{9}{2}\right\}$

c) $\{x \in \mathbf{R} | x > -15\}$

d) $\{x \in \mathbf{R} | x > 0\}$

8. a) Domain of $f(x)$: $\{x \in \mathbf{R} | x > -1\}$
Range of $f(x)$: $\{y \in \mathbf{R} | y > 0\}$
Domain of $g(x)$: $\{x \in \mathbf{R}\}$
Range of $g(x)$: $\{y \in \mathbf{R} | y \geq 3\}$

b) $$\begin{aligned}f(g(x)) &= f(x^2 + 3)\\ &= \frac{1}{\sqrt{x^2 + 3 + 1}}\\ &= \frac{1}{\sqrt{x^2 + 4}}\end{aligned}$$

c) $$\begin{aligned}g(f(x)) &= g\left(\frac{1}{\sqrt{x + 1}}\right)\\ &= \left(\frac{1}{\sqrt{x + 1}}\right)^2 + 3\\ &= \frac{1}{x + 1} + 3\\ &= \frac{1}{x + 1} + \frac{3x + 3}{x + 1}\\ &= \frac{3x + 4}{x + 1}\end{aligned}$$

d) $$\begin{aligned}f(g(0)) &= \frac{1}{\sqrt{0^2 + 4}}\\ &= \frac{1}{\sqrt{4}}\\ &= \frac{1}{2}\end{aligned}$$

e) $$\begin{aligned}g(f(0)) &= \frac{3(0) + 4}{0 + 1}\\ &= \frac{4}{1}\\ &= 4\end{aligned}$$

f) For $f(g(x))$: $\{x \in \mathbf{R}\}$
For $g(f(x))$: $\{x \in \mathbf{R} | x > -1\}$

9. a) $$\begin{aligned}(f \circ f)(x) &= f(x - 3)\\ &= x - 3 - 3\\ &= x - 6\end{aligned}$$

b) $$\begin{aligned}(f \circ f \circ f)(x) &= f(x - 6)\\ &= x - 6 - 3\\ &= x - 9\end{aligned}$$

c) $$\begin{aligned}(f \circ f \circ f \circ f)(x) &= f(x - 9)\\ &= x - 9 - 3\\ &= x - 12\end{aligned}$$

d) f composed with itself n times $= x - 3(1 + n)$

10. a) $A(r) = \pi r^2$

b) $r(C) = \dfrac{C}{2\pi}$

c) $$\begin{aligned}A(r(C)) &= A\left(\frac{C}{2\pi}\right)\\ &= \pi\left(\frac{C}{2\pi}\right)^2\end{aligned}$$

$$= \pi\left(\frac{C^2}{4\pi^2}\right)$$

$$= \frac{C^2}{4\pi}$$

d) $\frac{C^2}{4\pi} = \frac{(3.6)^2}{4\pi}$

$\doteq 1.03$ m

11. Use the graph to find the solutions.

$f(x) < g(x)$: $-1.2 < x < 0$ or $x > 1.2$

$f(x) = g(x)$: $x = -1.2, 0,$ or 1.2

$f(x) > g(x)$: $x < -1.2$ or $0 < x < 1.2$

12. a) $-3 \csc x = x$

Try $x = 3$: $-3 \csc 3 = 3$

$-21.3 = 3$

Try $x = 4$: $-3 \csc 4 = 4$

$3.9 = 4$

So, $x \doteq 4.0$

b) $\cos^2 x = 3 - 2\sqrt{x}$

Try $x = 1$: $\cos^2 1 = 3 - 2\sqrt{1}$

$0.29 = 3 - 2(1)$

$0.29 = 1$

Try $x = 2$: $\cos^2 2 = 3 - 2\sqrt{2}$

$0.17 = 0.17$

So, $x \doteq 2.0$

c) $8^x = x^8$

Try $x = -0.7$: $8^{-0.7} = (-0.7)^8$

$0.23 = 0.06$

Try $x = -0.8$: $8^{-0.8} = (-0.8)^8$

$0.19 = 0.17$

So, $x \doteq -0.8$.

d) $7 \sin x = \frac{3}{x}$

Try $x = 0.6$: $7 \sin (0.6) = \frac{3}{0.6}$

$3.95 = 5$

Try $x = 0.7$: $7 \sin (0.7) = \frac{3}{0.7}$

$4.5 = 4.3$

So, $x \doteq 0.7$.

13. a) The rate of change is $(3200 - 2000) \div (7 - 5)$ or 600 frogs per year. So, the equation is:

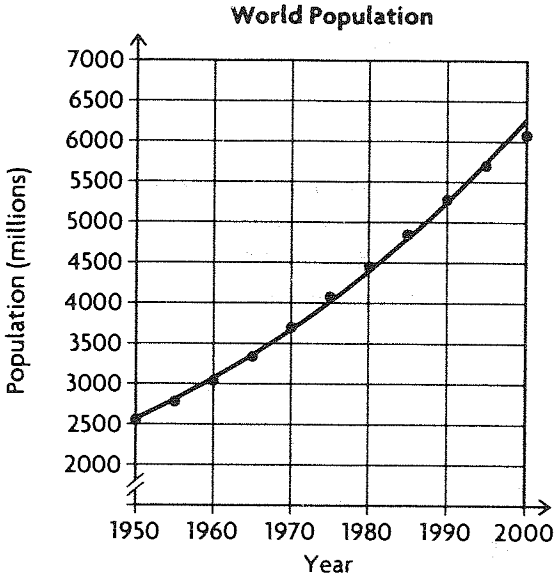
$P(t) = 600t - 1000$

The slope is the rate at which the population is changing.

b) $P(t) = 617.6(1.26)^t$

617.6 is the initial population and 1.26 represents the growth.

14. Use a graphing calculator to graph the points and the model, $P(t) = 2570.99(1.018)^t$.


World Population

Use the graph to estimate the values for $t = 13, 23,$ and 90.

When $t = 13$, $P(t) = 3242$.

When $t = 23$, $P(t) = 3875$.

When $t = 90$, $P(t) = 12\,806$.

Chapter Self-Test, p. 578

1. a) $A(r) = 4\pi r^2$

b) $V = \frac{4}{3}\pi r^3$

$\frac{3V}{4\pi} = r^3$

$\sqrt[3]{\frac{3V}{4\pi}} = r$

So, $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$

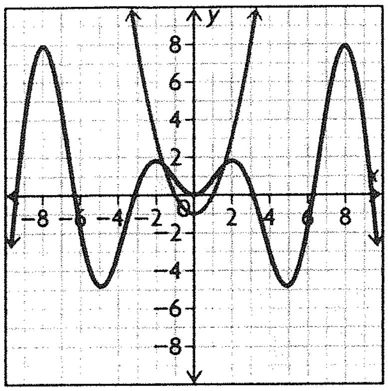
c) $A(r(V)) = A\left(\sqrt[3]{\frac{3V}{4\pi}}\right)$

$= 4\pi\left(\sqrt[3]{\frac{3V}{4\pi}}\right)^2$

$= 4\pi\left(\frac{3V}{4\pi}\right)^{\frac{2}{3}}$

d) $4\pi\left(\frac{3(0.75)}{4\pi}\right)^{\frac{2}{3}} \doteq 4$ m^2

2. Draw the graph of each function and use it to determine when $x \sin x \geq x^2 - 1$.



From the graph, the solution is $-1.62 \le x \le 1.62$.

3. Answers may vary. For example, $g(x) = x^7$ and $h(x) = 2x + 3$

$g(x) = (x + 3)^7$ and $h(x) = 2x$

4. a) Use a graphing calculator to determine the regression equation.

$N(n) = 1n^3 + 8n^2 + 40n + 400$

b)
$$\begin{aligned} N(3) &= 1(3)^3 + 8(3)^2 + 40(3) + 400 \\ &= 27 + 72 + 120 + 400 \\ &= 619 \end{aligned}$$

5.
$$\begin{aligned} f(x) &= 6x + b \\ -3 &= 6(2) + b \\ -3 &= 12 + b \\ -15 &= b \end{aligned}$$

So, $f(x) = 6x - 15$

$$\begin{aligned} g(x) &= 5(x + 8)^2 - 1 \\ (f \times g)(x) &= (6x - 15)(5(x + 8)^2 - 1) \\ &= (6x - 15)(5(x^2 + 16x + 64) - 1) \\ &= (6x - 15)(5x^2 + 80x + 320 - 1) \\ &= (6x - 15)(5x^2 + 80x + 319) \\ &= 30x^3 + 405x^2 + 714x - 4785 \end{aligned}$$

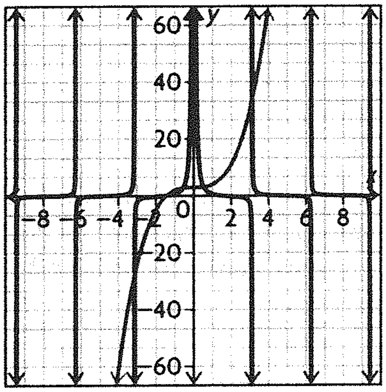
6. a) There is a horizontal asymptote of $y = 275$ cm. This is the maximum height this species will reach.

b)
$$\begin{aligned} 150 &= \frac{275}{1 + 26(0.85)^t} \\ 150(1 + 26(0.85)^t) &= 275 \\ 26(0.85)^t &= (275 \div 150) - 1 \\ 26(0.85)^t &\doteq 0.8333 \\ (0.85)^t &\doteq 0.032\,05 \\ t \log 0.85 &= \log 0.032\,05 \\ t &\doteq 21.2 \text{ months} \end{aligned}$$

7. Find when $C(x) = R(x)$
$$\begin{aligned} 5x + 18 &= 2x^2 \\ 0 &= 2x^2 - 5x - 18 \\ 0 &= (2x - 9)(x + 2) \end{aligned}$$
$x = 4.5$ or -2

Negative answers do not make sense in this context, so the answer is $x = 4.5$ or 4500 items.

8. Graph both sides on the equation and use the graph to find the solutions.



The solutions are $x = -3.1, -1.4, -0.6, 0.5,$ or 3.2.

9. Division will turn it into a tangent function that is not sinusoidal.

Chapters 7–9: Cumulative Review, pp. 580–583

1. Since $\sin \theta = \cos\left(\theta - \frac{\pi}{2}\right)$,
$$\begin{aligned} \sin \frac{2\pi}{5} &= \cos\left(\frac{2\pi}{5} - \frac{\pi}{2}\right) \\ &= \cos\left(\frac{4\pi}{10} - \frac{5\pi}{10}\right) \\ &= \cos\left(-\frac{\pi}{10}\right). \end{aligned}$$

Since $\cos \theta = \cos(2\pi - \theta)$,
$$\begin{aligned} \cos\left(-\frac{\pi}{10}\right) &= \cos\left(2\pi - \left(-\frac{\pi}{10}\right)\right) \\ &= \cos\left(2\pi + \frac{\pi}{10}\right). \end{aligned}$$

Since the period of the cosine function is 2π, $\cos\left(2\pi + \frac{\pi}{10}\right) = \cos \frac{\pi}{10}$. Therefore, answer (a) is correct. Also, since $\cos(\pi - \theta) = -\cos \theta$,
$$\begin{aligned} \cos \frac{\pi}{10} &= -\cos\left(\pi - \frac{\pi}{10}\right) \\ &= -\cos\left(\frac{10\pi}{10} - \frac{\pi}{10}\right) \\ &= -\cos \frac{9\pi}{10}. \end{aligned}$$

Therefore, answer (c) is correct. Also, since $\sin(\pi - \theta) = \sin\theta$,

$$\begin{aligned}\sin\frac{2\pi}{5} &= \sin\left(\pi - \frac{2\pi}{5}\right)\\ &= \sin\left(\frac{5\pi}{5} - \frac{2\pi}{5}\right)\\ &= \sin\frac{3\pi}{5}.\end{aligned}$$

Therefore, answer (b) is correct. Since answers (a), (b), and (c) are all correct, the correct answer is **d)**.

2. Since $\cos(a - b) = \cos a\cos b + \sin a\sin b$,

$$\begin{aligned}\cos\frac{\pi}{12} &= \cos\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right)\\ &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{2} + \sqrt{6}}{4}.\end{aligned}$$

Therefore, the correct answer is **b)**.

3. Since $\sin\alpha = \frac{12}{13}$, the leg opposite the angle α in a right triangle has a length of 12, while the hypotenuse of the right triangle has a length of 13. For this reason, the other leg of the right triangle can be calculated as follows:

$$\begin{aligned}x^2 + 12^2 &= 13^2\\ x^2 + 144 &= 169\\ x^2 + 144 - 144 &= 169 - 144\\ x^2 &= 25\\ x &= 5\end{aligned}$$

Since $\tan\alpha = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan\alpha = \frac{12}{5}$. In addition, $\sin\beta = \frac{8}{17}$, the leg opposite the angle β in a right triangle has a length of 8, while the hypotenuse of the right triangle has a length of 17. For this reason, the other leg of the right triangle can be calculated as follows:

$$\begin{aligned}x^2 + 8^2 &= 17^2\\ x^2 + 64 &= 289\\ x^2 + 64 - 64 &= 289 - 64\\ x^2 &= 225\\ x &= 15\end{aligned}$$

Since $\tan\beta = \frac{\text{opposite leg}}{\text{adjacent leg}}$, $\tan\beta = \frac{8}{15}$.

Since $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a\tan b}$,

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\\ &= \frac{\frac{12}{5} + \frac{8}{15}}{1 - \left(\frac{12}{5}\right)\left(\frac{8}{15}\right)}\\ &= \frac{\frac{36}{15} + \frac{8}{15}}{1 - \frac{96}{75}}\\ &= \frac{\frac{44}{15}}{\frac{75}{75} - \frac{96}{75}}\\ &= \frac{\frac{44}{15}}{-\frac{21}{75}}\\ &= \frac{44}{15} \times \left(-\frac{75}{21}\right)\\ &= -\frac{3300}{315}\\ &= -\frac{220}{21}\end{aligned}$$

Therefore, the correct answer is **a)**.

4. Since $\sin\theta = \frac{3}{8}$, the leg opposite the angle θ in a right triangle has a length of 3, while the hypotenuse of the right triangle has a length of 8. For this reason, the other leg of the right triangle can be calculated as follows:

$$\begin{aligned}x^2 + 3^2 &= 8^2\\ x^2 + 9 &= 64\\ x^2 + 9 - 9 &= 64 - 9\\ x^2 &= 55\\ x &= \sqrt{55}\end{aligned}$$

Since $\tan\theta = \dfrac{\text{opposite leg}}{\text{adjacent leg}}$,

$$\tan\theta = -\frac{3}{\sqrt{55}} = -\frac{3\sqrt{55}}{55}.$$

(The reason the sign is negative is because angle θ is in the second quadrant.)

Since $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$,

$$\begin{aligned}\tan 2\theta &= \frac{(2)\left(-\frac{3\sqrt{55}}{55}\right)}{1 - \left(-\frac{3\sqrt{55}}{55}\right)^2}\\ &= \frac{-\frac{6\sqrt{55}}{55}}{1 - \frac{495}{3025}}\\ &= \frac{-\frac{6\sqrt{55}}{55}}{\frac{3025}{3025} - \frac{495}{3025}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{-\frac{6\sqrt{55}}{55}}{\frac{2530}{3025}} \\
&= \frac{6\sqrt{55}}{55} \times \frac{3025}{2530} \\
&= -\frac{18\,150\sqrt{55}}{13\,9150} \\
&= -\frac{3\sqrt{55}}{23}
\end{aligned}
$$

Therefore, the correct answer is **a**).

5. Since $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$,

$$
\begin{aligned}
\cos\frac{\pi}{8} &= \pm\sqrt{\frac{1 + \cos\frac{\pi}{4}}{2}} \\
&= \pm\sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} \\
&= \pm\sqrt{\frac{\frac{2}{2} + \frac{\sqrt{2}}{2}}{2}} \\
&= \pm\sqrt{\frac{\frac{2 + \sqrt{2}}{2}}{2}} \\
&= \pm\sqrt{\frac{2 + \sqrt{2}}{4}} \\
&= \pm\frac{\sqrt{2 + \sqrt{2}}}{2}
\end{aligned}
$$

Since the angle $\frac{\pi}{8}$ is in the first quadrant, the sign of $\cos\frac{\pi}{8}$ is positive. Therefore, the correct answer is **d**).

6. The expression $\frac{2 - \sec^2\left(\frac{1}{2}x\right)}{\sec^2\left(\frac{1}{2}x\right)}$ can be simplified as follows:

$$
\begin{aligned}
\frac{2 - \sec^2\left(\frac{1}{2}x\right)}{\sec^2\left(\frac{1}{2}x\right)} &= \frac{2}{\sec^2\left(\frac{1}{2}x\right)} - \frac{\sec^2\left(\frac{1}{2}x\right)}{\sec^2\left(\frac{1}{2}x\right)} \\
&= 2\cos^2\left(\frac{1}{2}x\right) - 1
\end{aligned}
$$

Since $\cos 2\theta = 2\cos^2\theta - 1$,

$$2\cos^2\left(\frac{1}{2}x\right) - 1 = \cos 2\left(\frac{1}{2}x\right) = \cos x.$$

Therefore, the correct answer is **c**).

7. The identity $\frac{2\tan x}{1 + \tan^2 x} = \sin 2x$ can be proven as follows:

$$
\begin{aligned}
\frac{2\tan x}{1 + \tan^2 x} &= \frac{2\frac{\sin x}{\cos}}{\sec^2 x} \\
&= 2\frac{\sin x}{\cos x} \times \cos^2 x \\
&= 2\sin x\cos x \\
&= \sin 2x.
\end{aligned}
$$

The identities $1 + \tan^2 x = \sec^2 x$, $\sin 2x = 2\sin x\cos x$, and $\tan x = \frac{\sin x}{\cos x}$ were all used in the proof.

Therefore, the correct answer is **d**).

8. The equation $5 + 7\sin\theta = 0$ can be solved as follows:

$$
\begin{aligned}
5 + 7\sin\theta &= 0 \\
5 + 7\sin\theta - 5 &= 0 - 5 \\
7\sin\theta &= -5 \\
\frac{7\sin\theta}{7} &= -\frac{5}{7} \\
\sin\theta &= -\frac{5}{7} \\
\sin^{-1}(\sin\theta) &= \sin^{-1}\left(-\frac{5}{7}\right) \\
\theta &= \sin^{-1}\left(-\frac{5}{7}\right)
\end{aligned}
$$

$\theta = -0.80$ or -2.35

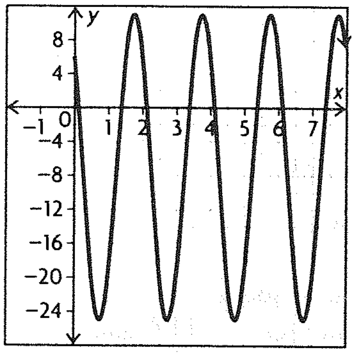
Therefore, the correct answer is **b**).

9. The blade tip is at least 30 m above the ground when $18\cos\left(\pi t + \frac{\pi}{4}\right) + 23 \geq 30$. This inequality can be simplified as follows:

$$
\begin{aligned}
18\cos\left(\pi t + \frac{\pi}{4}\right) + 23 &\geq 30 \\
18\cos\left(\pi t + \frac{\pi}{4}\right) + 23 - 30 &\geq 30 - 30 \\
18\cos\left(\pi t + \frac{\pi}{4}\right) - 7 &\geq 0
\end{aligned}
$$

The inequality $18\cos\left(\pi t + \frac{\pi}{4}\right) - 7 \geq 0$ can be solved by graphing the function $h(t) = 18\cos\left(\pi t + \frac{\pi}{4}\right) - 7$ and determining where the graph is at or above the x-axis.

The graph is as follows:



The graph is at or above the x-axis in the intervals $1.37 \le x \le 2.12$, $3.37 \le x \le 4.12$ and $5.37 \le x \le 6.12$. Therefore, the correct answer is **c)**.

10. The equation $(2 \sin x + 1)(\cos x - 1) = 0$ is true when either $2 \sin x + 1 = 0$ or $\cos x - 1 = 0$ (or both). If $2 \sin x + 1 = 0$, x can be solved for as follows:

$$\begin{aligned} 2 \sin x + 1 &= 0 \\ 2 \sin x + 1 - 1 &= 0 - 1 \\ 2 \sin x &= -1 \\ \frac{2 \sin x}{2} &= -\frac{1}{2} \\ \sin x &= -\frac{1}{2} \end{aligned}$$

The solutions to the equation $\sin x = -\frac{1}{2}$ occur at $x = 210°$ or $330°$.

If $\cos x - 1 = 0$, x can be solved for as follows:

$$\begin{aligned} \cos x - 1 &= 0 \\ \cos x - 1 + 1 &= 0 + 1 \\ \cos x &= 1 \end{aligned}$$

The solutions to the equation $\cos x = 1$ occur at $x = 0°$ or $360°$. Therefore, the correct answer is **d)**.

11. Since the solutions to the equation are $\theta = 0$, $\frac{\pi}{3}$, $\frac{5\pi}{3}$, or 2π, either $\cos \theta = 1$ or $\cos \theta = \frac{1}{2}$. (This is because $\cos 0 = 1$, $\cos \frac{\pi}{3} = \frac{1}{2}$, $\cos \frac{5\pi}{3} = \frac{1}{2}$, and $\cos 2\pi = 1$.) For this reason, either $\cos \theta - 1 = 0$ or $\cos \theta - \frac{1}{2} = 0$ (or both). If the left sides of these two equations are considered factors of a quadratic equation and multiplied together, the result is as follows:

$$\begin{aligned} (\cos \theta - 1)\left(\cos \theta - \frac{1}{2}\right) &= 0 \\ \cos^2 \theta - \cos \theta - \frac{1}{2} \cos \theta + \frac{1}{2} &= 0 \\ \cos^2 \theta - \frac{3}{2} \cos \theta + \frac{1}{2} &= 0 \end{aligned}$$

If both sides of the equation are multiplied by 2, the result is as follows:

$$\begin{aligned} (2)\left(\cos^2 \theta - \frac{3}{2} \cos \theta + \frac{1}{2}\right) &= (2)(0) \\ 2 \cos^2 \theta - 3 \cos \theta + 1 &= 0 \end{aligned}$$

Since $2 \cos^2 \theta - 1 = \cos 2\theta$, the equation can be rewritten as follows:

$$\begin{aligned} 2 \cos^2 \theta - 1 + 1 - 3 \cos \theta + 1 &= 0 \\ \cos 2\theta + 1 - 3 \cos \theta + 1 &= 0 \\ \cos 2\theta - 3 \cos \theta + 2 &= 0 \end{aligned}$$

Therefore, the correct answer is **a)**.

12. In the equation $y = \log_7 x$, y is the power to which 7 must be raised in order to produce x. Therefore, the exponential form of $y = \log_7 x$ is $x = 7^y$, and the correct answer is **b)**.

13. In logarithmic functions of the form $f(x) = a \log_{10}[k(x - d)] + c$, if a is negative, $f(x)$ is reflected in the x-axis. Also, If $0 < |k| < 1$, a horizontal stretch of factor $\left|\frac{1}{k}\right|$ occurs, and if $f(x) = \log_{10} x$ a vertical translation of c units up occurs. Therefore, if $b = 1.0117$ is reflected in the x-axis, stretched horizontally by a factor of 3, and translated 2 units up, the resulting function is $f(x) = -\log_{10}\left(\frac{1}{3}x\right) + 2$, and the correct answer is **d)**.

14. Since the value of $\log_7 49$ is the power to which 7 must be raised to produced 49, $\log_7 49 = 2$. Therefore, $7^{\log_7 49} = 7^2 = 49$, so the correct answer is **d)**.

15. The length of the planet's year in days can be calculated as follows:

$$\begin{aligned} \log_{10} T &= 1.5 \log_{10} d - 0.45 \\ \log_{10} T &= 1.5 \log_{10} 11 - 0.45 \\ \log_{10} T &= \log_{10} 11^{1.5} - 0.45 \\ \log_{10} T - \log_{10} 11^{1.5} &= \log_{10} 11^{1.5} - 0.45 \\ &\quad - \log_{10} 11^{1.5} \\ \log_{10} T - \log_{10} 11^{1.5} &= -0.45 \\ \log_{10} \frac{T}{11^{1.5}} &= -0.45 \\ 10^{-0.45} &= \frac{T}{11^{1.5}} \\ 10^{-0.45} \times 11^{1.5} &= \frac{T}{11^{1.5}} \times 11^{1.5} \\ T &= 10^{-0.45} \times 11^{1.5} \\ T &= 0.3548 \times 36.4829 \\ T &= 12.9 \end{aligned}$$

Therefore, the correct answer is **c)**.

16. The equation $\log_4 x + 3 = \log_4 1024$ can be solved as follows:

$$\begin{aligned}
\log_4 x + 3 &= \log_4 1024 \\
\log_4 x + 3 - \log_4 x &= \log_4 1024 - \log_4 x \\
\log_4 1024 - \log_4 x &= 3 \\
\log_4 \frac{1024}{x} &= 3 \\
4^3 &= \frac{1024}{x} \\
64 &= \frac{1024}{x} \\
64 \times x &= \frac{1024}{x} \times x \\
64x &= 1024 \\
\frac{64x}{64} &= \frac{1024}{64} \\
x &= 16
\end{aligned}$$

Therefore, the correct answer is **a)**.

17. $\log_5 25x = \log_5 25 + \log_5 x$

$= 2 + \log_5 x$

So $g(x)$ is a vertical translation of $f(x)$ 2 units up, and the correct answer is **b)**.

18. The equation $x = \log_3 27\sqrt{3}$ can be rewritten as $3^x = 27\sqrt{3}$. Since $27 = 3^3$ and $\sqrt{3} = 3^{1/2}$, the equation $3^x = 27\sqrt{3}$ can be rewritten as $3^x = 3^3 \times 3^{1/2}$. By adding the exponents on the right side of the equation, the equation becomes $3^x = 3^{3\frac{1}{2}}$. Therefore, x must equal $3\frac{1}{2}$, and the correct answer is **b)**.

19. Since the formula for compound interest is $A = P(1 + i)^n$, the length of time it will take for the investment to be worth more than \$6400 can be calculated as follows:

$$\begin{aligned}
6400 &< 1600(1 + 0.01)^n \\
\frac{6400}{1600} &< \frac{1600(1 + 0.01)^n}{1600} \\
4 &< (1 + 0.01)^n \\
\log_{10} 4 &< \log_{10}((1 + 0.01)^n) \\
\log_{10} 4 &< n \log_{10}(1 + 0.01) \\
n &> \frac{\log_{10} 4}{\log_{10}(1 + 0.01)} \\
n &> \frac{\log_{10} 4}{\log_{10} 1.01} \\
n &> \frac{0.6021}{0.0043} \\
n &> 140
\end{aligned}$$

Since n represents the number of months it will take for the investment to be worth more than \$6400, and since there are 12 months in a year, the number of years it will take for the investment to be worth more than \$6400 is 11 years and 8 months. Therefore, the correct answer is **c)**.

20. Since the formula for the loudness of sound is $L = 10\log\left(\frac{I}{I_0}\right)$, the intensity of the sound of a jet taking off with a loudness of 133 dB can be calculated as follows:

$$\begin{aligned}
133 &= 10\log\left(\frac{I}{10^{-12}}\right) \\
\frac{133}{10} &= \frac{10\log\left(\frac{I}{10^{-12}}\right)}{10} \\
\frac{133}{10} &= \log\left(\frac{I}{10^{-12}}\right) \\
10^{\frac{133}{10}} &= \frac{I}{10^{-12}} \\
10^{13.3} &= \frac{I}{10^{-12}} \\
10^{13.3} \times 10^{-12} &= \frac{I}{10^{-12}} \times 10^{-12} \\
I &= 10^{1.3} \\
I &= 20.0 \text{ W/m}^2
\end{aligned}$$

Therefore, the correct answer is **d)**.

21. The equation $\log_a(x - 3) + \log_a(x - 2) = \log_a(5x - 15)$ can be rewritten as follows:

$$\begin{aligned}
\log_a(x - 3) + \log_a(x - 2) &= \log_a(5x - 15) \\
\log_a(x - 3)(x - 2) &= \log_a(5x - 15)
\end{aligned}$$

For this reason, $(x - 3)(x - 2) = 5x - 15$. This equation can be solved as follows:

$$\begin{aligned}
(x - 3)(x - 2) &= 5x - 15 \\
x^2 - 3x - 2x + 6 &= 5x - 15 \\
x^2 - 5x + 6 &= 5x - 15 \\
x^2 - 5x + 6 - 5x + 15 &= 5x - 15 - 5x + 15 \\
x^2 - 10x + 21 &= 0 \\
(x - 7)(x - 3) &= 0 \\
x = 7 \text{ or } x &= 3
\end{aligned}$$

Since it's impossible to find the log of 0, $x = 3$ is not a valid answer, because if 3 is substituted back into the original equation, both sides of the equation would have a term of $\log_a 0$. Therefore, the correct answer is **b)**.

22. Since Carbon–14 has a half-life of 5730 years, the following equation holds true:

$$0.017 = (3.9)\left(\frac{1}{2}\right)^{\frac{x}{5730}}$$

This equation can be solved as follows:

$$0.017 = (3.9)\left(\frac{1}{2}\right)^{\frac{x}{5730}}$$

$$\frac{0.017}{3.9} = \frac{(3.9)(\frac{1}{2})^{\frac{x}{5730}}}{3.9}$$

$$0.0044 = \left(\frac{1}{2}\right)^{\frac{x}{5730}}$$

$$\log 0.0044 = \log\left(\frac{1}{2}\right)^{\frac{x}{5730}}$$

$$\log 0.0044 = \frac{x}{5730}\log\frac{1}{2}$$

$$-2.3606 = \left(\frac{x}{5730}\right)(-0.3010)$$

$$-2.3606 = -0.000\,05x$$

$$\frac{-2.3606}{-0.000\,05} = \frac{-0.000\,05x}{-0.000\,05}$$

$$x = 44\,933$$

The closest answer is 45 000 years, so the correct answer is **a)**.

23. In each of the next 5 years, the cost of goods and services will increase by 3.1 percent. In other words, in each of the next 5 years, the cost of goods and services will be 103.1 percent of what it was the previous year. Since 103.1 percent written as a decimal is 1.031, the cost of goods and services now should be multiplied by $(1.031)^t$ to find the cost of goods and services after t years. Since the cost of good and services now is P, the correct answer is **c)**.

24. If the population of a city is currently 150 000 and is increasing at 2.3 percent per year, the population of the city in 6 years can be calculated as follows:

$P = (150\,000)(1.023)^6$

$P = (150\,000)(1.1462)$

$P = 171\,927$

Since the population in 6 years will be 171 927, and since the population is increasing at 2.3 percent per year, the number of people by which the population will increase in the 7th year from now will be $(171\,927)(0.023) = 3954$. The closest answer is 4000, so the correct answer is **c)**.

25. It's apparent from the graph that as x moves away from 0 in the negative direction, y becomes smaller and smaller, while as x moves away from 0 in the positive direction, y becomes larger and larger. For this reason, (a) cannot be the correct answer, since as x moves away from 0 in the negative direction, x^2 becomes larger and larger. Also, (b) cannot be the correct answer, since the domain of $\log x$ is $\{x \in \mathbf{R} \mid x > 0\}$, and (d) cannot be the correct answer, since as x moves away from 0 in the positive direction, 0.5^x becomes smaller and smaller. Therefore, the correct answer is **c)**.

26. The domain of $f - g$ is the intersection of the domain of f and the domain of g. Since the domain of f is $\{x \in \mathbf{R} \mid x > 0\}$, and the domain of g is $\{x \in \mathbf{R} \mid x \neq 3\}$, the domain of $f - g$ is $\{x \in \mathbf{R} \mid x > 0, x \neq 3\}$. Therefore, the correct answer is **b)**.

27. The sum or difference of an odd function and an even function is neither even nor odd, unless one of the functions is identically zero, so neither (b) nor (c) can be the correct answer. The difference of two even functions is even, so (d) cannot be the correct answer. Since the sum of two odd functions is always an odd function, the correct answer is **a)**.

28. If $f(x) = g(x) = \sec x$, then $f(x) \times g(x) = \sec^2 x$. Since the range of both $f(x) = \sec x$ and $g(x) = \sec x$ is $\{y \in \mathbf{R} \mid -1 \geq y \geq 1\}$, the range of $f(x) \times g(x) = \sec^2 x$ must be $\{y \in \mathbf{R} \mid y \geq 1\}$. This is because a negative number squared is always positive, and $(-1)^2 = 1$. Therefore, the correct answer is **a)**.

29. If $f(x) = ax^2 + 3$ and $g(x) = bx - 1$, then $(f \times g)(x)$ can be calculated as follows:

$(f \times g)(x) = (ax^2 + 3)(bx - 1)$

$(f \times g)(x) = abx^3 - ax^2 + 3bx - 3$

Since $(f \times g)(x)$ passes through the point $(-1, -3)$, the function can be rewritten as follows:

$$-3 = ab(-1)^3 - a(-1)^2 + 3b(-1) - 3$$

$$-3 = -ab - a - 3b - 3$$

$$-3 + 3 = -ab - a - 3b - 3 + 3$$

$$0 = -ab - a - 3b$$

Also, since $(f \times g)(x)$ passes through the point $(1, 9)$, the function can be rewritten as follows:

$$9 = ab(1)^3 - a(1)^2 + 3b(1) - 3$$

$$9 = ab - a + 3b - 3$$

$$9 + 3 = ab - a + 3b - 3 + 3$$

$$12 = ab - a + 3b$$

If the equations $0 = -ab - a - 3b$ and $12 = ab - a + 3b$ are added together, the resulting equation is $12 = -2a$, or $a = -6$.

If -6 is substituted for a into the equation $0 = -ab - a - 3b$, the equation can be rewritten as follows:

$$0 = -(-6)b - (-6) - 3b$$
$$0 = 6b + 6 - 3b$$
$$0 = 3b + 6$$
$$0 - 6 = 3b + 6 - 6$$
$$3b = -6$$
$$b = -2$$

Therefore, the correct answer is **d)**.

30. Since $f(x) = \log x$ and $g(x) = |x - 2|$, $(f \div g)(x) = \frac{\log x}{|x - 2|}$. Since the denominator can never equal 0, x can never equal 2. Also, since it's impossible to find the log of a number less than or equal to 0, x must be greater than 0. Therefore, the domain of $(f \div g)(x)$ is $\{x \in \mathbf{R} | x > 0, x \neq 2\}$, and the correct answer is **d)**.

31. The domain of $f(x) = \sqrt{3 - x}$ is $\{x \in \mathbf{R} | x \leq 3\}$. For this reason, the range of $g(x) = 3x^2$ that is permissible is $\{y \in \mathbf{R} | y \leq 3\}$, and the domain of $f \circ g$ can be calculated as follows:

$$3x^2 \leq 3$$
$$\frac{3x^2}{3} \leq \frac{3}{3}$$
$$x^2 \leq 1$$
$$-1 \leq x \leq 1$$

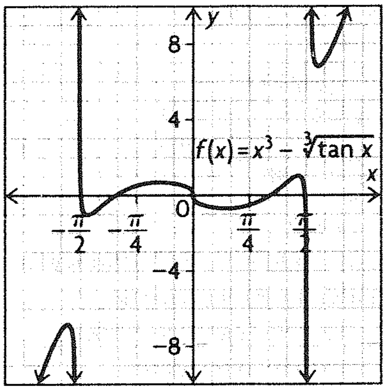
Therefore, the domain of $f \circ g$ is $\{x \in \mathbf{R} | -1 \leq x \leq 1\}$, and the correct answer is **c)**.

32. It's clear from the graph that two points on the line are (0, 3) and (4, -5). Since it is already known from the point (0, 3) that the y-intercept of the line is 3, all that is needed to determine the equation of the line is its slope. The slope of the line can be calculated as follows:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{-5 - 3}{4 - 0}$$
$$m = \frac{-8}{4}$$
$$m = -2$$

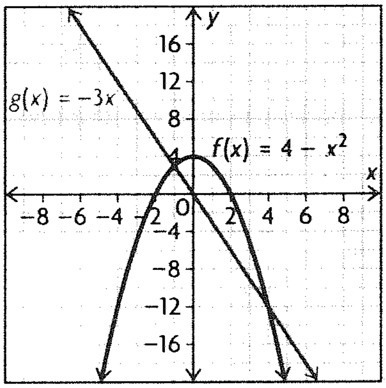
Since the y-intercept of the line is 3 and the slope of the line is -2, the equation of the line is $y = -2x + 3$. Because $(h \circ f)(x) = 3 - 2x = -2x + 3$, the correct answer is **d)**.

33. To solve the equation $x^3 = \sqrt[3]{\tan x}$, first subtract $\sqrt[3]{\tan x}$ from both sides of the equation to produce the equation $x^3 - \sqrt[3]{\tan x} = 0$. Then graph the function $f(x) = x^3 - \sqrt[3]{\tan x}$ and determine the x-coordinates of the points where the graph crosses the x-axis—these are the solutions to the equation. The graph of $f(x) = x^3 - \sqrt[3]{\tan x}$ is as follows:



The graph crosses the x-axis at about $x = -1.55, -1.07, 0, 1.07$, and 1.55. Therefore, the correct answer is **d)**.

34. The functions $f(x) = 4 - x^2$ and $g(x) = -3x$ graphed on the same coordinate grid are as follows:



It's apparent from the graph that $f(x) < g(x)$ when $x < -1$ and when $x > 4$. Therefore, the correct answer is **b)**.

35. Since the horizontal distance that a football can be thrown can be modelled by the function $d = \frac{v^2}{9.8}\sin 2\theta + 1.8$, the angle at which the football was thrown can be calculated as follows:

$$d = \frac{v^2}{9.8}\sin 2\theta + 1.8$$
$$35 = \frac{20^2}{9.8}\sin 2\theta + 1.8$$
$$35 = \frac{400}{9.8}\sin 2\theta + 1.8$$

$$
\begin{aligned}
35 - 1.8 &= \frac{400}{9.8}\sin 2\theta + 1.8 - 1.8\\
33.2 &= \frac{400}{9.8}\sin 2\theta\\
33.2 \times \frac{9.8}{400} &= \frac{400}{9.8}\sin 2\theta \times \frac{9.8}{400}\\
\sin 2\theta &= 0.8134\\
\sin^{-1}(\sin 2\theta) &= \sin^{-1}(0.8134)\\
2\theta &= 54.4295^\circ \text{ or } 125.5705^\circ\\
\frac{2\theta}{2} &= \frac{54.4295^\circ}{2} \text{ or } \frac{125.5705^\circ}{2}\\
\theta &= 27^\circ \text{ or } 63^\circ
\end{aligned}
$$

Therefore, the football could have been thrown at either 27° or 63° relative to the horizontal.

36. a) Answers may vary. For example, since population growth is usually exponential, suitable models for the population of Niagara and Waterloo could be exponential functions in the form $P(x) = ab^x$. If the year 1996 is considered $t = 0$, the model for Niagara can be developed as follows:

$$
\begin{aligned}
P(x) &= ab^x\\
414.8 &= ab^0\\
414.8 &= a(1)\\
a &= 414.8
\end{aligned}
$$

Since $a = 414.8$, the model can now be developed as follows:

$$
\begin{aligned}
P(x) &= (414.8)(b^x)\\
476.8 &= (414.8)(b^{32})\\
\frac{476.8}{414.8} &= \frac{(414.8)(b^{32})}{414.8}\\
b^{32} &= 1.1495\\
b &= 1.0044
\end{aligned}
$$

Since $b = 1.0044$, the model for Niagara is $P(x) = (414.8)(1.0044^x)$.

The model for Waterloo can be developed as follows:

$$
\begin{aligned}
P(x) &= ab^x\\
418.3 &= ab^0\\
418.3 &= a(1)\\
a &= 418.3
\end{aligned}
$$

Since $a = 418.3$, the model can now be developed as follows:

$$
\begin{aligned}
P(x) &= (418.3)(b^x)\\
606.1 &= (418.3)(b^{32})\\
\frac{606.1}{418.3} &= \frac{(418.3)(b^{32})}{418.3}\\
b^{32} &= 1.4490\\
b &= 1.0117
\end{aligned}
$$

Since $b = 1.0117$, the model for Waterloo is $P(x) = (418.3)(1.0117^x)$.

b) Answers may vary. For example, to estimate the doubling time for Niagara, the model $P(x) = (414.8)(1.0044^x)$ could be used as follows:

$$
\begin{aligned}
P(x) &= (414.8)(1.0044^x)\\
829.6 &= (414.8)(1.0044^x)\\
\frac{829.6}{414.8} &= \frac{(414.8)(1.0044^x)}{414.8}\\
1.0044^x &= 2\\
\log 1.0044^x &= \log 2\\
(x)(\log 1.0044) &= \log 2\\
\frac{(x)(\log 1.0044)}{\log 1.0044} &= \frac{\log 2}{\log 1.0044}\\
x &= \frac{0.3010}{0.0019}\\
x &= 159 \text{ years}
\end{aligned}
$$

To estimate the doubling time for Waterloo, the model $P(x) = (418.3)(1.0117^x)$ can be used as follows:

$$
\begin{aligned}
P(x) &= (418.3)(1.0117^x)\\
836.6 &= (418.3)(1.0117^x)\\
\frac{836.6}{418.3} &= \frac{(418.3)(1.0117^x)}{418.3}\\
1.0117^x &= 2\\
\log 1.0117^x &= \log 2\\
(x)(\log 1.0117) &= \log 2\\
\frac{(x)(\log 1.0117)}{\log 1.0117} &= \frac{\log 2}{\log 1.0117}\\
x &= \frac{0.3010}{0.0050}\\
x &= 60 \text{ years}
\end{aligned}
$$

c) Answers may vary. For example, to calculate the rate at which Niagara's population will be growing in 2025, first it's necessary to find the projected population in 2024, and then it's necessary to find the projected population in 2025. The projected population in 2024 can be found as follows:

$$
\begin{aligned}
P(x) &= (414.8)(1.0044^{28})\\
P(x) &= (414.8)(1.1308)\\
P(x) &= 469.1
\end{aligned}
$$

The projected population in 2025 can be found as follows:

$$
\begin{aligned}
P(x) &= (414.8)(1.0044^{29})\\
P(x) &= (414.8)(1.1358)\\
P(x) &= 471.1
\end{aligned}
$$

Therefore, the rate at which Niagara's population will be growing in 2025 is 471.1 − 469.1 = 2 thousand people per year. To calculate the rate at which Waterloo's population will be growing in 2025, first it's necessary to find the projected population in 2024, and then it's necessary to find the projected population in 2025. The projected population in 2024 can be found as follows:

$P(x) = (418.3)(1.0117^{28})$
$P(x) = (418.3)(1.3850)$
$P(x) = 579.3$

The projected population in 2025 can be found as follows:

$P(x) = (418.3)(1.0117^{29})$
$P(x) = (418.3)(1.4012)$
$P(x) = 586.1$

Therefore, the rate at which Waterloo's population will be growing in 2025 is $586.1 - 579.3 = 6.8$ thousand people per year. Since Niagara's population will be growing at 2 thousand people per year and Waterloo's population will be growing at 6.8 thousand people per year, Waterloo's population will be growing faster.

37. Since the mass of the rocket just before launch is 30 000 kg, and since its mass is decreasing at 100 kg/s, $m(t) = 30\,000 - 100t$. Since $T - 10m = ma$, a can be isolated as follows:

$$T - 10m = ma$$
$$\frac{T - 10m}{m} = \frac{ma}{m}$$
$$a = \frac{T - 10m}{m}$$
$$a = \frac{T}{m} - 10$$

Since $m(t) = 30\,000 - 100t$, $a(t)$ can be determined as follows:

$$a = \frac{T}{m} - 10$$
$$a(t) = \frac{T}{30\,000 - 100t} - 10$$

Since $m = 30\,000(2.72)^{-v-gt}$, v can be isolated as follows:

$$m = 30\,000(2.72)^{-v-gt}$$
$$\frac{m}{30\,000} = \frac{30\,000(2.72)^{-v-gt}}{30\,000}$$
$$\frac{m}{30\,000} = (2.72)^{-v-gt}$$
$$\log\frac{m}{30\,000} = \log((2.72)^{-v-gt})$$
$$\log\frac{m}{30\,000} = (-v - gt)(\log 2.72)$$
$$\frac{\log\frac{m}{30\,000}}{\log 2.72} = \frac{(-v - gt)(\log 2.72)}{\log 2.72}$$
$$-v - gt = \frac{\log\frac{m}{30\,000}}{\log 2.72}$$
$$-v - gt + gt = \frac{\log\frac{m}{30\,000}}{\log 2.72} + gt$$
$$-v = \frac{\log\frac{m}{30\,000}}{\log 2.72} + gt$$
$$-v \times -1 = \left(\frac{\log\frac{m}{30\,000}}{\log 2.72} + gt\right) \times -1$$
$$v = -\frac{\log\frac{m}{30\,000}}{\log 2.72} - gt$$

Since $m(t) = 30\,000 - 100t$, $v(t)$ can be determined as follows:

$$v = -\frac{\log\frac{m}{30\,000}}{\log 2.72} - gt$$
$$v(t) = -\frac{\log\frac{30\,000 - 100t}{30\,000}}{\log 2.72} - gt$$
$$v(t) = -\frac{\log\left(1 - \frac{t}{300}\right)}{\log 2.72} - gt$$

In order for the rocket to get off the ground, $a(0)$ must be greater than 0. Therefore, the constraints on the value of T can be determined as follows:

$$a(t) = \frac{T}{30\,000 - 100t} - 10$$
$$0 < \frac{T}{30\,000 - 100(0)} - 10$$
$$0 < \frac{T}{30\,000} - 10$$
$$0 + 10 < \frac{T}{30\,000} - 10 + 10$$
$$10 < \frac{T}{30\,000}$$
$$10 \times 30\,000 < \frac{T}{30\,000} \times 30\,000$$
$$T > 300\,000 \text{ N}$$